7. NUMERICAL EXAMPLES

7.1 INTRODUCTION:

The various theoretical aspects of Fractional Ratio Algorithms have been discussed in the earlier chapters and all these algorithms are illustrated with worked examples, in this chapter.

7.2 LINEAR FRACTIONAL PROGRAMMING PROBLEM WITH UPPER BOUND CONSTRAINTS:

7.2.1 Example 1:

Maximize 
\[
Z = \frac{5x + 3x}{1} = \frac{5x + 2x + 1}{2}
\]
subject to
\[
\begin{align*}
3x + 5x &\leq 15 \\
5x + 2x &\leq 10 \\
x, x &\geq 0
\end{align*}
\]
Step 1: The problem can be represented in the tabular form as given below:

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>Z</th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Z - C</td>
<td>1</td>
<td>0</td>
<td>-5</td>
<td>-3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>Z - C</td>
<td>0</td>
<td>1</td>
<td>-5</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>j</td>
<td>R =</td>
<td>j</td>
<td>j</td>
<td>j</td>
<td>j</td>
<td>j</td>
<td>j</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1.5</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>0</td>
<td>15</td>
</tr>
<tr>
<td>1</td>
<td>S</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Iteration 1:

Step 2: The selection of the entering variable is done by computing the ratio \( R = \frac{Z_c}{Z_j} \) for all non-basic variables. Since \( Z \) is positive and all \( R \)'s are positive the maximum \( R \) should be greater than the objective function ratio. Here the maximum \( R \) is 1.5 and this value is greater than the objective function value 0. Therefore the corresponding variable \( x \) is the entering variable.
Step 3: (Selection of leaving variable)

\[ \theta = \min \begin{bmatrix} -1 \\ B \ p \\ 0 \\ -1 \\ B \ p \end{bmatrix} \ \text{where} \ p \ \text{is the column} \ \ \ i = 1, 2 \]

\[ \begin{bmatrix} S \\ S \\ 1 \\ 2 \\ \text{s} \end{bmatrix} = \min \begin{bmatrix} 3 \\ 5 \end{bmatrix} \]

\[ = 3 \ \text{corresponding to variable} \ S \]

ie, S leaves the basis.

pivot element is 5.

Step 4: Perform the simplex iteration and the table at the end of this operation is as given below.

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>Z</th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Z - C</td>
<td>1</td>
<td>0</td>
<td>-16/5</td>
<td>0</td>
<td>3/5</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>Z - d</td>
<td>0</td>
<td>1</td>
<td>-19/5</td>
<td>0</td>
<td>2/5</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Z - C</td>
<td>-</td>
<td>-</td>
<td>16/19</td>
<td>-</td>
<td>3/2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Z - d</td>
<td>-</td>
<td>-</td>
<td>19/5</td>
<td>-</td>
<td>-2/5</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>x</td>
<td>0</td>
<td>0</td>
<td>3/5</td>
<td>1</td>
<td>1/5</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>S</td>
<td>0</td>
<td>0</td>
<td>19/5</td>
<td>0</td>
<td>-2/5</td>
<td>1</td>
</tr>
</tbody>
</table>
Iteration 2:

Step 2:  Ratio (R_{j}) is positive for x and S; Z is positive. However Z - c_{j} is negative for x & positive for S. Then for a variable to be promising either the maximum ratio corresponding to negative Z - c_{j} should be greater than the objective function value 9/7 or positive minimum corresponding to S should be less than the objective function value 9/7. None of these situations is true. Therefore there is no promising variable and hence, the optimal solution is reached.

Step 5:  The optimal solution is

\[
\begin{align*}
    x &= 0 \\
    Z &= 9/7 \\
    x &= 3 \\
    Z &= 9/7
\end{align*}
\]

7.2.2 Example 2:

Maximize

\[
Z = \begin{array}{c}
-8x - 7x + 16 \\
\frac{1}{2}
\end{array}
\]

subject to

\[
\begin{align*}
2x + \frac{3x}{2} &\leq 2 \\
\frac{x}{2} + 5x &\leq 5 \\
x, x &\geq 0
\end{align*}
\]
Step 1: The problem can be represented in the tabular form as given below:

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>Z</th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z - c</td>
<td>1</td>
<td>0</td>
<td>8</td>
<td>7</td>
<td>0</td>
<td>0</td>
<td>16</td>
</tr>
<tr>
<td>Z - d</td>
<td>0</td>
<td>1</td>
<td>12</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>17</td>
</tr>
<tr>
<td>Z - c</td>
<td>0</td>
<td>1</td>
<td>12</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>17</td>
</tr>
<tr>
<td>Z - d</td>
<td>0</td>
<td>1</td>
<td>12</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>17</td>
</tr>
</tbody>
</table>

Iteration 1

Step 2: (Selection of entering variable).

This is done by computing the ratio \( R = \frac{Z - c}{Z - d} \) for all non-basic variables. Since \( Z \) is positive and all \( R \)'s are positive the minimum \( R \) should be less than the objective function ratio. Here \( R \) is minimum for \( x \) and this value 8/12 is less than 16/17.

Therefore \( x \) is the entering variable.
Step 3  (Selection of leaving variable)

Compute \( \theta = \min \begin{bmatrix} -1 & B & p & 0 \\ -1 & 0 & 1 \\ -1 & B & p & 1 \end{bmatrix} \) where \( p \) is the column corresponding to the variable \( x \)

\[
\begin{align*}
S & \quad S \\
1 & \quad 2
\end{align*}
\]

\[
= \min \begin{bmatrix} 1 & 5 \end{bmatrix}
\]

= 1 corresponding to \( S \)

i.e., \( S \) is the leaving variable and the pivot element is \( 1 \) \( 2 \).

Step 4: The table obtained at the end of the simplex iteration is given below.

<table>
<thead>
<tr>
<th>( Z )</th>
<th>( Z )</th>
<th>( x )</th>
<th>( x )</th>
<th>( S )</th>
<th>( S )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z - C )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-5</td>
<td>-4</td>
<td>0</td>
</tr>
<tr>
<td>( Z - d )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-12</td>
<td>-6</td>
<td>0</td>
</tr>
<tr>
<td>( Z - C )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>5/12</td>
<td>4/6</td>
<td>0</td>
</tr>
<tr>
<td>( Z - d )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( x )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3/2</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>( S )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>7/2</td>
<td>-1/2</td>
<td>1</td>
</tr>
</tbody>
</table>
Iteration 2

Step 2: Ratio $R$ is positive for $x$ and $S$. $Z$ is positive and $Z - c$ is negative for both $x$ and $S$. Then for a variable to be promising the maximum ratio corresponding to $Z - c$ negative should be greater than the objective function value $8/5$. There is no such case. Hence the optimal solution is reached.

Step 5: The optimal solution is

\[
\begin{align*}
x &= 1 \\
1 & \quad 8 \\
0 & \quad Z = - \\
2 &
\end{align*}
\]

7.2.3. Example 3:

Maximize $Z = \frac{-8x - 6x + 12}{1 + 2}$

subject to

\[
\begin{align*}
x + x & \leq 1 \\
1 & \quad 2 \\
3x + x & \leq 4 \\
1 & \quad 2 \\
x, x & \geq 0 \\
1 & \quad 2
\end{align*}
\]

For this problem the optimal solution is

\[
\begin{align*}
x &= x = 0 \text{ and } Z = 12/10.
\end{align*}
\]

since for any positive value for either $x$ or $x$ the numerator value of the objective function decreases and the denominator value increases and hence there is no scope for improvement.
7.2.4. Example 4:

Maximize \( Z = \frac{8x + 6y + 11}{2} \)

subject to

\[
\begin{align*}
x + x & \leq 2 \\
1 & \ 2 \\
3x + 5y & \leq 7 \\
1 & \ 2 \\
x, x & \geq 0 \\
1 & \ 2
\end{align*}
\]

**Step 1:** The problem can be represented in the tabular form as given below:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>2</th>
<th>S</th>
<th>S</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z - c )</td>
<td>1</td>
<td>0</td>
<td>-8</td>
<td>-6</td>
<td>0</td>
<td>0</td>
<td>11</td>
</tr>
<tr>
<td>( Z - d )</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>( Z - c )</td>
<td>-</td>
<td>-</td>
<td>-2</td>
<td>-3</td>
<td>-</td>
<td>-</td>
<td>11/9</td>
</tr>
<tr>
<td>( Z - d )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( Z - c )</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>7</td>
</tr>
</tbody>
</table>
Step 2: (Selection of entering variable)

The entering variable is selected based on the ratio

$$ R = \frac{Z_c}{Z - d_j} $$

for all non-basic variables.

Since $Z$ is positive and all $R$'s are negative the maximum negative $R$ should be less than the objective function value. Therefore the entering variable is $x_1$.

Step 3: (Selection of leaving variable)

Compute

$$ \theta = \min \begin{bmatrix} -1 \\ B \ p \ 0 \\ -1 \\ B \ p \ 1 \end{bmatrix} \begin{bmatrix} S \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} S \\ 1 \\ 2 \end{bmatrix} = \min \begin{bmatrix} 2 \\ 7/3 \end{bmatrix} = 2, \ i.e., \ S \ is \ the \ leaving \ variable. $$

The pivot element is 1.
Step 4: Performing the simplex iteration, the table obtained is as given below.

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>Z</th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z - C</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>8</td>
<td>0</td>
<td>27</td>
</tr>
<tr>
<td>Z - d</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>-4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Z - C</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>-2</td>
<td>0</td>
<td>27</td>
</tr>
<tr>
<td>Z - d</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>-3</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Iteration 2:

Step 2: Ratio R is negative for x and S; Z is positive and Z - c is positive. Therefore there will not be any improvement in the objective function. Hence the optimal solution is reached.

Step 5: The optimal solution is

\[ x = 2 \]
\[ 1 \]
\[ Z = 27 \]
\[ x = 0 \]
\[ 2 \]
7.2.5. Example 5

Maximize \[ Z = \frac{-2x - 3x + 5}{1 - 2} \]
subject to
\[ x + \frac{x}{1 - 2} \leq 2 \]
\[ 2x + \frac{3x}{1 - 2} \leq 8 \]
\[ x, \frac{x}{1 - 2} \geq 0 \]

Here for any value of \( x \) or \( \frac{x}{1 - 2} \) the numerator value decreases in the negative direction and the denominator in the positive direction. Hence the ratio will be decreasing. Therefore this will not improve the objective function and the solution is \( x = 0; \ x = 0; \ z = 5/10 \).

7.2.6. Example 6

Maximize \[ Z = \frac{3x + 2x - 20}{1 - 2} \]
subject to
\[ 2x + \frac{x}{1 - 2} \leq 6 \]
\[ 3x + \frac{4x}{1 - 2} \leq 18 \]
\[ x, \frac{x}{1 - 2} \geq 0 \]
Step 1: The problem can be represented in the tabular form as given below:

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>Z</th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z - c</td>
<td>1</td>
<td>0</td>
<td>-3</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>-20</td>
</tr>
<tr>
<td>Z - d</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>-4</td>
<td>-5</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>Z - c</td>
<td>1</td>
<td>0</td>
<td>-4</td>
<td>3/4</td>
<td>2/5</td>
<td>-</td>
<td>-4</td>
</tr>
<tr>
<td>Z - d</td>
<td>2</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>s</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>s</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>18</td>
</tr>
</tbody>
</table>

Step 2 (Selection of entering variable)

The entering variable is selected based on the ratio

\[ R = \frac{Z - c}{Z - d} \]

for all non-basic variables. Since Z is negative and R's are positive the maximum possible \( R \) should be greater than ---. For \( x \), the ratio is maximum \( R \) and is greater than the objective function value. Therefore, the entering variable is \( x \).
Step 3: (Selection of leaving variable)

Compute \( \theta = \min \left[ \begin{array}{c}
-1 \\
B p \\
0 \\
-1 \\
B p \\
1 \\
^i
\end{array} \right] \) where \( p \) is the column corresponding to the variable \( x_i \)

\[ i = 1, 2 \]

\[ = \min \left[ \begin{array}{c}
3 \\
6
\end{array} \right] = 3 \]

ie, \( S \) is the leaving variable and the pivot is 2.

Step 4: Performing the simplex iteration the table obtained is as given below:

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>Z</th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z - C )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1/2</td>
<td>3/2</td>
<td>0</td>
<td>-11</td>
</tr>
<tr>
<td>( Z - d )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-3</td>
<td>2</td>
<td>0</td>
<td>17</td>
</tr>
<tr>
<td>( Z - C )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1/6</td>
<td>3/4</td>
<td>0</td>
<td>-11</td>
</tr>
<tr>
<td>( Z - d )</td>
<td>1/6</td>
<td>0</td>
<td>0</td>
<td>5/2</td>
<td>-3/2</td>
<td>1</td>
<td>9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>S</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Iteration 2:

Step 2: (Selection of entering variable)

Ratio \( R \) is positive for \( x \) and \( S \). \( Z - c \) is positive for \( S \) and \( Z \) is negative for \( x \) and \( S \). Z - c is negative for \( x \) and positive for \( S \) and \( Z \) is negative. Hence \( s \) cannot enter the basis. But in case of \( x \) the ratio is greater than the objective function value and therefore \( x \) is the entering variable.

Step 3: (Selection of leaving variable)

Column \( P \) corresponds to the variable \( x \).

\[
\theta = \min \begin{bmatrix}
-1 \\
B_p \\
0 \\
-1 \\
B_p \\
2
\end{bmatrix}
\]

Column \( P \) corresponds to the variable \( x \).

\[
x \quad S \\
1 \\
2
\]

\[
= \min \begin{bmatrix}
6 \\
18/5
\end{bmatrix} = 18/5, \text{ } S \text{ } is \ the \ leaving \ variable
\]

The pivot element is 5/2.
Step 4: Performing the simplex iteration the table obtained is as given below:

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>Z</th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z - C</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>12/10</td>
<td>1/5</td>
<td></td>
</tr>
<tr>
<td>1j</td>
<td>1j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z - d</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1/5</td>
<td>6/5</td>
<td></td>
</tr>
<tr>
<td>2j</td>
<td>2j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z - C</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>6</td>
<td>1/6</td>
<td></td>
</tr>
<tr>
<td>1j</td>
<td>1j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z - d</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>139</td>
<td></td>
</tr>
<tr>
<td>2j</td>
<td>2j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Iteration 3:

Step 2: R's are positive for S and S since Z is negative and Z - c is positive for both S and S there will not be any improvement if either S or S enters. Hence the optimum is reached.

Step 5: The optimal solution is

\[
x = \frac{6}{5} \quad Z = -46
\]

\[
x = \frac{18}{5} \quad Z = 139
\]
### Example 7

Maximize \( Z = \frac{-3x - 4x - 16}{1 \ 2} \)

subject to

\[
\begin{align*}
2x + 3x & \leq 8 \\
1 & \quad 2 \\
3x + 4x & \leq 15 \\
1 & \quad 2 \\
x, x & \geq 0 \\
1 & \quad 2
\end{align*}
\]

**Step 1**: The problem can be represented in the tabular form as given below:

<table>
<thead>
<tr>
<th></th>
<th>( Z )</th>
<th>( Z )</th>
<th>( x )</th>
<th>( x )</th>
<th>( S )</th>
<th>( S )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( Z - C )</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>( Z - d )</td>
<td>0</td>
<td>1</td>
<td>-4</td>
<td>-3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( Z - Z )</td>
<td>-</td>
<td>-</td>
<td>-3/4</td>
<td>-4/3</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>( Z - d )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( Z - d )</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Iteration 1:

Step 2: (Selection of entering variable)

It is done by computing the ratio

$$c_{j} - c_{1j}$$

for all non-basic variables.

$$Z = \frac{c_{j}}{c_{1j}} \quad \text{for all non-basic variables.}$$

$$Z = \frac{d}{j}$$

$Z$ is negative, all $R'_s$ are negative and $Z - c$ is positive. Hence the maximum $R$ should be greater than the $j$th objective function value if the $j$ variable is to become the entering variable. Here for the variable $x_1$, $R$ is $-3/4$ and is greater than $-4/5$. Therefore $x_1$ is the entering variable.

Step 3: (Selection of leaving variable)

Compute $\theta = \min_{i=1,2} \left[ \begin{array}{cc} -1 & 0 \\ B_p & 1 \\ 1 & B_p \\ \end{array} \right]$ column $p$ corresponds to the variable $x_i = 1, 2$

$$S = \min \left[ \begin{array}{c} 4 \\ 5 \\ 1 \\ \end{array} \right] = 4$$

ie., $S$ leaves the basis and the pivot element is 2.
Step 4: Performing the simplex iteration the table obtained at the end is given below.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>2</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z - C$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1/2</td>
<td>-3/2</td>
<td>0</td>
<td>-28</td>
</tr>
<tr>
<td>$Z - d$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>36</td>
</tr>
<tr>
<td>$Z - C$</td>
<td>-</td>
<td>-</td>
<td>-1/6</td>
<td>-3/4</td>
<td>-</td>
<td>-7</td>
<td></td>
</tr>
<tr>
<td>$Z - d$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>3/2</td>
<td>1/2</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>$x$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3/2</td>
<td>1/2</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>$S$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1/2</td>
<td>-3/2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Iteration 2:

Step 2: (Selection of entering variable)

R's are negative for $x$ and $S$ and $Z$ is negative. For either $x$ or $S$ to be promising the minimum negative ratio should be less than the objective function value. The minimum negative ratio corresponding to $S$ has value $-3/4$. and this is not less than the objective function value. Therefore, there is no more promising variable and the optimal solution is reached.
Step 5: The optimal solution is
\[ x = 4 \quad -7 \]
\[ 1 \quad Z = \quad 9 \]
\[ x = 0 \quad 2 \]

7.2.8 Example 8.

Maximize
\[
Z = \frac{3x + 4x - 16}{1} \quad \frac{1}{2}
\]
\[
-4x - 3x + 20
\]
\[ 1 \quad 2 \]
subject to
\[
2x + 3x \leq 9
\]
\[ 1 \quad 2 \]
\[
3x + 4x \leq 15
\]
\[ 1 \quad 2 \]
\[
x, x \geq 0
\]
\[ 1 \quad 2 \]

Step 1: The problem can be represented in the tabular form as given below.

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>Z</th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Z - C</td>
<td>1</td>
<td>0</td>
<td>-3</td>
<td>-4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>Z - d</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>Z - C</td>
<td>-</td>
<td>-</td>
<td>-3/4</td>
<td>-4/3</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>Z - d</td>
<td>-</td>
<td>-</td>
<td>3/4</td>
<td>4/3</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>s</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>s</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Iteration 1:

Step 2: (Selection of entering variable)

It is done by computing the ratio

\[
\frac{Z - c_{ij}}{Z - d_{2j}}
\]

for all non-basic variables.

Here \( Z \) is negative, \( Z - c \) is negative and \( R_j \)'s are all negative. Hence for a variable to be promising the minimum ratio should be less than the objective function value. Here the minimum value is \(-4/3\) and this corresponds to \( x_2 \). This is less than the objective function value. Hence \( x_2 \) is the entering variable.

Step 3: (Selection of leaving variable):

Compute \( \theta = \min \frac{-1}{B_{2p}} \) column \( P \) corresponds to the variable \( x_2 \)

\[
\begin{bmatrix}
-1 & 0 \\
-1 & -1 \\
2 & 2
\end{bmatrix}
\]

\( i = 1, 2 \)
Step 4: Performing the simplex iteration the table obtained is as given below:

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>Z</th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z - c</td>
<td>1</td>
<td>0</td>
<td>-1/3</td>
<td>0</td>
<td>4/3</td>
<td>0</td>
<td>-4</td>
</tr>
<tr>
<td>1j  j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z - d</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>11</td>
</tr>
<tr>
<td>2j  j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>------</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z - c</td>
<td>0</td>
<td>0</td>
<td>2/3</td>
<td>1</td>
<td>1/3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1j  j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z - d</td>
<td>0</td>
<td>0</td>
<td>1/3</td>
<td>0</td>
<td>-4/3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2j  j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Iteration 2

Step 2: Z is negative, R's are negative for x & S. Since Z - c
1 1 1
j j
is negative for x, x can enter the basis if the ratio
1 1

\[
\frac{Z - c}{1j - j} = \frac{-1}{6}
\]

is less than the objective function value \(-4/11\).

This is not true. Hence x is not promising. For S to be promising the ratio \((-4/3)\) must be greater than the objective function value \((-4/11)\). This is also not
true. Therefore there is no more promising variable, and optimal solution is reached.

Step 5: The optimal solution is:

\[
\begin{align*}
x &= 0 & -4 \\
1 & & \\
x &= 3 & 11 \\
2 & & \\
\end{align*}
\]

7.3 LINEAR FRACTIONAL PROGRAMMING PROBLEM IN MATRIX FORM

Consider the LFPP of the form

Maximize \[
Z = \frac{5x + 3x}{1 + 2}
\]
subject to

\[
\begin{align*}
3x + 5x & \leq 15 \\
1 & 2 \\
5x + 2x & \leq 10 \\
1 & 2 \\
x , x & \geq 0 \\
1 & 2 \\
\end{align*}
\]

This can be written in matrix form as follows.

Maximize \[
Z = \frac{5x + 3x}{1 + 2} + 1
\]
subject to

\[
\begin{bmatrix}
3 & 5 \\
5 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\leq
\begin{bmatrix}
15 \\
10
\end{bmatrix}
\]

\[
T^nC = [5 \ 3]^T \quad T^\top d = [5 \ 2]
\]

\[
A = \begin{bmatrix}
3 & 5 \\
5 & 2
\end{bmatrix} \quad P = 15
\]

Step 1: Initial basic feasible solution is

\[
X = \begin{bmatrix}
S \\
1 \\
S \\
2
\end{bmatrix}
= \begin{bmatrix}
15 \\
10
\end{bmatrix};
\]

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Iteration 1:

Step 2: \( Z = 0 \) \( Z_1 = 1 \) \( Z_2 = 0/1 = 0 \)

Step 3: Compute

\[
\begin{bmatrix}
Z - c_j \\
1j \\
Z - d_j \\
2j
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0
\end{bmatrix}\begin{bmatrix}
-5 & -3 \\
-5 & -2
\end{bmatrix} = \begin{bmatrix}
-5 & -3 \\
-5 & -2
\end{bmatrix}
\]

\[
\begin{bmatrix}
5 & 2
\end{bmatrix}
\]
Step 4: Compute the ratio \( \frac{Z - c}{1_j J} \) for all non basic \( Z - d \) variables

\[
\begin{array}{c c c c c c}
  x & x & x & x \\
  1 & 2 & 1 & 2 \\
\end{array}
\]

\[ [R R J] = [1, 1.5] \]

Step 5: (Selection of entering variable):

Since both \( Z - c \) and \( Z - c \) are negative for both \( 1_j J \) the variables, the variable which corresponds to the maximum ratio is the entering variable. The maximum ratio is 1.5 and it corresponds to \( x \) \( 2 \)

Step 6: Since the ratio 1.5 is greater than the objective function value at step 2, \( x \) is the entering variable.

Step 7: Compute \( B_p \) and \( B_{p0} \)

\[
B_{p0} = \begin{bmatrix}
  1 & 0 & 15 \\
  0 & 1 & 10 \\
\end{bmatrix} = \begin{bmatrix}
  15 \\
  10 \\
\end{bmatrix}
\]

\[
B_{p2} = \begin{bmatrix}
  1 & 0 & 5 \\
  0 & 1 & 2 \\
\end{bmatrix} = \begin{bmatrix}
  5 \\
  2 \\
\end{bmatrix}
\]
Step 8: Compute θ using the relation

\[
\theta = \min \left( \begin{bmatrix}
-1 \\
B_p^0 \\
0 \\
\end{bmatrix}, \begin{bmatrix}
-1 \\
B_p^1 \\
2 \\
\end{bmatrix} > 0 \right)
\]

\[
S = \min \left[ \begin{array}{cc}
1 & 2 \\
3 & 5 \\
\end{array} \right] = 3
\]

ie., S is the leaving variable.

Step 9a) Computation of θ vector

The θ vector is computed as below

\[
\begin{bmatrix}
Z - c_j \\
12 \\
Z - d_j \\
22 \\
-1 \\
(B_p)^2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-3 \\
-2 \\
5 \\
2 \\
\end{bmatrix}
\]

Since S is the leaving variable 3 element 5 is the pivot element.
9b) Compute $\nabla$ using the pivot as given below

$$\nabla = \begin{bmatrix} 3/5 \\ 2/5 \\ 1/5 \\ -2/5 \end{bmatrix}$$

Step 10: $E$ matrix is given by

$$E = \begin{bmatrix} 1 & 0 & 3/5 & 0 \\ 0 & 1 & 2/5 & 0 \\ 0 & 0 & 1/5 & 0 \\ 0 & 0 & -2/5 & 1 \end{bmatrix}$$

$$M_{\text{new}} = E^{-1} M_{\text{old}}$$

$$= \begin{bmatrix} 1 & 0 & 3/5 & 0 \\ 0 & 1 & 2/5 & 0 \\ 0 & 0 & 1/5 & 0 \\ 0 & 0 & -2/5 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3/5 & 0 \\ 0 & 1 & 2/5 & 0 \\ 0 & 0 & 1/5 & 0 \\ 0 & 0 & -2/5 & 1 \end{bmatrix}$$

Iteration 2:

Step 2: $Z = 3 Z_{\text{old}} = 3/2$
Step 3: Compute

\[
\begin{bmatrix}
Z - c \\
1j j
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 3/5 & 0 \\
0 & 1 & 2/5 & 0
\end{bmatrix}
\begin{bmatrix}
-5 \\
-2
\end{bmatrix}
\]

\[
\begin{bmatrix}
Z - d \\
2j j
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 2/5 & 0 \\
3 & 5
\end{bmatrix}
\begin{bmatrix}
-5 \\
2
\end{bmatrix}
\]

\[
x_1 x_2 S_1 S_2
\begin{bmatrix}
1 & 2 & 1 & 2 \\
-16/5 & 0 & 3/5 & 0 \\
-19/5 & 0 & 2/5 & 0
\end{bmatrix}
\]

Step 4:

Compute the ratio \( \frac{Z - c}{1j j} \) for all non basic \( Z - d \) variables \( 2j j \)

\[
x_1 S
\begin{bmatrix}
1 & 1 \\
16/19 & 3/2 \\
1 & 2
\end{bmatrix}
\]

Step 5: For \( x \) both \( Z - c \) and \( Z - d \) are negative and \( 1 \)

\( 1j j 2j j \)

hence the maximum ratio is \( 16/19 \) (\( r_1 \))

For \( S \) both \( Z - c \) and \( Z - d \) are positive and \( 1 \)

\( 1j j 2j j \)

hence the minimum ratio is \( 3/2 \) (\( r_k \))

Step 6: Since \( r_1 \leq Z \) and \( r_k \leq Z \)

\[
\begin{bmatrix}
16 & 3 \\
29 & 2
\end{bmatrix}
\begin{bmatrix}
3 & 3 \\
2 & 2
\end{bmatrix}
\]
there is no more promising variable and hence the optimal solution is reached.

**Step 11:** The solution is

\[
\begin{bmatrix}
    x \\
    S \\
\end{bmatrix} =
\begin{bmatrix}
    1/5 & 0 & 15 \\
    -2/5 & 1 & 10 \\
\end{bmatrix}
\begin{bmatrix}
    3 \\
    4 \\
\end{bmatrix}
\]

\[Z = 3/2.\]

7.4 **MIXED TYPE OF CONSTRAINTS**

7.4.1 **SOLUTION BY PHASE I - PHASE II TECHNIQUE**

Maximize \(Z = \frac{4x + 3y + 1}{1 + 2}\)

subject to

\[
\begin{align*}
    x + x & \leq 3 \\
    1 & \quad 2 \\
    2x + x & \geq 4 \\
    1 & \quad 2 \\
    3x + 2x & \geq 7 \\
    1 & \quad 2 \\
    x, x & \geq 0 \\
    1 & \quad 2
\end{align*}
\]

The solution procedure using Phase I, Phase II Technique is illustrated below. For that the problems for Phase I and Phase II have to be defined and are given below.

**For Phase I**

The objective is to minimize the artificial variables.

\[\text{ie., Minimize } Z = R + R\]

\[0 \quad 1 \quad 2\]
subject to
\[
\begin{align*}
    x + x + S &= 3 \\ 1 & 2 & 1 \\
    2x + x - S + R &= 4 \\ 1 & 2 & 2 & 1 \\
    3x + 2x - S + R &= 7 \\ 1 & 2 \\
    x, x & \geq 0 \\ 1 & 2 \\
    S, S, S & \geq 0 \\ 1 & 2 & 3
\end{align*}
\]

Step 1

This problem can be represented in tabular form as given below.

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>S</th>
<th>R</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>j</td>
<td>j</td>
<td>j</td>
<td>j</td>
<td>j</td>
<td>j</td>
<td>j</td>
<td>j</td>
</tr>
<tr>
<td>Z - c</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>S 1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>R 1</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>R 2</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Since the Z - c coefficients of the variables staying in the basis are to be zero, by adding the rows corresponding to R and R to Z - c the new Z - c row can be obtained. The modified table is as shown below.
Step 2: Selection of entering variable:

As per the simplex procedure the variable which has the most positive $Z - c$ coefficient will enter the basis. Here $x$ has the most positive $Z - c$ coefficient ($5$). Therefore $x$ enters the basis.

Step 3: Selection of leaving variable:

The leaving variable is computed using the RHS value and the entering variable column elements by finding the $\theta$.

$$\begin{align*}
S & \quad R & \quad R \\
1 & \quad 1 & \quad 2 \\
\theta = \min (3, 2, 2.33) & = 2 \\
R & \quad \text{leaves the basis} \\
1 & \quad \\
\end{align*}$$

The pivot element is $2$. 

<table>
<thead>
<tr>
<th>$Z - c_{ji}$</th>
<th>$x_{1}$</th>
<th>$x_{2}$</th>
<th>$S_{1}$</th>
<th>$S_{2}$</th>
<th>$S_{3}$</th>
<th>$R_{1}$</th>
<th>$R_{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$j$</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$S_{1}$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$R_{1}$</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$R_{2}$</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>
Step 4: Performing the simplex iteration the following table is obtained after iteration 1.

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>S</th>
<th>R</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>3/2</td>
<td>-1</td>
<td>-5/2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>1</td>
<td>1/2</td>
<td>0</td>
<td>-1/2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>1/2</td>
<td>0</td>
<td>-1/2</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>3/2</td>
<td>-1</td>
<td>-3/2</td>
<td>1</td>
</tr>
</tbody>
</table>

Iteration 2:

Step 2: The entering variable is S since it has the most positive \( Z - c \) value.

Step 3: The leaving variable is R based on \( \theta \) value computed as follows.

\[
\theta = \min \{ 2 - 2/3 \} = 2/3
\]

The pivot element is 3/2.
Step 4  The table obtained at the end of this iteration is as given below.

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>S</th>
<th>R</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z - c</td>
<td>j</td>
<td>j</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>S 1</td>
<td>0</td>
<td>0</td>
<td>1/3</td>
<td>1</td>
<td>0</td>
<td>1/3</td>
<td>0</td>
<td>-1/3</td>
</tr>
<tr>
<td>x 1</td>
<td>0</td>
<td>1</td>
<td>2/3</td>
<td>0</td>
<td>0</td>
<td>-1/3</td>
<td>0</td>
<td>1/3</td>
</tr>
<tr>
<td>S 2</td>
<td>0</td>
<td>0</td>
<td>1/3</td>
<td>0</td>
<td>1</td>
<td>-2/3</td>
<td>-1</td>
<td>2/3</td>
</tr>
</tbody>
</table>

Iteration 3:

Step 2 Since there is no more promising variable phase I is completed.

Phase II:

For phase II, the objective function is the original objective function of the problem itself. The starting tableau
for the Phase II is as given below.

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>Z</th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1  2</td>
<td>1  2</td>
<td>1  2</td>
<td>1  2</td>
<td>1  2</td>
<td>1  2</td>
<td>1  2</td>
</tr>
<tr>
<td>Z - c</td>
<td>1  0</td>
<td>-4  -3</td>
<td>0  0</td>
<td>0  0</td>
<td>0  0</td>
<td>1  0</td>
<td></td>
</tr>
<tr>
<td>1j  j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z - d</td>
<td>0  1</td>
<td>-2  -1</td>
<td>0  0</td>
<td>0  0</td>
<td>0  0</td>
<td>4  0</td>
<td></td>
</tr>
<tr>
<td>2j  j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R =</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1j  j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>j Z - d</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2j  j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Since \( x \) stays in the basis the coefficients of \( Z - c \) and \( Z - d \) for \( x \) must be zero. By multiplying the \( x \) row with the \( Z - c \) and \( Z - d \) coefficients of \( x \) and subtracting from the \( Z - c \) and \( Z - d \) coefficients of \( x \) and subtracting from the
The respective row the new tableau obtained is given below.

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>Z</th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1/3</td>
<td>0</td>
<td>0</td>
<td>-4/3</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1/3</td>
<td>0</td>
<td>0</td>
<td>-2/3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Z - c</th>
<th>Z - d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1j j</td>
<td>2j j</td>
</tr>
<tr>
<td>-1/3</td>
<td>1/3</td>
</tr>
</tbody>
</table>

\[ R = \frac{Z - c}{Z - d} \]
\[ = \frac{1}{2} \]
\[ j 2j j \]
\[ -1/3 2/3 \]
\[ \]
\| S | S | S |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Iteration 1:**

**Step 2: Selection of entering variable:**

For \( x \), \( Z - c \) is negative and \( Z - d \) is positive. For \( Z - c \) \( 1j j \) \( 2j j \)

For \( S \), both \( Z - c \) and \( Z - d \) are negative and the respective ratios are -1 and 2.

\[ R = \frac{Z - c}{Z - d} \]
\[ = \frac{1}{2} \]
\[ j 2j j \]
\[ -1/3 2/3 \]
\[ \]

Since for \( S \), \( \frac{Z - c}{Z - d} \) is greater than the objective function there will be improvement if it enters into the basis.
**Step 3:** Selection of Leaving variable

\[
\begin{align*}
S & \quad \times & S \\
1 & \quad 1 & 2 \\
\theta = \min \{2 - - \} &= 2 \\
\end{align*}
\]

ie, \( S \) leaves the basis.

**Step 4:** Performing the simplex iteration the resulting tableau is as given below.

\[
\begin{array}{cccccccc}
Z & Z & x & x & S & S & S \\
1 & 2 & 1 & 2 & 1 & 2 & 3 \\
\hline
Z - c & 1j & j & 1 & 0 & 0 & 1 & 4 & 0 & 0 & 13 \\
\hline
Z - d & 2j & j & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 10 \\
\hline
R = & \frac{Z - c}{1j} & j & - & - & - & 1 & 2 & - & - & 13/10 \\
& \frac{Z - d}{2j} & j & - & - & - & 1 & 2 & - & - & 13/10 \\
\hline
S & 3 & 0 & 0 & 0 & 1 & 3 & 0 & 1 & 2 \\
\hline
x & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 3 \\
\hline
S & 2 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 2 \\
\end{array}
\]

**Iteration 2:**

**Step 2:** Selection of entering variable

For both \( x \) and \( S \), both \( Z - c \) and \( Z - d \) are positive. For a variable to be promising the minimum of the positive ratios should be less than the
objective function value. Here for $x$ the ratio is less than the objective function value. Hence $x$ is the promising variable.

**Step 3: Selection of leaving variable**

$$S \times S$$

$$\theta = \min \left\{ 2 \ 3 \ 2 \right\} = 2$$

$S$ leaves the basis.

**Step 4: The table obtained at the end of this iteration is as given below.**

<table>
<thead>
<tr>
<th></th>
<th>$Z$</th>
<th>$Z$</th>
<th>$x$</th>
<th>$x$</th>
<th>$S$</th>
<th>$S$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z - c_{1j}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$Z - d_{2j}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$R = \frac{Z - c_{1j}}{j}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$R = \frac{Z - d_{2j}}{j}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$x$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$S$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$S$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>
Iteration 3 :

Step 2: Selection of entering variable

Ratio exists for $S_1$ and $S_3$. For $S_1$ and $S_3$ to be promising, the ratio $\frac{Z - c_{1j}}{Z - d_{2j}}$ should be greater than the objective function value. It is not true for both the cases. Hence the optimal solution is reached.

Step 3: The optimal solution is

$$x = 1$$
$$X = 2$$
$$Z = 1.375.$$  

7.4.2 Solution Using Dual-Ratio Algorithm:

Maximize $Z = \frac{4x + 3x + 4}{2x + x + 1}$ subject to

$$x + x \leq 3$$
$$2x + x \geq 4$$
$$3x + 2x \geq 7$$
$$x, x \geq 0$$
The starting tableau obtained by adding slack variables to all the constraints irrespective of the type of the constraints is as given below:

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>1</td>
<td>0</td>
<td>-4</td>
<td>-3</td>
<td>0</td>
</tr>
<tr>
<td>x</td>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>z – c</td>
<td>-</td>
<td>-</td>
<td>2</td>
<td>3</td>
<td>-</td>
</tr>
<tr>
<td>z – d</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>S</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>S 1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>S 2</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>S 3</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

**Iteration 1:**

**Step 2: Selection of entering variable:**

Maximum of the positive ratios whose \( z - c \) and \( z - d \) are negative, are found out. For both the \( x \) variables \( x \) and \( x \) both \( z - c \) and \( z - d \) are negative. The value 3 corresponds to \( x \) and it is less than the objective function value 4. Hence the optimality condition is satisfied. It has to be checked for feasibility.
Step 5: Check for feasibility.

Only the slack variables stay in the basis. The slack variable $S$ corresponding to upper bound constraint stays with positive value, whereas $S$ and $S$ corresponding to lower bound constraints also stay with positive values. i.e., the solution is infeasible.

Step 6: Selection of leaving variable.

Either $S$ or $S$ may leave the basis. However, the variable which is having minimum value (including negative sign) for the slack variables corresponding to lower bound constraints will be selected to leave the basis. Here between $-4$ and $-7$, $-7$ is the least. Hence $S$ leaves the basis.

Step 7: Selection of entering variable

The ratio $\frac{Z - c_{ij}}{Z - d_{ij}}$ is computed for all the variables whose $(B_p)$ is greater than zero, since $j$ corresponds to the leaving variable row.

\[
\begin{bmatrix}
Z - c_{ij} \\
Z - d_{ij}
\end{bmatrix}
= \begin{bmatrix}
1 & 2 & 1 & 2 & 3 \\
2 & 3 & - & - & -
\end{bmatrix}
\]
Maximum of the positive ratios is 3 and this corresponds to \( x \). Hence \( x \) enters the basis. Go to step 4.

**Step 4:** The table obtained after performing simplex operation at the end of iteration 1 is as given below:

<table>
<thead>
<tr>
<th></th>
<th>( Z ) - c</th>
<th>( Z ) - d</th>
<th>( Z ) - e</th>
<th>( Z ) - f</th>
<th>( Z ) - g</th>
<th>( Z ) - h</th>
<th>( Z ) - i</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( x )</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>3/2</td>
</tr>
<tr>
<td>( s )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
</tr>
<tr>
<td>( y )</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1/2</td>
</tr>
<tr>
<td>( z )</td>
<td>0</td>
<td>0</td>
<td>3/2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
</tr>
</tbody>
</table>

**Step 5:** Check for feasibility

Both \( s \) and \( s \) (lowerbound constraint variable) are having negative values. Hence the solution is infeasible.

**Step 6:** Selection of leaving variable

Both \( s \) and \( s \) have the value -1/2. \( s \) leaves the basis.
Step 7: Selection of entering variable

Compute \( \frac{Z - c}{ij} \) for all the variables whose \( \frac{Z - d}{ij} \) is negative since the leaving variable corresponds to the slack variable of upper bound constraint. The values are

\[
\begin{bmatrix}
Z - c \\
ij \\
\hline 
Z - d \\
ij
\end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 2 & 3 \\
-1 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 14/5
\end{bmatrix}
\]

Hence \( x \) enters the basis. Go to step 4.

Step 4: The following table is obtained after performing the simplex iteration.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z ) - ( c ) 1j</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( Z ) - ( d ) 2j</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>( Z ) - ( c ) 1j</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-1</td>
<td>-</td>
</tr>
<tr>
<td>( Z ) - ( d ) 2j</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>( x ) 1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( -2 )</td>
<td>0</td>
</tr>
<tr>
<td>( S ) 2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( x ) 2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( +3 )</td>
<td>0</td>
</tr>
</tbody>
</table>
Step 5: Check for feasibility

Both $x_1, x_2$ remain with positive values and $S_2$, the slack variable corresponding to the lower bound constraint stays with zero value. Hence the solution is feasible. The optimal solution is

$$x_1 = 1$$
$$x_2 = 2$$
$$z = 14/5$$

7.5 FRACTIONAL TRANSPORTATION PROBLEM:

Consider a transportation problem where there are 2 source centres and 3 distribution centres. The cost of transporting one unit from source $i$ to distribution centre $j$ is $c_{ij}$ and $p_{ij}$ is the corresponding profit obtained. The capacities of source centres, distribution centres and the profits are as given below:

```
<table>
<thead>
<tr>
<th></th>
<th>i</th>
<th>j</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td></td>
<td></td>
</tr>
<tr>
<td>c</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```
7.5.1 **RATIO TRANSPORTATION ALGORITHM**

The steps involved in the Ratio Transportation Algorithm to obtain the optimal solution are described below:

**Step 1:**

The initial solution is found out by using North West corner method and is as given below:

```
<table>
<thead>
<tr>
<th></th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>10</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>S2</td>
<td>15</td>
<td>16</td>
<td>7</td>
</tr>
</tbody>
</table>
```

The profit for this transportation \( P \)

\[
= 40 \times 10 + 10 \times 8 + 30 \times 16 + 30 \times 7
= 400 + 80 + 480 + 210
= 1170
\]
The cost of transportation \((C)\) 
\[
= 40 \times 2 + 10 \times 4 + 30 \times 4 + 30 \times 1 \\
= 80 + 40 + 120 + 30 \\
= 270 
\]

\[
z = \frac{P}{C} = \frac{1170}{270} = 4.33 
\]

**Iteration 1**

**Step 2:**

The \(u\) and \(v\) values based on profit and cost coefficients and the relative cost factors for non basic cells are computed and shown below:

<table>
<thead>
<tr>
<th>Based on Profit Coefficients</th>
<th>Based on Cost Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v = 10)</td>
<td>(v = 2)</td>
</tr>
<tr>
<td>(v = 8)</td>
<td>(v = 4)</td>
</tr>
<tr>
<td>(v = -1)</td>
<td>(v = 1)</td>
</tr>
</tbody>
</table>

| \(u = 0\) | \(u = 0\) |
| \(u = 1\) | \(u = 1\) |
| \(u = 2\) | \(u = 2\) |

**Step 3:** Ratio values \(r\) for non basic cells are as given below:

\[
r_{ij} = \frac{C1_{ij}}{C2_{ij}} 
\]
Step 4:

a) The ratio $r$ is positive and $C_1$ is negative. For this variable to be promising the ratio $r$ must be greater than the objective function value. Here $r$ is greater than the objective function value. I.e. $5 > 4.33$.

Therefore the corresponding entering variable is $x_{13}$.

Step 5: Formation of loop

The loop that is formed and the value to be added to or subtracted from the loop corners are as given below:

```
  40  -10
   10 +10
```

```
  30  +10 -10
```
Step 6: The next solution is,

\[
\begin{array}{cccc}
10 & 40 & 8 & 2 \\
40 & 2 & 4 & 10 \\
15 & 16 & 40 & 7 \\
3 & 4 & 20 & 1 \\
\end{array}
\]

Objective function value = \[
\frac{40 \times 10 + 10 \times 9 + 40 \times 16 + 20 \times 7}{40 \times 2 + 10 \times 3 + 40 \times 4 + 20 \times 1}
\]

= \[
\frac{400 + 90 + 640 + 140}{80 + 30 + 160 + 20}
\]

= \[
\frac{1270}{290}
\]

= 4.3793

Iteration 2

Step 2: The u and v values based on profit and cost coefficients and the relative cost factors for non basic cells are given below:
Based on Profit

\[
\begin{array}{ccc}
  v &=& 10 \\
  u &=& 0 \\
  &1& \\
  &10& \\
  &8& \\
  &10& \\
  &40& \\
  &-7& \\

  v &=& 18 \\
  u &=& 1 \\
  &2& \\
  &1& \\
  &16& \\
  &7& \\
  &40& \\
  &20& \\

  v &=& 9 \\
  u &=& 2 \\
  &3& \\
  &2& \\
  &10& \\
  &0& \\
  &20& \\
  &50& \\
\end{array}
\]

Based on Cost

\[
\begin{array}{ccc}
  v &=& 2 \\
  u &=& 0 \\
  &1& \\
  &40& \\
  &2& \\
  &10& \\
  &-3& \\

  v &=& 6 \\
  u &=& 1 \\
  &2& \\
  &2& \\
  &2& \\
  &20& \\

  v &=& 3 \\
  u &=& 2 \\
  &3& \\
  &4& \\
  &1& \\
  &40& \\
  &60& \\
\end{array}
\]

Step 3: The Ratio values $r_{ij}$ for non basic cells are as given below:

\[
\begin{align*}
  r_{12} &= \frac{10}{2} = 5 \\
  r_{21} &= \frac{-7}{-3} = 2.33
\end{align*}
\]
Step 4:

(a) Since $r$ is positive and $C_1$ is also positive this variable will be promising if the ratio is less than the objective function value. But $r$ is not less than the objective function value ($5 \geq 4.3793$). $r$ is also positive, but $C$ is negative. For this variable to be promising the ratio should be greater than the objective function value. Here $2.33$ is not greater than $4.3793$. Therefore there is no more promising variable. Hence the optimal solution is reached.

7.5.2 DEL-RATIO ALGORITHM

The various steps involved in arriving at the optimal solution using the algorithm are given below:

Step 1:

Find out an initial basic solution using North-West corner method. The initial solution obtained is,

\[
\begin{array}{ccc}
10 & 40 & 20 \\
8 & 14 & 30 \\
4 & 30 & 1 \\
50 & 60 & \\
\end{array}
\]
Objective function value

\[ Z = \frac{\sum_{i} \sum_{j} c_{ij} x_{ij}}{\sum_{i} \sum_{j} c_{ij} x_{ij}} \]

\[ Z = \frac{40 \times 10 + 10 \times 8 + 30 \times 16 + 30 \times 7}{1170} = \frac{1170}{270} \]

**Iteration 1:**

**Step 2:** The \( u \) and \( v \) values computed based on profit and cost coefficients are shown below:

**Based on Profit**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>10</td>
<td>-10</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>16</td>
<td>7</td>
</tr>
</tbody>
</table>

**Based on Cost**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>40</td>
<td>10</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>30</td>
<td>1</td>
</tr>
</tbody>
</table>

**Step 3:** The Del values are computed using the formula

\[ \Delta_{ij} = C_{ij} \cdot C_{i1} - P_{ij} \cdot C_{2j} \] for all non-basic cells.

\[ \Delta_{13} = 270(-10) - 1170(-2) = 2700 + 2340 = 5040 \]

\[ \Delta_{21} = 270(3) - 1170(-1) = 810 + 1170 = 1980 \]
Step 4:
If all Del values are positive the optimal solution is reached. Otherwise, the variable corresponding to the maximum negative ratio is the entering variable. Here \( x \) is the entering variable.

Step 5: (Formation of loop to decide leaving variable)
The formation of the loop and the resulting leaving variable are given below:

\[
\begin{array}{c|c|c}
40 & -10 & +10 \\
\hline
10 & \Rightarrow & \\
\hline
30+10 & \Leftarrow & 30-10 \\
\end{array}
\]

Step 6: The following table gives the solution obtained at the end of iteration 1

\[
\begin{array}{cccccccc}
10 & 40 & 2 & 8 & 4 & 10 & 3 \\
15 & 16 & 40 & 7 & 20 & 1 & \\
3 & 4 & & & & & \\
\end{array}
\]

Objective function value \( Z = \frac{P}{C} = \frac{1270}{290} = 4.3793 \)
Iteration 2

Step 2: The u and v values calculated based on Profit and cost coefficients and the relative cost factors obtained for non-basic cells are as follows:

Based on Profit

<table>
<thead>
<tr>
<th></th>
<th>v = 10</th>
<th>v = 18</th>
<th>v = 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>16</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>-7</td>
<td>40</td>
<td>20</td>
</tr>
</tbody>
</table>

Based on Cost

<table>
<thead>
<tr>
<th></th>
<th>v = 2</th>
<th>v = 6</th>
<th>v = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>40</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Step 3: The Del are computed for non-basic cells as given below:

\[\text{Del}_{ij} = 290 \times 10 - 1270 \times 2\]
\[= 2900 - 2540\]
\[= 360\]

\[\text{Del}_{ij} = 290 \times (-7) - 1270 \times (-3)\]
\[= -2030 + 3810\]
\[= 1780\]

Step 4: Since all Del values are positive the optimal solution is reached.
7.6 CONCLUSION

Various numerical examples have been worked out, illustrating the application of Ratio Algorithm, for solving the problems in the following methods.

i) L.F.P.P. in tabulation form

ii) L.F.P.P. in Matrix form

iii) L.F.P.P. with mixed type of constraints

iv) L.F.P.P. using dual ratio algorithm

v) Fractional Transportation Problem using Ratio Algorithm

vi) Fractional Transportation Problem using Del-Ratio Algorithm.