3. COMPUTER ORIENTED FRACTIONAL ALGORITHM

3.1 INTRODUCTION

In the previous chapter the various methods available for solving Linear Fractional Programming problems were discussed. In this chapter a new algorithm which is more suitable to computers, is discussed. A special feature of this method is that the ratios of contribution coefficients of the decision variables in the numerator and the denominator of the objective function are used to select the entering variable. The following paragraphs explain in detail the various steps involved in this algorithm.

3.2 BASIC RULE FOR THE SELECTION OF ENTERING VARIABLE

Statement:

Given that

\[
\begin{align*}
    c_0 & : c_1 & : c_2 & \ldots & : c_n & : d_0 & : d_1 & : d_2 & \ldots & : d_n \\
    0 & : 1 & : 2 & \ldots & : n & : 0 & : 1 & : 2 & \ldots & : n
\end{align*}
\]

take positive values in the ratio
\[
\begin{align*}
&c + c \times c + c \times c + \ldots + c \times c \\
&\downarrow \quad 0 \quad 1 \quad 1 \quad 2 \quad 2 \quad n \quad n \\
&d + d \times d + d \times d + \ldots + d \times d \\
&\downarrow \quad 0 \quad 1 \quad 1 \quad 2 \quad 2 \quad n \quad n
\end{align*}
\]

then

\begin{enumerate}
\item \( a ) \quad c + c \times c \\
\begin{align*}
0 &\quad 1 \quad 1 \quad 0 \\
\downarrow &\quad \downarrow \\
d &\quad d \\
0 &\quad 1 \quad 1 \\
\end{align*}
\text{iff} \quad -- \quad \downarrow \quad -- \\
\begin{align*}
c &\quad 1 \quad 0 \\
d &\quad 1 \quad 0
\end{align*}
\text{for all positive values of } x

\item \( b ) \quad c + c \times c + c \times c \\
\begin{align*}
0 &\quad 1 \quad 1 \quad 2 \quad 2 \\
\downarrow &\quad \downarrow \quad \downarrow \\
d &\quad d + d \times d \\
0 &\quad 1 \quad 1 \quad 2 \quad 2
\end{align*}
\text{iff} \quad -- \quad \downarrow \quad \downarrow \\
\begin{align*}
c &\quad 2 \quad 0 \quad 1 \quad 1 \\
d &\quad d + d \times d \\
0 &\quad 2 \quad 0 \quad 1 \quad 1
\end{align*}
\text{for all positive values of } x , x

\item \( n ) \quad c + c \times c + c \times c + \ldots + c \times c \\
\begin{align*}
0 &\quad 1 \quad 1 \quad 2 \quad 2 \quad n \quad n \\
\downarrow &\quad \downarrow \quad \downarrow \\
d &\quad d + d \times d + d \times d + \ldots + d \times d \\
0 &\quad 1 \quad 1 \quad 2 \quad 2 \quad n \quad n
\end{align*}
\text{iff} \quad -- \quad \downarrow \quad \downarrow \\
\begin{align*}
c &\quad c + c \times c + \ldots + c \times c \\
\downarrow &\quad 0 \quad 1 \quad 1 \quad n - 1 \quad n - 1 \quad n - 1 \\
d &\quad d + d \times d + \ldots + d \times d \\
\downarrow &\quad 0 \quad 1 \quad 1 \quad n - 1 \quad n - 1 \quad n - 1
\end{align*}
\text{for all positive values of } x , x , \ldots , x
\end{enumerate}
Proof:

\[
\begin{align*}
&\frac{c + cx}{0 1 1} > \frac{c}{0} \\
i) \text{Let } \frac{d + dx}{0 1 1} > \frac{d}{0}\end{align*}
\]

Cross multiplying

\[
\begin{align*}
&cd + cdx > cd + cdx \\
&0 1 1 0 0 1 1 1
\end{align*}
\]

or

\[
\begin{align*}
&cdx > cdx \\
&1 0 1 0 1 1
\end{align*}
\]

\[
\begin{array}{ccc}
c & c & 1 \\
1 & 0 & 0
\end{array}
\]

\[
\begin{array}{ccc}
d & d & 1 \\
1 & 0 & 0
\end{array}
\]

ie. \(c \times c < d \times d\) since \(x\) is positive

Conversely,

\[
\begin{align*}
&\frac{c}{1} \times \frac{C}{0} \\
&\text{let } \frac{d}{1} \times \frac{d}{0}\end{align*}
\]

Cross multiplying

\[
\begin{align*}
&cd > cd \\
&1 0 1 0 1
\end{align*}
\]

Multiplying both sides by \(x\) where \(x > 0\), it becomes

\[
\begin{align*}
&cdx > cdx \\
&1 0 1 0 1 1
\end{align*}
\]

Adding a constant term \(cd\) to both sides the relation becomes

\[
\begin{align*}
&cdx + cd > cdx + cd \\
&1 0 1 0 0 1 1 0 0
\end{align*}
\]

ie. \((c + c)x + cd > (d + d)x + cd\)

\[
\begin{align*}
&1 1 0 0 1 1 0 0
\end{align*}
\]
\[
\begin{align*}
\text{cross multiplying} & \\
(c + c \times d) + c \times (d + d \times x) & \geq 0 \quad 1 \quad 1 \\
(c + c \times x) + d \times (c + c \times x) & \geq 0 \quad 1 \quad 1 \quad 2 \quad 2
\end{align*}
\]

or \[
(c \times (d + d \times x)) > d \times (c + c \times x)
\]

\[
\begin{align*}
\text{cross multiplying} & \\
(d + d \times x) + c \times (d + d \times x) & > 0 \quad 1 \quad 1 \\
(c + c \times x) + (d + d \times x) + d \times (c + c \times x) & > 0 \quad 1 \quad 1 \quad 2 \quad 2
\end{align*}
\]

\[
\begin{align*}
\text{or} & \\
c \times (d + d \times x) & > d \times (c + c \times x)
\end{align*}
\]

\[
\begin{align*}
\text{conversely} & \\
c + c \times x & \geq 0 \quad 1 \quad 1 \\
\text{Let} & \\
d + d \times x & \geq 0 \quad 1 \quad 1
\end{align*}
\]

\[
\begin{align*}
\text{ie., } c \times (d + d \times x) & > d \times (c + c \times x) \\
\text{since } x & \text{ is positive}
\end{align*}
\]

\[
\begin{align*}
\text{ie., } c \times (d + d \times x) & > d \times (c + c \times x) \\
\text{since } x & \text{ is positive}
\end{align*}
\]
Adding \((c +c x)(d +d x)\) to both sides the relation becomes:

\[
c x (d +d x) + (c +c x)(d +d x) > d x (c +c x) +
\]

\[
(c +c x)(d +d x)
\]

i.e., \((c +c x + c x)(d +d x) > (d +d x + d x)(c +c x)\)

\[
\]

\[
c +c x +c x
\]

\[
0 1 1 2 2
\]

\[
c +c x
\]

\[
0 1 1
\]

or

\[
-------
\]

\[
d +d x +d x
\]

\[
0 1 1 2 2
\]

\[
d +d x
\]

\[
0 1 1
\]

Similarly the other relations can be proved.

### 3.3 Ratio Algorithm:

The step by step procedure to obtain the optimal solution to the given Linear Fractional Programming problem is explained below.

The general Linear Fractional Programming problem is of the form:

\[
Z = \frac{c +c x +c x + \ldots + c x}{d +d x +d x + \ldots + d x}
\]

Maximize

\[
Z = \frac{1}{0 1 1 2 2 n n}
\]

\[
Z = \frac{d +d x +d x + \ldots + d x}{0 1 1 2 2 n n}
\]
It is to be noted here that in case the problem is given as

Minimize \( Z = \frac{1}{Z} \) then it has to be changed to the standard form as

Maximize \( Z = \frac{2}{Z} \)

subject to

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & \leq p_1 \\
& \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & \leq p_m
\end{align*}
\]

Note: Check whether all \( p_j \) (\( j = 1, 2, \ldots, m \)) are non-negative. If not multiply both sides by \(-1\) and change them to positive.

This problem can be converted to standard form by adding slack variables to the constraints. 
The standard form of the problem is

Maximize \[ Z = \sum_{i=0}^{n} c_i X_i \] subject to

\[ \sum_{i=1}^{n} a_{ij} X_i + S_j = p_j, \quad j = 1, 2, \ldots, m \]

\[ x_i \geq 0, \quad i = 1, 2, \ldots, n \]

\[ S_j \geq 0, \quad j = 1, 2, \ldots, m \]
Step 1:
The standard problem can be represented in tabular form as given below.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>Z</td>
<td>Z</td>
<td>x</td>
<td>x</td>
<td>...</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>...</td>
</tr>
<tr>
<td>Z - c</td>
<td>1j</td>
<td>0</td>
<td>-c</td>
<td>-c</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>n</td>
<td></td>
</tr>
<tr>
<td>Z - d</td>
<td>2j</td>
<td>0</td>
<td>-d</td>
<td>-d</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>n</td>
<td></td>
</tr>
</tbody>
</table>

ratio R

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
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<tr>
<td>Z - c</td>
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<td>-c</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>n</td>
<td></td>
</tr>
<tr>
<td>Z - d</td>
<td>2j</td>
<td>0</td>
<td>-d</td>
<td>-d</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>n</td>
<td></td>
</tr>
</tbody>
</table>

Step 2: Selection of entering variable

The selection of entering variable is done by computing the Z - c

ratio R = ------------ for all non-basic variables. Depending

J Z - d

upon the sign of Z - c and Z - d the R may be positive or

1j j 2j j

negative. The following table gives the value of the numerator

(Z ) of the objective function, Z - c , Z - d and R and the

1 1j j 2j j 2j j
condition under which improvement in the solution can be obtained. An important assumption is that \( Z \) must be always positive.

<table>
<thead>
<tr>
<th>Sign of ( Z )</th>
<th>Sign of ( Z - c )</th>
<th>Sign of ( Z - d )</th>
<th>Sign of ( R )</th>
<th>Condition for improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Positive</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>Minimum positive ( R &lt; \frac{Z}{j} )</td>
</tr>
<tr>
<td>2. Positive</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>Maximum positive ( R &gt; \frac{Z}{j} )</td>
</tr>
<tr>
<td>3. Positive</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>No improvement</td>
</tr>
<tr>
<td>4. Positive</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>Maximum negative ( R &lt; \frac{Z}{j} )</td>
</tr>
<tr>
<td>5. Negative</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>No improvement</td>
</tr>
<tr>
<td>6. Negative</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>Maximum positive ( R &gt; \frac{Z}{j} )</td>
</tr>
<tr>
<td>7. Negative</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>Maximum negative ( R &gt; \frac{Z}{j} )</td>
</tr>
<tr>
<td>8. Negative</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>Minimum negative ( R &lt; \frac{Z}{j} )</td>
</tr>
</tbody>
</table>
(Maximum negative means the largest of the negative values.
Minimum negative means the smallest of the negative values).

Step 2a)
   i) Among the positive $R^*$'s for which $Z - c$ is positive
      find the minimum $R$. Let it be $\text{POSITIVE}$. 
         \[ j \min \]
   ii) Among the positive $R^*$'s for which $Z - c$ is negative
        find the maximum $R$. Let it be $\text{POSITIVE}$. 
           \[ j \max \]
   iii) Among the negative $R^*$'s for which $Z - c$ is positive
        find the maximum $R$. Let it be $\text{NEGATIVE}$. 
           \[ j \max \]
   iv) Among the negative $R^*$'s for which $Z - c$ is negative
        find the minimum $R$ if $Z < 0$ otherwise find the 
        maximum $R$. Let it be $\text{NEGATIVE}$. 
           \[ j \min \]

Step 2b) Check whether $\text{POSITIVE} \ j \min$ is less than the current 
objective function value. If so, the entering 
variable is the variable corresponding to this positive 
ratio, and let it be $x$ then go to step 3; else step 
2(c)

Step 2c) Check whether $\text{POSITIVE} \ j \max$ is greater than the current 
objective function value. If so, the entering 
variable is the variable corresponding to this positive 
ratio and let it be $x$ then go to step 3; else step 2(d).
Step 2d) Check whether NEGATIVE is greater than the current max objective function value. If so, the entering variable is the variable corresponding to this negative ratio, and let it be \( x \), then go to step 3; else step 2(e).

Step 2e) Check whether NEGATIVE is less than the current min objective function value. If so, the entering variable is the variable corresponding to this negative ratio, and let it be \( x \), then go to step 3; else step 2(f).

Step 2f) Since no promising variable could be selected, the optimal solution has been reached. Go to step 5.

Step 3 Selection of leaving variable:

Find out the ratio using the right hand side (RHS) value and the corresponding updated constraint coefficient of the \( x \) th variable for which the constraint coefficient is greater than zero and select the minimum out of these ratios. The row variable corresponding to this minimum is the leaving variable. Go to step 4. If a minimum could not be found out then the problem has unbounded solution. Stop.

Step 4 Perform the row operation such that the entering variable column becomes a unity column. Go to step 2.
Step 5 The optimal solution for the problem is reached. The objective function value is the value given under RHS for the Rth row, and the variables staying in the basis are given in column 1 and their values are given in the RHS column against the respective variables.

3.4 RATIO ALGORITHM IN MATRIX FORM

The general Linear Fractional Programming problem can be represented in matrix form as given below.

\[
\begin{align*}
\text{Maximize } & \quad R = \frac{\mathbf{c}^T \mathbf{X} + c}{\mathbf{d}^T \mathbf{X} + d} \\
\text{subject to } & \quad \mathbf{A} \mathbf{X} \leq \mathbf{p} \\
& \quad \mathbf{X} \geq 0
\end{align*}
\]

where \( \mathbf{A} \) is an \( m \times n \) matrix (constraint matrix)

\( \mathbf{p} \) is an \( m \times 1 \) vector (resource vector)

\( c \) and \( d \) are Contribution Coefficient vectors of order \( n \times 1 \)

\( c \) and \( d \) are scalar constants

The additional assumption is that the denominator is positive for all feasible solutions (to avoid considering a host of possibilities in the exposition)
Let $X$ be the initial basic feasible solution such that

$$BX = \mathbf{p} \quad \text{(3.4.2)}$$

$$X = \mathbf{B} P \quad \text{(3.4.3)}$$

$$X \geq 0$$

$B = (P_1 P_2 \ldots \ldots P_m)$ is the $m \times m$ basis matrix where $P_i$ is the column of matrix $A$, corresponding to the $i$th variable.

Let

$$Z = c^T x + c^T \quad \text{(3.4.4)}$$

$$Z = d^T x + d^T \quad \text{(3.4.5)}$$

$$Z = \begin{array}{c}
\text{c}^T x + c^T \\
\text{d}^T x + d^T \\
\end{array} \quad \text{(3.4.6)}$$

where $c$ and $d$ are the vectors having their components as the coefficients associated with the basic variables in the numerator and the denominator of the objective function respectively.

To find another basic feasible solution which will improve the value of the objective function, as in the case of revised simplex method, the modified contribution coefficients of the non-basic variables have to be found out.
This can be done using the relations

\[
[Z - c_j] = \begin{bmatrix} T^{-1} & -c \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ c_j \\ B \end{bmatrix} \begin{bmatrix} -c \\ j \\ P \\ J \end{bmatrix} \quad \ldots \quad (3.4.7)
\]

\[
[Z - d_j] = \begin{bmatrix} T^{-1} & -d \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ d_j \\ B \end{bmatrix} \begin{bmatrix} -d \\ j \\ P \\ J \end{bmatrix} \quad \ldots \quad (3.4.8)
\]

In the new basic solution only one column of \( B \) given above is changed. In other words, if \( \bar{X} \) is the new basic feasible solution then

\[
\bar{X} = \bar{B}^{-1} P
\]

where \( \bar{B} = (\bar{P}_1 \bar{P}_2 \ldots \ldots \bar{P}_m) \) is the new basis matrix obtained from \( B \) by replacing \( P \) with \( P_r \).

Then, \( \bar{P}_i = P_i \quad (i \neq r) \)

\( \bar{P}_r = P_j \quad (i \neq r) \)

and \( \bar{Z} \) be the improved value of the objective function. It has been proved in section 3.2 that there would be improvement in the value of the objective function only if the ratio of the coefficients of the entering variable is greater than the current value of \( Z \).
3.5 **CONDITION FOR OPTIMALITY**

The conditions to be satisfied in order that \( \overline{Z} > Z \) can be derived as given below.

\[
\begin{align*}
Z & = \frac{1}{Z} \\
Z & = \frac{Z - \theta(Z - c)}{1} \\
\overline{Z} & = \frac{Z - \theta(Z - d)}{2} \\
\end{align*}
\]

where \( \theta \) is the value of the entering variable and is equal to

\[
-1 \quad \text{(B, p)} \\
\text{or} \\
-1 \quad \text{(B, p)} \\
\]

There will be improvement in the value of the objective function only if \( \overline{Z} > Z \)

That is if

\[
\begin{align*}
\frac{Z - \theta(Z - c)}{1} & > \frac{Z}{1} \\
\frac{Z - \theta(Z - d)}{2} & > \frac{Z}{2} \\
\end{align*}
\]

cross multiplying and simplifying

\[
\begin{align*}
Z(Z - \theta(Z - c)) & > Z(Z - \theta(Z - d)) \\
1Z & > 1Z \\
1Z & > 1Z \\
\end{align*}
\]

\[
\begin{align*}
\theta Z(Z - c) & > \theta Z(Z - d) \\
2 & > 2 \\
1j & > 2j \\
\end{align*}
\]

\[
\begin{align*}
Z(Z - c) & < Z(Z - d) \\
2 & < 2 \\
1j & < 2j \\
\end{align*}
\]

\[
\text{....(3.5.1)}
\]
But Z is always positive as per the assumption. Therefore (3.5.1) will be true if any one of the following conditions are satisfied, depending on the values of Z, (Z - c) and (Z - d).

**Case 1:** Z, (Z - c) and (Z - d) are all positive

Now the relation (3.5.1) yields the condition to be satisfied as

\[
\frac{Z - c}{Z} \quad \frac{1}{1j} < \quad \frac{1}{j} \quad \frac{Z - d}{Z} \quad \frac{2}{2j} \quad \frac{2}{j} \quad \text{(3.5.2)}
\]

**Case 2:** Z is positive, (Z - c) and (Z - d) are negative.

In this case (3.5.1) gives the condition

\[
\frac{Z - c}{Z} \quad \frac{1}{1j} > \quad \frac{1}{j} \quad \frac{Z - d}{Z} \quad \frac{2}{2j} \quad \frac{2}{j} \quad \text{(3.5.3)}
\]

**Case 3:** Z and (Z - d) are positive, (Z - c) negative.

According to this, (3.5.1) yields the condition.

\[
\frac{Z - c}{Z} \quad \frac{1}{1j} < \quad \frac{1}{j} \quad \frac{Z - d}{Z} \quad \frac{2}{2j} \quad \frac{2}{j} \quad \text{(3.5.4)}
\]

**Case 4:** Z, (Z - c) and (Z - d) are all negative.

\[
\frac{Z - c}{Z} \quad \frac{1}{1j} < \quad \frac{1}{j} \quad \frac{Z - d}{Z} \quad \frac{2}{2j} \quad \frac{2}{j} \quad \text{(3.5.5)}
\]
Under this, the condition

\[
\begin{align*}
Z - c & \quad Z \\
1j & \quad j & 1 \\
\hline
Z - d & \quad Z \\
2j & \quad j & 2
\end{align*}
\]

is obtained.

**Case 5:** \( Z \) and \((Z - c)\) are negative, \((Z - d)\) positive.

\[
\begin{align*}
Z - c & \quad Z \\
1j & \quad j & 1 \\
\hline
Z - d & \quad Z \\
2j & \quad j & 2
\end{align*}
\]

This gives the result

\[
\begin{align*}
Z - c & \quad Z \\
1j & \quad j & 1 \\
\hline
Z - d & \quad Z \\
2j & \quad j & 2
\end{align*}
\]

**Case 6:** \( Z \) and \((Z - d)\) are negative, \((Z - c)\) positive.

\[
\begin{align*}
Z - c & \quad Z \\
1j & \quad j & 1 \\
\hline
Z - d & \quad Z \\
2j & \quad j & 2
\end{align*}
\]

This case gives rise to the result

\[
\begin{align*}
Z - c & \quad Z \\
1j & \quad j & 1 \\
\hline
Z - d & \quad Z \\
2j & \quad j & 2
\end{align*}
\]

**Case 7:** \( Z \) and \((Z - c)\) are positive, \((Z - d)\) negative.

In this case the relationship \(3.5.1\) will not be true always since positive quantity is subtracted from the numerator and negative quantity is subtracted from the denominator.

**Case 8:** \( Z \) negative, \((Z - c)\) and \((Z - d)\) are positive.

In this case also the relationship \(3.5.1\) will not be true always since positive quantity is added to a
negative number leaving negative numerator and a positive quantity is added to the denominator. Hence the resulting left hand side value will be greater than the right hand side value.

If a variable, which satisfies any one of the conditions 1 to 6, can be selected then there would be improvement in the objective function value. If it is not possible to select such a variable (satisfying any one of these six conditions) there is no chance for improvement in the value of the objective function and the optimal solution has been reached.

3.6. STEP BY STEP PROCEDURE

The optimal solution to a general Linear Fractional Programming Problem, if it exists, can be obtained by using the following steps.

Step 1 Select an initial basic feasible solution say \( X_B \).

Step 2 Compute the values \( Z_1 \) and \( Z_2 \) of the numerator and the denominator of the objective function.

Step 3 Compute \([Z - c_j]_{1j}\) and \([Z - d_j]_{2j}\) for all the variables which are not in the basis, using the formula
Step 4. Compute the ratios

\[ \frac{Z - c}{1j} \] for all non-basic variables

Step 5

a) Among the non-basic variables for which \( \frac{Z - c}{1j} \) is positive and at the same time \( \frac{Z - d}{2j} \) and \( \frac{Z - c}{1j} \) are both positive select the minimum ratio \( r \) and let \( x \) be the corresponding variable.

\[ \frac{Z - c}{1j} \]

b) Among the non-basic variables for which \( \frac{Z - d}{2j} \) is positive and at the same time \( \frac{Z - c}{1j} \) and \( \frac{Z - d}{2j} \) are both positive select the minimum ratio \( r \) and let \( x \) be the corresponding variable.

\[ \frac{Z - c}{1j} \]
are both negative select the maximum ratio \( r \) and let \( x \) be the corresponding variable

\[
\frac{(Z - c)_{1j}}{1_j}
\]

c) Among the non-basic variables for which \( - \) is negative and \( (Z - d)_{2j} \) positive

select the maximum ratio \( r \) if \( Z > 0 \) else select

the minimum ratio \( r \) and let \( x \) be the corresponding variable

\[
\frac{(Z - c)_{1j}}{1_j}
\]

d) Among the non-basic variables for which \( - \) is negative select the

maximum ratio \( r \) and let \( x \) be the corresponding variable

\[
\frac{(Z - c)_{1j}}{1_j}
\]

**Step 6**

a) check whether \( r \) is less than \( - \). If so \( k \)

\[
\frac{Z}{2}
\]

\( x \) is the entering variable. Go to step 7.

\( k \)

b) check whether \( r \) is greater than \( - \). If so \( l \)

\[
\frac{Z}{2}
\]

\( x \) is the entering variable. Go to step 7.

\( l \)
c) check whether \( r \) is less than \(-\frac{Z}{p}\). If so  
\[ \frac{Z}{p} \]
\( x \) is the entering variable. Go to step 7.

\[ \frac{Z}{p} \]

d) check whether \( r \) is greater than \(-\frac{Z}{q}\). If so  
\[ \frac{Z}{q} \]
\( x \) is the entering variable. Go to step 7.

\[ \frac{Z}{q} \]

e) If no such variable could be selected then the  
current solution is the optimal solution. Go to  
Step 12.

Step 7 Let the variable selected by anyone of the steps 6(a)  
to 6(d) be \( x \). Compute \( B P \) and \( B P \).  
\[ j \]

Step 8 (Selection of leaving variable). Compute \( \theta \) using the  
relation

\[ \theta = \min_{i=1,2,3,...,m} \]

\[ \begin{bmatrix}
-1 \\
B P \\
o i \\
\vdots \\
-1 \\
B P \\
\end{bmatrix} \]

\[ \begin{bmatrix}
-1 \\
B P \\
j i \\
\vdots \\
-1 \\
B P \\
\end{bmatrix} \]

Let the minimum correspond to the \( r \)th row. Then  
the variable corresponding to the \( r \)th row is the  
leaving variable.
If \((B P)_{ji} < 0\) for all \(i\) then there is unbounded solution to the given problem.

**Step 9 a) (Computation of M matrix)**

The matrix corresponding to the new basic solution can be obtained by using the product form of inverse. As a first step \(\mathbf{N}\) vector is computed using the relations,

\[
\mathbf{N} = \mathbf{Z} - c_{1j} \quad \mathbf{N} = \mathbf{Z} - d_{2j} \quad \mathbf{N} = (B P)_{ji} - 1
\]

\[i = 1, 2, 3, ..., m\]

b) **Computation of \(N_{new}\)**

Since the variable corresponding to the \(r\)th row is the leaving variable, \((r+2)\) element in the \(\mathbf{N}\) vector is the pivot element. \(\mathbf{N}_{new}\) can be obtained using the following relation

\[
-(\mathbf{N}_{i old}) \quad i = 1, 2, 3, ..., m+2
\]

\[
(\mathbf{N}_{i new}) = \frac{-------}{i < r+2}
\]

\[
1 \quad i = 1, 2, 3, ..., m\]

\[
(\mathbf{N}_{i new}) = \frac{-------}{r+2 new}
\]

\[
(\mathbf{N}_{i old}) = \frac{-------}{r+2 old}
\]
Step 10:

$-1\quad M$ can be found out by pre multiplying the current
next $-1 \quad M$ by a matrix $E$ in which all columns except one are unit
columns. Suppose that $P$ in the current basis is replaced by a
new vector $P$

Let $a = B_j P$ where $a_j$ is the $k$th element of $a_j$ then

$-1 \quad M$ can be computed as
next $-1 \quad M = E \quad M$ \hspace{1cm} (3.6.4)
next $\quad$ old

Here $E = (e_j, e_2, e_3, \ldots e_r, e_r, e_{r+1}, \ldots e_m)$

$1 \ 2 \ 3 \ \ r-1 \ \ r+1 \ \ m$

and $a =$

\[
\begin{bmatrix}
    j & j & -a_j/a_j & & & & \\
    1 & r & & & & & \\
    j & j & -a_j/a_j & & & & \\
    2 & r & & & & & \\
    & & & & & \ddots & & \\
    j & j & -a_j/a_j & & & & \\
    1 & r & & & & & \\
    . & & & \ddots & & & & \\
    . & & & & \ddots & & & \\
    . & & & & & \ddots & & & \\
    . & & & & & & \ddots & & \\
    & & & & & & & \ddots & & \\
    & & & & & & & & \ddots & & \\
    & & & & & & & & & \ddots & & \\
    & & & & & & & & & & \ddots & & \\
    & & & & & & & & & & & \ddots & & \\
    & & & & & & & & & & & & \ddots & & \\
    & & & & & & & & & & & & & \ddots & & \\
    & & & & & & & & & & & & & & \ddots & & \\
    & & & & & & & & & & & & & & & \ddots & \rightarrow r \text{ place} \\
\end{bmatrix}
\]

provided that $a_j \neq 0$
Let $M$ represent the inverted matrix.

1. The matrix is updated in each iteration. If $le$ denotes the leaving element then

   $$M_{1} = M_{1} + \n(i) \ast \text{pivot} \quad \text{for all } i \neq le+1$$

   $$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (i,j)$$

   and

   $$M_{1} = \n(i) \ast \text{pivot} \quad \text{if } i = le + 1$$

   $$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (i,j)$$

for columns having elements entered already. All other columns are left as they are, except the one corresponding to $(r+2)$ position which is changed as $\n$ column.

Step 11: Repeat steps 3 to 10 until an optimal solution is reached or there is an indication of unbounded solution.

Step 12: Compute the optimal value and the solution vector using the equation,

$$\begin{bmatrix}
Z \\
1 \\
Z \\
2 \\
X \\
B
\end{bmatrix} \begin{bmatrix}
-1 \\
c \\
op \\
d \\
op \\
op
\end{bmatrix} = M$$

3.7 Procedure to Solve Problems with Mixed Type of Constraints

The step by step procedure explained in 3.4 describes the method of finding the optimal solution when all the constraints are of upperbound type (less than or equal to form).
If mixed type of constraints are given then the procedure has to be slightly modified. A two-phase technique can be used. In the first phase a feasible solution is obtained as is done in phase 1 of Two-phase Simplex method for Linear Programs. Once a feasible solution is obtained, then in phase 2 the coefficients of the variables in the original objective function are taken and steps 3 to 10 of the previous section 3.6 are used to obtain the optimal solution.

Consider a Linear Fractional programming problem of the following form

\[
\begin{align*}
\text{Max } Z &= \frac{T}{c X + c_0} \\
\text{Subject to } &\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} X \end{bmatrix} \leq P \\
&\begin{bmatrix} K \end{bmatrix} \begin{bmatrix} X \end{bmatrix} \geq K \\
&\begin{bmatrix} L \end{bmatrix} \begin{bmatrix} X \end{bmatrix} = L
\end{align*}
\]

where

\[
X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
\]
A is m x n upperbound constraint matrix
K is k x n lowerbound constraint matrix
L is 1 x n equality constraint matrix.

To obtain the solution to this problem the two phase technique is used by converting into two problems. For phase I the objective function and the constraints would be

\[
\begin{align*}
\text{Minimize} \quad & \sum_{i=1}^{k+1} R_i \\
\text{subject to} \quad & AX \leq P \\
& KX \geq K \\
& LX = L
\end{align*}
\]

where \( k \) and \( 1 \) are number of lowerbound and equality constraints. \( R \) is the artificial variable added to each \( i \) lowerbound and equality constraint.

The aim of phase I is to find a feasible solution.

The feasible solution for this phase can be obtained by applying either the simplex method or the revised simplex method. Once the feasible solution is obtained, phase II is applied by considering the original objective function and the constraints given. With this initial solution, the optimality of the solution is found out using the ratio algorithm given in 3.6.
3.8 CONCLUSION:

An efficient method of solving Linear Fractional Programming problem has been presented in this chapter. Since more number of optimality conditions are to be tested in each iteration, this algorithm is best suited for computers. Method of solving Problems using matrix form has been discussed in detail. Two phase method for solving problems with mixed type of constraints has also been explained in this chapter.