Chapter 3

Self-Consistent Theory of Localization

Since the discovery of scaling theory, the physics of weak localization became a subject of intense interest in disordered solids. As mentioned earlier, the new features of the theory, like localization in two dimensions at any disorder have been understood as being due to interference between time-reversed paths that return to the same point. This constructive interference leads to scale dependent correction to conductivity, which imply localization of the entire band in dimensions two and below [11, 12, 10]. In this chapter, we examine the robustness of the subtle interference effects, by considering localization in films of finite thickness. We will study cross-over from two to three dimensions in thin films.

There has been a lot of interest in the two-dimensional localization problem since the observation of metal-insulator transition (MIT) in MOSFETs with the variation of electron concentration [31, 32]. This MIT also exhibits remarkable behaviour in magnetic fields parallel and perpendicular to the film [33, 32]. Since the zero-field observation is contrary to the conclusions of the scaling theory, other physical factors like electron-electron interactions need to be invoked to account for this transition [34, 35]. However, till now no satisfactory theory that explains all the features associated with this transition has been given. We feel that one might possibly gain some insights into this unresolved situation by studying the phenomena with the variation of the thickness of the film, though here we have not considered electron-electron interactions. There have been a few experimental studies in which the electronic transport has been studied as function of the film thickness [50, 51, 52]. One does see here some evidence of the delocalization of states.

In the previous chapter, we have studied the disorder-induced localization in thin films numerically, and found evidence for a delocalization transition as function of thickness. In this chapter, we examine the problem analytically using the self-consistent theory of localization due to Vollhardt and Wölfle [39, 40]. This allows us to understand how the back-scattering corrections mentioned above, operate as we go from 2 to 3 dimensions by increasing the thickness of the film.

In the first section, we describe the self-consistent theory in detail when the system has
time reversal invariance. The method of calculation of probability of return in restricted geometries within the path integral formalism is discussed in the next section. Then we extended this theory to the case of thin films of finite thickness in the absence of magnetic field. We end by comparing the numerical results obtained in the previous chapter and the analytical results.

3.1 Self-Consistent Theory

We give an outline of the self-consistent theory of localization developed by Vollhardt and Wölfle [39, 40]. The basic idea of this theory is to study the, density response function, $\chi(\vec{q}, \omega)$, which can be obtained from Linear Response Theory. Here, $\vec{q}$ is wave vector and $\omega$ is the frequency of the density fluctuation. The linear response theory describes the response of the system when a weak external force like electric or magnetic field etc. is applied to the system. For electrons moving in a field of randomly distributed impurities, the one electron Hamiltonian is given by,

$$H = \int d\vec{r} \psi^\dagger(\vec{r}) \left[ -\frac{\hbar^2}{2m} \nabla^2 + \sum_i V(\vec{r} - \vec{R}_i) \right] \psi(\vec{r})$$  \hspace{1cm} (3.1)$$

where, $V(\vec{r} - \vec{R}_i)$ is the potential of an impurity at the random position $\vec{R}_i$. $\psi^\dagger(\vec{r})$ and $\psi(\vec{r})$ are creation and annihilation operators. The aim of this theory is to calculate conductivity by using the density response function which is given by,

$$\chi(\vec{r}, t; \vec{r}', 0) = i \theta(t) \langle \phi_0 | [\rho(\vec{r}, t), \rho(\vec{r}', 0)] | \phi_0 \rangle$$  \hspace{1cm} (3.2)$$

where $\rho(\vec{r}, t)$ is the density operator, $\psi^\dagger(\vec{r}, t)\psi(\vec{r}, t)$, $\theta(t)$ is step function in $t$, $[A, B]$ is the commutator between two dynamical variables $A$ and $B$ and $|\phi_0\rangle$ is the ground state of the system. Now by taking Fourier transform of the response function in frequency space, we have,

$$\chi(\vec{r}, \vec{r}'; \omega) = i \int_0^\infty dt \ e^{i(\omega + i\delta)t} \langle \phi_0 | [\rho(\vec{r}, t), \rho(\vec{r}', 0)] | \phi_0 \rangle$$  \hspace{1cm} (3.3)$$

When we average this quantity over the distribution of randomly placed impurities, the translational invariance is recovered. Then taking Fourier transform in momentum space, we have,

$$\chi(\vec{q}, \omega) = i \int e^{i\vec{q} \cdot (\vec{r} - \vec{r}')} \langle \chi(\vec{r}, \vec{r}'; \omega) \rangle d^3 r d^3 r'$$  \hspace{1cm} (3.4)$$

here, $\langle \ldots \rangle$ is the average over distribution of positions of the impurities. The density operator, $\rho(\vec{r}, t)$ and the current operator, $\vec{j}(\vec{r}, t)$ obey the continuity equation $\partial_t \rho(\vec{r}, t) + \nabla \cdot \vec{j}(\vec{r}, t) =$
From this continuity equation one can recover frequency dependent conductivity, $\sigma(\omega)$ as,

$$\sigma(\omega) = -\lim_{q \to 0} \frac{ie^2\omega}{q^2} \chi(q, \omega)$$  \hspace{0.5cm} (3.5)$$

where, $e$ is the charge of the electron. Thus, one is able to calculate the conductivity if we know $\chi(q, \omega)$. Since one wants to study Anderson transition, it is required to analyse the behaviour of $\sigma(\omega)$ or diffusion constant, $D(\omega)$ near the transition. To describe both the conducting and insulating phases in this formalism, it is necessary to understand how the calculation of $\sigma(\omega)$ can describe both the phases. For the metallic phase, as $\omega \to 0$, $\sigma(\omega)$ has a non-zero positive limit, $\sigma(0) = \sigma_{dc}$. For the insulating phase, on the other hand $\sigma(\omega)$ describes a purely capacitative response and is thus purely imaginary at small frequencies. As $\omega \to 0$, $\sigma(\omega) \propto i\omega \xi^2$, where $\xi$ denotes the localization length. Thus in the insulating phase the quantity $\sigma(\omega)/i\omega$ must have a positive limit as $\omega \to 0$.

The density response function $\chi(q, \omega)$ has a simple structure due to the conservation of density. This can be understood from the following considerations. A conserved probability density $P(q, t)$ obeys diffusion equation. A fluctuation in this quantity decays towards equilibrium through a time dependence of the following form,

$$P(q, t) \sim e^{-Dq^2t}P(q, 0) \hspace{0.5cm} t \geq 0$$  \hspace{0.5cm} (3.6)$$

where, $q$ is the wave vector of the fluctuation. The Fourier transform (FT) of this $P(q, t)$ gives the following diffusion pole,

$$P(q, \omega) \sim \frac{1}{-i\omega + Dq^2}P(q, 0)$$  \hspace{0.5cm} (3.7)$$

Thus $P(q, \omega)$ has singularity at $q, \omega \to 0$. This singularity is due to particle conservation during diffusion.

The same kind of $\omega, q$ dependence also occurs for $\chi(q, \omega)$ as it is related to correlation function through fluctuation-dissipation theorem. So one can write $\chi(q, \omega)$ with generalized diffusion constant, $D(q, \omega)$ as in the following,

$$\chi(q, \omega) = N_F \frac{D(q, \omega)q^2}{-i\omega + D(q, \omega)q^2}$$  \hspace{0.5cm} (3.8)$$

where, $N_F$ is the density of states at Fermi level. Substituting this $\chi(q, \omega)$ in equation (3.5), one gets the dynamical conductivity as,

$$\sigma(\omega) = e^2N_F D(\omega)$$  \hspace{0.5cm} (3.9)$$
which is known as Einstein's relation and relates $\chi(0, \omega)$ to $D(\omega)$.

The first step of this analysis is to express $\chi(\vec{q}, \omega)$ in terms of the single particle Green's functions. One defines the following one particle Green's function,

$$g^{R,A}(\vec{p}, \vec{p}'; E) = \sum_n \frac{\phi_n(\vec{p})\phi_n^*(\vec{p}')}{E - \epsilon_n \pm i\delta}$$  \hspace{1cm} (3.10)

where, $\phi_n(\vec{p})$ is an eigenstate with eigenvalue $\epsilon_n$ in the momentum representation. The superscripts R,A indicate the retarded and advanced Green's functions respectively. If one averages this Green's function over impurity positions then the translational invariance is restored and one has,

$$\langle g^{R,A}(\vec{p}, \vec{p}'; E) \rangle = \delta_{\vec{p}\vec{p}'} G^{R,A}(\vec{p}, E)$$  \hspace{1cm} (3.11)

where,

$$G^{R,A}(\vec{p}, E) = \frac{1}{E - \epsilon_n - \Sigma^{R,A}(\vec{p}, E)}$$  \hspace{1cm} (3.12)

here, $\Sigma^{R,A}(\vec{p}, E)$ is self-energy of the particle. In the following we shall use the weak-disorder approximation and take,

$$\Sigma^{R,A}(\vec{p}, E) = \mp \frac{i}{2\tau}$$  \hspace{1cm} (3.13)

where, $\tau$ is the finite mean free time or collision time which is defined as the time over which the particle survives in the momentum state $\vec{p}$.

Now using these definitions (3.10) and (3.11) one can express $\chi(\vec{q}, \omega)$ in equation (3.3) in terms of one-particle Green's functions as in the following,

$$\chi(\vec{q}, \omega) = \frac{1}{2\pi i} \sum_{\vec{p}, \vec{p}'} \int dE \left[ (f(E + \omega) - f(E))
\times \langle g^R(\vec{p}_+, \vec{p}_+'; E + \omega)g^A(\vec{p}_-, \vec{p}_-'; E) \rangle
+ f(E)\langle g^R(\vec{p}_+, \vec{p}_+'; E + \omega)g^R(\vec{p}_-, \vec{p}_-'; E) \rangle
- f(E + \omega)\langle g^A(\vec{p}_+, \vec{p}_+'; E + \omega)g^A(\vec{p}_-, \vec{p}_-'; E) \rangle \right]$$  \hspace{1cm} (3.14)

here, we have the definitions, $\vec{p}_{\pm} = \vec{p} \pm \vec{q}/2$. The $f(E)$ is the Fermi distribution function. One can prove that for $q, \omega \to 0$, $\langle g^R g^R \rangle^* = (g^A g^A)$. Since one is interested the behaviour of $\chi(\vec{q}, \omega)$ for small values of $\vec{q}$ and $\omega$, the function $f(E + \omega)$ can be expanded as, $f(E + \omega) \sim f(E) - \omega\delta(E)$. Then the last two terms in equation (3.14) can be evaluated giving rise the following results,

$$\chi^T \sim \frac{1}{\tau} \sum_{\vec{p}, \vec{p}'} \int dE \ f(E) \ Im\langle g^R g^R \rangle
= N_F$$  \hspace{1cm} (3.15)
Equation (3.14) involves the following two-particle propagator,
\[
\phi_{p,p'}(\vec{q}, \omega) = -(g^R(p^+_+, p^-_+; E + \omega)g^A(p^-_-, p^+_-; E))
\]
(3.16)
We also introduce the following function,
\[
\phi(q, \omega) = \sum_{p,p'} \phi_{p,p'}(q, \omega)
\]
(3.17)
Now using (3.15) and (3.17), equation (3.14) arrive at the following equation in the limit, \(q, \omega \to 0\),
\[
\chi(q, \omega) = -\frac{\omega}{2\pi i} \phi(q, \omega) + N_F + O(q^2, \omega)
\]
(3.18)
To calculate \(\phi_{p,p'}(q, \omega)\), averaging over impurities is to be done. Then the the two-particle propagator, \(\phi_{p,p'}(q, \omega)\) in equation (3.16) may be written as,
\[
\phi_{p,p'}(q, \omega) = -G^R(p^+_+, E_F + \omega)G^A(p^-_-, E_F) \\
\times \left[ \delta_{p,p'} + \Gamma_{p,p'}(q, \omega)G^R(p^+_+, E_F + \omega)G^A(p^-_-, E_F) \right]
\]
(3.19)
The diagrammatic representation of \(\phi_{pp'}(q, \omega)\) in terms of averaged single particle Green’s functions \(G^R\) and \(G^A\) is shown in the Fig.(3.1). The solid lines in these diagrams denote Green’s functions and the dotted lines indicate interaction with impurities. The lines going right denote \(G^R\) while those going left denote \(G^A\). These diagrams contain reducible as

\[\phi = \]

\[
\begin{array}{cccccc}
  p_+ & p'_+ & & & & p_+ & p'_+ \\
  p_- & p'_- & & & & p_- & p'_-
\end{array}
\]

\[= \delta_{pp'} + \]

\[
\begin{array}{c}
  p_+ \\
  p_-
\end{array}
\]

\[
\begin{array}{c}
  p'_+ \\
  p'_-
\end{array}
\]

Figure 3.1: Diagrammatic representation of \(\phi(q, \omega)\).

As well as irreducible diagrams. Reducible diagrams are those which can be splitted into two diagrams when the electron and hole lines are cut (eg. third diagram in Fig.3.1). Whereas, irreducible diagrams are those which do not split (eg. fourth diagram in Fig.3.1). The vertex
Figure 3.2: Diagrammatic representation of the reducible vertex $\Gamma_{pp'}(\vec{q},\omega)$.

function, $\Gamma_{pp'}(\vec{q},\omega)$ has both types of the diagrams mentioned above. But one can construct the reducible diagrams from the irreducible diagrams. The reducible ladder diagrams can be sum up as in the Fig.3.2. Denoting the irreducible vertex by $\Lambda_{pp'}(\vec{q},\omega)$ one can easily get the expression for $\Gamma_{pp'}$ as in the following,

$$\Gamma_{pp'}(\vec{q},\omega) = \Lambda_{pp'}(\vec{q},\omega) + \sum_k \Lambda_{pk}(\vec{q},\omega)G^R(k_+,E+\omega)G^A(k_-,E)\Gamma_{kp'}(\vec{q},\omega) \quad (3.20)$$

In the first approximation the irreducible part is taken to be just the second diagram of Fig.3.1 with $\Lambda_0 = n_i V(|\vec{p} - \vec{p}'|)^2$, where, $n_i$ is density of impurities. This corresponds to the ladder sum for $\Gamma_{pp'}$, as shown in Fig.3.3. The function, $\chi_0 = \sum_p G^R(p) G^A(p)$ can be approximated in the limits of small values of $\vec{q}$ and $\omega$. In this limit, it is reasonable to take the following approximations,

(i) The single particle quantities like density of states etc. are smoothly varying quantities across the Anderson transition. So they can be regarded as essentially unchanged as we increase impurity concentration to go from extended to localized regimes. So it is reasonable to approximate the self-energy $\Sigma$, by the lowest order result $\sim \Lambda_0(p - p')$ by Born’s approximation for future calculations,

$$\gamma = \frac{1}{\tau} = Im\Sigma^A_{pp'}(E_F) = \pi N_F \int \frac{d\Omega_{p'}}{4\pi} \Lambda_0(p - p') = \pi N_F \Lambda_0 \quad (3.21)$$

here, $\Lambda_0$ is a constant when we consider point like potentials, and the integral in equation (3.21) is evaluated at Fermi level.
(ii) If $\gamma << E_F$, then, $G_p^R(E_F)$ is strongly peaked at $p = p_F$ with width $\gamma$ and height $\gamma^{-1}$. These forms of $G^R$ and $G^A$ are used to analyse and approximate $\Gamma_{pp'}$ and $\phi_{pp'}$ equations.

With these approximations and after some algebra, one can get $\chi_0 = (1 + i\omega\tau - D_0\tau q^2)/\Lambda_0$. Here, $D_0$ is bare diffusion constant given by $D_0 = v_F^2\tau/d$, with $d$ as the dimension of the system. Then equation (3.20) can written as,

$$\Gamma_{pp'}(q, \omega) = \frac{2\Lambda_0\tau^{-1}}{-i\omega + D_0q^2} \quad (3.22)$$

Substituting this equation (3.22) to equation (3.18), one arrives at the following result

$$\chi(q, \omega) = N_F \frac{D_0q^2}{-i\omega + D_0q^2} \quad (3.23)$$

Generally, the vertex function $\Gamma_{pp'}$ is known as bare diffusion propagator. Next is to solve for $\phi_{pp'}$ for small $q$ and $\omega$.

### 3.1.1 Equation of motion for $\phi_{pp'}(q, \omega)$

To go beyond the above approximation for $\phi_{pp'}$, we make explicit use of having time reversal invariance to note the following property of $\Gamma_{pp'}$

$$\Gamma_{pp'}(q, \omega) = \Gamma_{e' + q', e - q}(\vec{p} + \vec{p}', \omega) \quad (3.24)$$

The simplest diagrammatical proof due to Vollhardt and Wölfle is given in the diagram of Fig.(3.4). In doing so, momentum has to be conserved and so one can change the indices $\vec{p}$, $\vec{p}'$ and $\vec{q}$ as in the equation (3.24). The first and the last diagrams where the particle and hole arrows point in opposite directions, are known as particle-hole channel whereas the second time reversed diagram where both the arrows point in the same direction is known as particle-particle channel. In time reversal symmetry, the set of diagrams in particle-particle and particle-hole channels are equal. But in case time reversal invariance is revoked the set of diagrams in these two channels are not equal. We will discuss this in detail in the next chapter.

Now using equation (3.20), one can write the equation for $\phi_{pp'}(q, \omega)$ in terms of irreducible vertex

$$\phi_{pp'}(q, \omega) = -G^R(p_+, E_F + \omega)G^A(p'_-, E_F) \left[ \delta_{pp'} + \sum_k \Lambda_{pk}(q, \omega) \phi_{kp'}(q, \omega) \right] \quad (3.25)$$

This equation is known as Bethe-Salpeter equation, and is many-body equivalent of two particle Schrödinger equation. To write simplified form of this equation, the following identity is used,

$$g^R(p_+, E_F + \omega)g^A(p'_-, E_F) = \frac{\Delta G_p}{\omega - \frac{\vec{p} \cdot \vec{p}'}{m} - \Delta \Sigma_p} \quad (3.26)$$
Figure 3.4: Diagrammatical proof of the relation of $\Gamma_{pp'}$ under time reversal symmetry.

where,

$$
\Delta G_p = G^R(\vec{p}_+; E_F + \omega) - G^A(\vec{p}_-; E_F)
$$

$$
\Delta \Sigma_p = \Sigma^R(\vec{p}_+; E_F + \omega) - \Sigma^A(\vec{p}_-; E_F)
$$

$$
= \sum_k \Lambda_{pk}(\vec{q}, \omega) \Delta G_k
$$

(3.27)

here, the relation for $\Delta \Sigma_p$ is a general ward identity. Then one is able to rewrite the equation (3.25) as,

$$
\left[ \omega - \frac{\vec{p} \cdot \vec{q}}{m} - \Delta \Sigma \right] \phi_{pp'}(\vec{q}, \omega) = \Delta G_p \left[ \delta_{pp'} + \sum_k \Lambda_{pk}(\vec{q}, \omega) \phi_{kp'}(\vec{q}, \omega) \right]
$$

(3.28)

This equation is known as kinetic equation and is an exact equation. Substituting the relations for $\Delta G_p$ and $\Delta \Sigma_p$ in equation (3.28) and summing over $p$ and $p'$ one can reach the following equation after some algebra in the limit of small $q$ and $\omega$,

$$
\omega \phi(\vec{q}, \omega) - q \phi_j(\vec{q}, \omega) = 2\pi i N_F
$$

(3.29)

where, the function, $\phi_j(\vec{q}, \omega)$ is defined as current relaxation function which is given by,

$$
\phi_j(\vec{q}, \omega) = \sum_{pp'} \left( \frac{\vec{p} \cdot \vec{q}}{m} \right) \phi_{pp'}(\vec{q}, \omega)
$$

(3.30)

If $q \to 0$, then we have, $\phi(0, \omega) = 2\pi i N(E_F)/\omega$. Substituting this to equation (3.18), one gets that $\chi(0, \omega) = 0$. The vanishing of the density response function indicates the conservation of particle number. Obviously the $\phi$-equation (3.29) incorporates the particle number conservation.
3.1.2 Self-consistent equation with time reversal invariance

The self-consistent equation can be derived from the study of the functions $\phi(\mathbf{q}, \omega)$ and $\phi_j(\mathbf{q}, \omega)$. We need to obtain a closure of the equations between these two functions. For this purpose, it is necessary to study the function $\phi_{pp'}$. The dependence of $\phi_{pp'}$ on the magnitude of the vectors $\mathbf{p}$ and $\mathbf{p}'$ comes from the structure of $G^R$ or $\text{Im} G^R \propto \Delta G_p$. The function is strongly peaked at $\mathbf{p} = \mathbf{p}' \approx \mathbf{p}_F$ due to the nature of $G^R$. The remaining angular dependence can be extracted using Legendre expansion in terms of angular variables. Then one can write,

$$
\sum_{p'} \phi_{pp'}(\mathbf{q}, \omega) = \Delta G_p \times \mathcal{F}(p_F, \Omega) \quad (3.31)
$$

where, $\mathcal{F}$ is a function of $p_F$ and angular variable $\Omega = (\theta, \phi)$, which can be measured by taking $\mathbf{q}$ as the Z-axis with $\cos \theta = (\mathbf{p}, \mathbf{q})$. Now $\mathcal{F}$ can be expanded in Legendre polynomials in $d = 3$ and Fourier series in $d = 2$ respectively. Retaining only the terms $l = 0$ and $l = 1$ in the limits of $q, \omega \to 0$, one can write $\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1(\mathbf{p}, \mathbf{q})$. Evaluating these constants $\mathcal{F}_0$ and $\mathcal{F}_1$, one can reach the following equation,

$$
\sum_{p'} \phi_{pp'}(\mathbf{q}, \omega) = -\frac{\Delta G_p}{2\pi i N(E_F)} \left[ \phi(\mathbf{q}, \omega) + \frac{m_d}{p_F^2}(\mathbf{p}, \mathbf{q}) \phi_j(\mathbf{q}, \omega) \right] \quad (3.32)
$$

In small $q$ and $\omega$, one can approximate $\Delta \Sigma_p \approx i/\tau$. Now summing Bethe-Salpeter equation (3.28) over $p'$ and then substituting the above equation (3.32) and again summing the result over $p$, one is able to collect terms for $\phi_j$ and $\phi$ to reach the following equation,

$$
iq D_0 \phi(\mathbf{q}, \omega) + \kappa(\mathbf{q}, \omega) \phi_j(\mathbf{q}, \omega) = 0 \quad (3.33)
$$

where, the kernel, $\kappa(\mathbf{q}, \omega)$ is given by,

$$
\kappa(\mathbf{q}, \omega) = 1 - \frac{m_d \tau}{2\pi N_F p_F^2} \sum_{pp'} \left( \frac{\mathbf{p}, \mathbf{q}}{m} \right) \Delta G_p \Lambda_{pp'}(\mathbf{q}, \omega) \Delta G_{p'} \left( \frac{\mathbf{p}', \mathbf{q}}{m} \right) \quad (3.34)
$$

Thus we get the closed set of the equations to be equations (3.29) and (3.33). From these equations one can solve for $\phi(\mathbf{q}, \omega)$ as,

$$
\phi(\mathbf{q}, \omega) = \frac{2\pi i N_F}{\omega + i D(\mathbf{q}, \omega) q^2} \quad (3.35)
$$

where, the $\mathbf{q}$ and $\omega$ dependent diffusion constant in equation (3.35), $D(\mathbf{q}, \omega)$ is identified as,

$$
\kappa(\mathbf{q}, \omega) = \frac{D_0}{D(\mathbf{q}, \omega)} \quad (3.36)
$$

41
Now the calculation of $D(q, \omega)$ requires calculation of the irreducible vertex $\Lambda_{pp'}(\bar{q}, \omega)$. Langer and Neal [47] showed that the maximally crossed diagrams makes the most significant contribution to $\Lambda_{pp'}$ as it has a diffusion pole. These diagrams can be summed up easily as we have done for pure ladder diagrams before. The contribution due to these diagrams in the limits $q, \omega \to 0$ is,

$$\chi_0(\bar{p} + \bar{p'}, \omega) \approx \frac{1}{\Lambda_0} \left[ 1 + i\omega \tau - D_0(\bar{p} + \bar{p'})^2 \right]$$

so that,

$$\Lambda_{pp'}(\bar{q}, \omega) = \frac{2\Lambda_0\tau^{-1}}{-i\omega + iD_0(\bar{p} + \bar{p'})^2} \quad (3.39)$$

This equation has singularity at the limit $\omega \to 0$, $\bar{p} = -\bar{p'}$. Now we substitute this equation (3.37) to the kernal equation (3.34) and put $\bar{p} + \bar{p'} = \bar{Q}$. In the self-consistent procedure one replaces $D_0$ on right hand side of equation (3.39) by $D(\bar{q}, \omega)$. $D(\bar{q}, \omega)$ here has come in the particle-particle channel and has been equated to $D(\bar{q}, \omega)$ of the particle-hole channel. So after doing simplification using equations (3.34), (3.36) ans (3.39), one can get the following self-consistent equation,

$$\frac{D_0}{D(\omega)} = 1 + \frac{1}{\pi h N_F(d)} \frac{1}{L^d} \sum_Q \frac{1}{-i\omega + D(\omega)Q^2} \quad (3.40)$$

where, $N_F(d)$ is density of state at Fermi level in $d$ dimensions. The summation in the equation can be replaced by integration using $\frac{1}{L^d} \sum_Q = \frac{1}{(2\pi)^d} \int_0^{1/l} d^d \! Q$ where, the integration is restricted to momenta smaller than the inverse of the mean free path, $1/l$ (hydrodynamic régime). Then the frequency dependent diffusion constant can be written in integral equation as,

$$\frac{D(\omega)}{D_0} = 1 - \lambda \pi k_F^2 \int_0^{1/l} dQ \frac{Q^{d-1}}{-i\omega + D(\omega)} + Q^2 \quad (3.41)$$
where, $k_F$ is Fermi wave vector. This equation will give rise to either insulating or conducting solutions. To study this, one defines a frequency dependent correlation length, $\tilde{\xi}(\omega)$

$$\tilde{\xi}(\omega) = -\frac{D(\omega)}{i\omega l^2}$$ (3.42)

With this definition, we study the equation (3.41) to get insulating and metallic solutions for small $Q$ and $\omega$.

(i) The equation (3.41) will give insulating solution when $\omega \to 0$ if we get $\xi(\omega) \to \xi(0) = \xi$, a real positive number. In this regime, $D(0) \to 0$. Using (3.42) and taking $\omega \to 0$, it is given by

$$1 = \lambda dk_F^{2-d} \int_0^{1/l} dQ \frac{Q^{d-1}}{\xi^{-2} + Q^2}$$ (3.43)

(ii) Whereas, it will give metallic solution if in the limit $\omega \to 0$, we obtain $D(\omega) \to D(0)$, a real positive number. In this regime, $\xi(0) \to \infty$. In this situation, one calculates $D(0)$

$$\frac{D(0)}{D_0} = 1 - \lambda dk_F^{2-d} \int_0^{1/l} dQ \frac{Q^{d-1}}{Q^2}$$ (3.44)

Vollhardt and Wölfle have studied the cases of $d = 1, 2, 3$ in detail. We discuss the main results in the following.

(i) $d=1$ case:
In one dimension, putting $d = 1$ in equation (3.43) and doing the integration, one obtains the insulating solution for $\xi$ as in the following,

$$\frac{\pi l}{\xi} = \tan^{-1} \left( \frac{\xi}{l} \right)$$ (3.45)

This equation always has a real positive solution for all values of $l$. So in one dimension there is no metallic solution which implies that all the states are localized.

(ii) $d=2$ case:
Similarly, the insulating solution for $\xi$ in two dimensional case is given by putting $d = 2$ in equation (3.43) and doing the integration,
(iii) d=3 case:
The situation is different in three-dimensional case. The insulating solution in three dimensions is given by,

$$\frac{1}{3\pi \lambda_3^3} = \frac{\xi - \tan^{-1}[\xi/l]}{\xi}$$

One can see that the right hand side of this equation is always smaller than unity. So the finite positive solution for $\xi$ is obtained only when $3\pi \lambda_3^3 > 1$. Whereas when $3\pi \lambda_3^3 < 1$, this equation does not have any solution. In this situation one gets metallic solution from the equation (3.44) in the limit $\omega \to 0$ and putting $\lambda_3 = 1/(\pi k_F l)$,

$$D(0) = D_0(1 - 3\pi \lambda_3^2)$$

With these considerations, we extend the theory in thin films of finite thickness.

### 3.2 Diffusion of particles in restricted geometries

Since experiments are always done on samples having restricted geometry like thin films, thin wires etc, the calculation of important physical quantities like conductivity etc has to be done in these samples. We will study Anderson metal-insulator transition in thin film which one can study experimentally.

The corrections arising from Langer and Neal diagrams can be equivalently obtained from the path integral formalism, by restricting the path summation to those on random walks. This in turn related to the solution of the diffusion equation. The conductivity corrections are related to the probability of return of the walk to its origin, which is easily obtained by solving the diffusion equation in the desired geometry.

Consider a thin film having lateral dimension $\vec{r} = (x, y)$ and a finite thickness $'b'$. To find the solution of the diffusion equation (1.22), it is subjected to the boundary condition that the normal component of the diffusion current vanishes at the surfaces of the sample. To calculate the probability of return, first one has to consider for probability $P(\vec{r}, t; \vec{r}_0, t_0)$ of the particle to be at $(\vec{r}, t)$ if it was at initial point $(\vec{r}_0, t_0)$. For this we solve the equation

$$\left( \frac{\partial}{\partial t} - \hat{L} \right) P(\vec{r}, t - t_0; \vec{r}_0) = \delta(\vec{r} - \vec{r}_0) \delta(t - t_0)$$

where, $\hat{L}$ is the operator given by

$$\hat{L} = D \nabla^2$$

The solution of the equation (3.49) is

$$P(\vec{r}, t - t_0; \vec{r}_0) = \sum_n \phi_n(\vec{r}) \phi^*_n(\vec{r}_0) e^{-\lambda_n(t - t_0)}$$
where, \( \lambda_n \) and \( \phi_n(\vec{r}) \) are eigenvalue and eigenfunction satisfying

\[
\hat{L}\phi_n(\vec{r}) = -\lambda_n \phi_n(\vec{r}) \tag{3.52}
\]

So what one has to worry about is to calculate \( \lambda_n \) from equation (3.52). Substituting this value of \( \lambda_n \) to equation (3.51) and equating \( r = r_0 \) and \( t_0 = 0 \), one can calculate the probability of return, \( P(\vec{r}, t) \).

The eigenfunction \( \phi_n(\vec{r}) \) has the following form for thin film,

\[
\phi_{m,q}(\vec{r}) = \sqrt{\frac{2}{A b}} \cos\left(\frac{\pi m}{b} z\right) e^{i\vec{q}\cdot\vec{r}} \quad , \quad m = 0, 1, 2, 3, \ldots \tag{3.53}
\]

where, \( A \) is the area of the thin film, \( z \) is the co-ordinate in the direction of the thickness and \( q \) denote two dimensional wave vector of the film. The boundary condition in the plane of the film is taken to be periodic. Using this eigenfunction \( \phi_{m,q}(\vec{r}) \), the eigenvalues, \( \lambda_n \) can be calculated from equation (3.52) as

\[
\lambda_n = q^2 + \frac{n^2\pi^2}{b^2} \quad , \quad n = 0, 1, 2, 3, \ldots \tag{3.54}
\]

The probability of return is calculated by substituting this eigenvalue in equation (3.51),

\[
P(\vec{r}_0, z; \vec{r}_0) = \frac{2}{A b} \sum_{n=0}^{\infty} \sum_{q} \cos^2\left(\frac{n\pi z}{b}\right) \exp\left[ -D_0 \left( q^2 + \frac{n^2\pi^2}{b^2} \right) t \right] \tag{3.55}
\]
Now we use this expression of probability in equation (1.28) to calculate the correction to conductivity and get,

$$\delta \sigma = -\frac{2e^2D_0}{\pi \hbar L^2 b} \sum_{n=0}^{\infty} \sum_q \frac{1}{D_0(q^2 + \frac{n^2 \pi^2}{L^2}) - i\omega}$$  \hspace{1cm} (3.56)

The term $i\omega$ in the denominator arises due to the fact that in the calculation of frequency dependent correction there will be a factor $e^{i\omega t}$ inside the integral of the equation (1.28).

### 3.3 Self-Consistent Theory in Thin Films

For self-consistency, $D_0$ in the denominator inside summation is replaced by frequency dependent diffusion constant, $D(\omega)$. One can use Einstein's relation (3.9) to express the above equation in terms of diffusion constant. On adding the Drude term, one freely obtains the following self-consistent equation for $D(\omega)$

$$\frac{D(\omega)}{D_0} = 1 - 2\lambda \int_0^{1/l} dq \sum_{n=0}^{\infty} \frac{1}{q^2 + \left(\frac{n\pi}{b}\right)^2} - \frac{i\omega}{D(\omega)}$$  \hspace{1cm} (3.57)

We use the following summation formula,

$$\sum_{n=0}^{\infty} \frac{1}{c^2 + n^2 \pi^2} = \frac{1}{2} \left( \coth(c) + \frac{1}{c^2} \right)$$  \hspace{1cm} (3.58)

and one can perform the wave-vector integration. So one can write the integral equation in terms of $\xi^2(\omega)$ as given below,

$$-\frac{i\omega \xi^2(\omega)l^2}{D_0} = 1 - \lambda \log \left[ \frac{\sinh \left( \tilde{b}\sqrt{1 + \xi^2(\omega)^{-2}} \right)}{\sinh \left( \tilde{b} \xi^2(\omega)^{-1} \right)} \right] - \frac{\lambda}{2} \log \left[ 1 + \xi^2(\omega) \right]$$  \hspace{1cm} (3.59)

where, $\xi(\omega) = \xi(\omega)/l$ and $\tilde{b} = b/l$.

We first analyse the equation (3.59) for insulating solutions. Taking the limit $\omega \to 0$, we look for a finite, positive solution for $\xi(0) = \xi$ of the following equation,

$$\frac{1}{\lambda} = \log \left[ \sqrt{1 + \xi^2} \times \frac{\sinh \left( \tilde{b}\sqrt{1 + \xi^2}^{-2} \right)}{\sinh \left( \tilde{b}\xi^{-1} \right)} \right]$$  \hspace{1cm} (3.60)

Now let us first see the behaviour in the two limits of small $\tilde{b}$ and large $\tilde{b}$, and examine if the corresponding $d = 2$ and $d = 3$ limits are obtained. Now we consider the two limits.
(i) In small $\tilde{b}$ limit i.e. $\tilde{b} \to 0$, one can get,

$$\frac{1}{\lambda} \sim \log(1 + \xi^2)$$  \hspace{1cm} (3.61)

which is the correct 2D answer.

(ii) In order to study the crossover from 2D to 3D, one has to keep in mind that the intensive parameters of 2D become the extensive parameters with respect to the film thickness. So the conductivity and density of states have different units in 2D and 3D whereas the diffusion constant has the same dimension. The density of states in 2D and 3D systems are given by $N_2 = 1/\hbar D_0 \lambda$ and $N_3 = 4/(3\pi \hbar D_0 \lambda^2)$ respectively. So the frequency dependent conductivities and densities of states in two and three dimensions can be connected as following,

$$\sigma_2(\omega, b) \to b \sigma_3(\omega); \hspace{0.5cm} N_2(\omega, b) \to b N_3(\omega)$$ \hspace{1cm} (3.62)

This is also consistent with the large $b$ limit of the sum over $n$. In this limit $n/b$ is small enough to behave as a continuous variable, and the limit of the sum is obtained as,

$$\int_0^{Q_0} d^2 q \sum_{n=0}^{\infty} \frac{1}{q^2 + \frac{n^2 \pi^2}{b^2}} \to b \int_0^{Q_0} d^2 q \int_0^{\infty} dq_b \frac{1}{q^2 + q_b^2}$$ \hspace{1cm} (3.63)

which is proportional to $b$. $q_b$ becomes additional third dimensional wave vector. Similarly the interaction parameter would also have to be scaled with $b$, as we we see below.

For large $\tilde{b}$, the sinh factor can be approximated by the exponential and so one can write equation (3.60) as

$$\frac{1}{\lambda} = \tilde{b} \left[ \sqrt{1 + \tilde{\xi}^{-2}} - \tilde{\xi}^{-1} \right] + \frac{1}{2} \log \left( 1 + \tilde{\xi}^2 \right)$$ \hspace{1cm} (3.64)

The first term on the right hand side is extensive with regard to $\tilde{b}$, while the second term that can be traced to be half the contribution from $n = 0$ mode, is not. This boundary condition is very important in small $\tilde{b}$ limit whereas this is not of much consequence in large $\tilde{b}$ limit. Thus the only way to reach the 3D limit is to consider only the $\tilde{b}$ extensive part of the above formula. With this the 3D limit of the equation (3.62) takes the following form,

$$\frac{4}{3\pi \lambda_3^2} = \sqrt{1 + \tilde{\xi}^{-2}} - \tilde{\xi}^{-1}$$ \hspace{1cm} (3.65)

This result has the same qualitative behaviour as in equation (3.22). So the equation (3.59) correctly qualify the crossover from two to three dimensions. This has the existence of insulating solutions for $3\pi \lambda_3^2/4 \geq 1$. 

47
For large \( \tilde{b} \) limit, one is allowed to examine the metallic solution from equation (3.57). It can be done by solving \( D(0) \) from this equation in the limit \( \omega \to 0 \) and \( \xi \to \infty \). After doing the summation with respect to the wave vector \( \tilde{q} \) using equation (3.58) and integration we get the following solution,

\[
\frac{D(0)}{D_0} = 1 - \lambda \log \left[ \sinh(\tilde{b}) \right]
\]

(3.66)

If one take for large \( \tilde{b} \) limit \( \sinh \) function in this equation can be approximated to exponential term whose argument is positive and \( \lambda \to \lambda_2 \). Then one can reach the following result for zero frequency diffusion constant as,

\[
D(0) = D_0 \left[ 1 - \tilde{b}\lambda_2 \right]
\]

(3.67)

This result shows the existence of metallic solution in infinite system for \( \tilde{b}\lambda_2 < 1 \). This again matches with the 3D solution with the scaling given in equation (3.48).

Now we consider finite \( \tilde{b} \) situation from the solution (3.28). It is easy to see that the right hand side goes from 0 to \( \infty \), monotonically as \( \tilde{\xi} \) goes from 0 to \( \infty \) for any finite thickness \( \tilde{b} \). So there is always a solution for any values of \( \tilde{b} \) and \( \lambda \). Thus there is no thickness induced delocalization transition in this case. This implies that for finite thickness films in infinite size system, the conductance scales to zero with system size just as in two dimensions.

Now it is important to reconcile this result with the 3D limit discussed above in equation (3.67). Equation (3.67) was obtained by taking first the limit \( \tilde{b} \to \infty \), in which case the function obtained has a limiting value as \( \tilde{\xi} \to \infty \), which in turn yields a threshold value of disorder for band centre localization. In the present case let us look at the large \( \tilde{\xi} \) behaviour for finite \( \tilde{b} \), the difference coming from values of \( \tilde{\xi} \) comparable to \( \tilde{b} \). The critical issue in dealing with equation (3.60) is how we treat the ratio \( \tilde{\xi}/\tilde{b} \) as \( \tilde{b} \to \infty \). If we let \( \tilde{b} \to \infty \) in the equation, without worrying about the growth of \( \tilde{\xi} \) both localized and extended solutions occur. Now Fig.(3.7) shows the solutions of equation (3.60) for various values of disorder parameter \( \lambda_3 \). From the curves, one can easily see that for \( \frac{2\pi}{\lambda_3^2} \gg 1 \), the value of \( \tilde{\xi} \) remain small and tends to saturate to a particular value with thickness \( \tilde{b} \) for a fixed \( \lambda_3 \). We show three different curves for three different values of \( \lambda_3 \) in this case (three lower curves in the Fig.(3.7)). On the other hand, for weak disorders, i.e. \( \frac{2\pi}{\lambda_3^2} \ll 1 \), \( \tilde{\xi} \) rises rapidly with \( \tilde{b} \), assuming values that are orders higher than \( \tilde{b} \) as shown in the figure. Here also we show three curves for three different values of \( \lambda_3 \) (upper three curves in the figure). The approximate solution in the regime of large \( \tilde{\xi} \) is given by,

\[
\tilde{\xi}(\tilde{b}, \lambda_3) \approx 2\tilde{b} \exp \left[ \tilde{b} \left( \frac{4}{3\pi\lambda_3^2} - 1 \right) \right]
\]

(3.68)
Figure 3.7: The variation of localization length as a function of $b$ for six different values of $\lambda_3$. The upper three curves are for $\lambda_3^2 < 4/3\pi$ and the lower three curves are for $\lambda_3^2 > 4/3\pi$.

So there is no thickness induced metal-insulator transition in this case. This result is in variance with the numerical results using tight binding model at zero temperature in the previous chapter. The numerical results show clear metal-insulator transition induced by thickness of the film.

### 3.3.1 Metal-Insulator Transition at Finite Temperature

For finite size of the system, we solve equation (3.57) with $1/L$ as the lower limit in the integration. Proceeding in the same way as in the case of infinite system size and, we reach the following result,

$$\frac{1}{\lambda} = \log \left[ \sqrt{\frac{1 + \tilde{\xi}^{-2}}{\tilde{L}^{-2} + \tilde{\xi}^{-2}}} \times \frac{\sinh \left( \tilde{b} \sqrt{1 + \tilde{\xi}^{-2}} \right)}{\sinh \left( \tilde{b} \sqrt{\tilde{L}^{-2} + \tilde{\xi}^{-2}} \right)} \right] \quad (3.69)$$

This situation is physically relevant at finite temperatures. Due to inelastic scattering, the electron loses its phase coherence over a distance $L_\phi$, the inelastic coherence length. So we replace $\tilde{L}$ in this equation by $L_\phi$. This $L_\phi$ depends upon temperature which can be taken as $L_\phi \approx L_0 (T/T_0)^{-p}$, where $p$ is a parameter depending on scattering mechanism, dimensionality, etc. $L_0$ and $T_0$ are constants having the dimensions of length and temperature.
ture respectively. In this situation, Metal-Insulator transition can be observed by changing temperature. We study it in two-dimensional case as well as in thin films respectively.

(a) Two-dimensional results:

We obtain two dimensional insulating solution by taking the limit $\bar{b} \rightarrow 0$ in equation (3.69) and replacing $\bar{L}$ by $L_\phi$. In this limit one can approximate sinh function as exponential and $\lambda \rightarrow \lambda_2$. Then solving for $\tilde{\xi}$, we get

$$\tilde{\xi}(\lambda_2, L_\phi) = L_\phi \sqrt{\frac{e^{1/\lambda_2} - 1}{L_\phi^2 - e^{1/\lambda_2}}}$$  \hspace{1cm} (3.70)

The numerator inside the square root of this equation is always positive number for any values of $\lambda_2$. The solution of $\xi$ with $L_\phi$ for two different values of $\lambda_2$ are shown in Fig.3.8. We get real positive solution of $\tilde{\xi}$ only when the denominator is greater than one. So we get insulating phase as long as,

$$\frac{L_\phi^2}{l^2} > e^{1/\lambda_2}$$  \hspace{1cm} (3.71)

With increasing temperature $T$, $L_\phi$ decreases and at a certain critical temperature $T_c(\lambda_2)$, this inequality is no longer satisfied and $\tilde{\xi}$ does not have solution. It happens when the inequality (3.71) becomes equality giving $T_c(\lambda_2)$,

$$T_c(\lambda_2) = T_0 \left( \frac{L_0}{l} \right)^{2/p} e^{-1/(p\lambda_2)} = a e^{-1/(p\lambda_2)}$$  \hspace{1cm} (3.72)

When $T > T_c$, the equation (3.70) does not have solutions. In another words, metallic solution in two-dimensional case is obtained when

$$\frac{L_\phi^2}{l^2} < e^{1/\lambda_2}$$  \hspace{1cm} (3.73)

Proceeding as we have done for infinite system size, we solve for real positive value of $D(0)$ in the limit $\omega \rightarrow 0$. The result is

$$D_2(\lambda_2, L_\phi) = D_0 \left[ 1 - 2\lambda_2 \log(\bar{L}_\phi) \right]$$
$$= D_0 \left[ 1 - \frac{\lambda_2}{\lambda_2(T)} \right]$$  \hspace{1cm} (3.74)

Then by substituting this relation to equation (3.9) we obtain two dimensional zero frequency conductivity, $\sigma_2$ as

$$\sigma_2(\lambda_2, T) = e^2 N_F D_0 \left[ 1 - \frac{\lambda_2}{\lambda_2(T)} \right]$$  \hspace{1cm} (3.75)
Thus obviously we get a transition induced by temperature as well as disorder in two-dimensional system size.

(b) Thin film results:

The insulating solution of thin film can be obtained by scaling $\lambda$ in equation (3.69) using the relation (3.62) as

$$\frac{1}{\lambda_3} = \frac{3\pi}{4b} \log \left[ \sqrt{\frac{1 + \tilde{\xi}^{-2}}{\tilde{L}^{-2} + \tilde{\xi}^{-2}}} \times \frac{\sinh \left( \tilde{b} \sqrt{1 + \tilde{\xi}^{-2}} \right)}{\sinh \left( \tilde{b} \sqrt{\tilde{L}^{-2} + \tilde{\xi}^{-2}} \right)} \right]$$

(3.76)

The behaviour of $\xi$ with respect to $b$ is shown in the Fig.(3.9). In this figure we find that, for a fixed value of $\lambda_3$, $\xi$ saturates to some value for large value of $L_\phi$. Whereas for small values of $L_\phi$, $\xi$ monotonically increases with $b$. If one look at this equation carefully one can realise that $\tilde{\xi}$ always does not have solution for any values of $b$ and $\lambda_3$. So there is a possibility of insulator to metal transition in this case.

To study the critical behaviour we proceed as follows. First we try to get critical disorder $\tilde{\xi}_c$ which can be defined as the disorder at which phase transition takes place. If we take $\tilde{\xi} \to \infty$, then obviously we can get critical disorder $\lambda_c$ as following.
Figure 3.9: Plots of localization length $\xi$ as a function of $b/l$ for two different values of $L_\phi$, for fixed value of $\lambda_3 = 0.66$.

\[
\lambda_c(b, L_\phi) = \frac{4\tilde{b}}{\sqrt{3\pi \log \left( \frac{\tilde{L}_\phi \times \frac{\sinh(\tilde{b})}{\sinh(b/L_\phi)}}{\sinh(b/L_\phi)} \right)}}
\]

The critical disorder is found to be as a function of $\tilde{b}$ and $\tilde{L}_\phi$. So for finite temperature, one has thickness induced delocalization of states. Fig.(3.10) shows the behaviour of $\lambda_c$ as thickness of the film is varied. From this figure we get that as $\tilde{L}_\phi$ decreases, the critical disorder increases. The curves clearly separates the localized and extended regimes. Since we have a transition in this case, it is possible to obtain zero frequency diffusion constant in extended regime. We can calculate it from equation (3.57) first by taking the limit $\omega \to 0$ and then doing the summation and integration respectively. The result is given by,

\[
D_3(\lambda_3, b, L_\phi) = D_0 \left[ 1 - \frac{3\pi}{4b} \lambda_3^2 \log \left( \frac{\sinh(\tilde{b})}{\sinh(b/L_\phi)} \right) \right]
\]

\[
= D_0 \left[ 1 - \left( \frac{\lambda_3}{\lambda_c(b, L_\phi)} \right)^2 \right]
\]

where, the finite thickness zero frequency diffusion constant, $D_3$ is found to be as a function of $\tilde{b}$, $\lambda_3$ and $\tilde{L}_\phi$. So we can get metallic solution only when $\lambda_3 < \lambda_c$. Fig(3.11) shows
the solution of $D_3$ with respect to $\tilde{b}$ for two values of $\tilde{L}_\phi$. Now straightforwardly using equation (3.78) and the definition (4.24) we can obtain zero frequency conductivity $\sigma_3$ as in the following,

$$\sigma_3(\lambda_3, b, L_\phi) = e^2 N_F D_0 \left[ 1 - \left( \frac{\lambda_3}{\lambda_c(b, L_\phi)} \right)^2 \right]$$

(3.79)

So clearly in the finite system size, there is a Metal-Insulator transition driven by temperature, thickness of the film and disorder.

3.4 Conclusion

We have reviewed the self-consistent theory of localization due to Vollhardt and Wölfle in this chapter. This theory is very convenient technique to study metal-insulator transition specially in weak disorder.

We extended the self-consistent theory in finite thickness thin film case in the absence of any field. We found that the equation nicely describe the crossover from two to three dimensions. The solution of this equation are obtained at zero temperature (infinite size solution) and finite temperature ($L_\phi$ dependent solution).
Figure 3.11: The variation of zero frequency diffusion constant as a function of $b$. Curves of two different values of $L_\phi$ at fixed value of $\lambda = 0.01$ are shown in the figure.
The infinite size solution can be obtained by taking the limit $L_\phi = \infty$. For large values of disorder, the localization length saturates to some value as a function of thickness for a particular value of disorder. But the solution of this equation does not give any metal-insulator transition even though the localization lengths increase very rapidly with thickness for small values of disorder. This is contrary to the numerical results obtained at zero temperature using tight binding model in the previous chapter.

We feel the apparent contradiction between numerical and analytical results is due to the following two reasons. First, in the analytical calculations, the localization length grows exponentially with weakness of disorder and the thickness. The numerical calculations perhaps cannot distinguish the localized states with localization lengths of the order of the system size from the current-carrying states. Thus one would doubt the numerical conclusions, which are inevitably limited by the size considerations in the present context.

Second, it is also possible that the backscattering quantum correction overestimates the tendency towards localization and is not adequate for dealing with the thickness-induced delocalization phenomenon. The consideration of magnetic field effects should throw further light on this matter. We will discuss it in the next chapter. In the above analysis the phase relaxation length has been taken to be infinite. For comparisons with the behaviour of real films this is obviously an important parameter and our conclusion cannot be valid for thicknesses greater than the phase relaxation length.

When phase relaxation length is finite we get different results. In two dimensions, we found localization length in insulating regime as a function of disorder as well as temperature. We obtain a metal-insulator transition in 2D. We are able to get a critical disorder as a function of temperature. We also calculated zero frequency diffusion constant and conductivity in metallic regime. In the case of thin films, we found a critical disorder which depends on thickness of the film and temperature. So we obtained a transition from insulator to metal driven by thickness and temperature. In insulating regime, the localization lengths increase rapidly as a function of thickness for small values $L_\phi$ at fixed value of disorder. But for large $L_\phi$, the localization length saturates to some value as we increase thickness. We calculated the zero frequency conductivity for the disorder smaller that the critical value. So we claim that for non-zero temperature, there is insulator to metal transition induced by thickness of the film as well as temperature.