6. SOME SPECIAL ASPECTS

6.1 INTRODUCTION:

Various authors like Charnes-Cooper [10], Martos [36], Isbell-Marlow[24], Wolf[22] and Kanti-Swarup[27] have evolved various methods to solve the linear fractional programming problems. In the previous chapter a new algorithm has been described to solve the linear fractional programming problem. Mond[38] has shown that if the constraint set is unbounded, then method of Martos may fail to reach optimal solution, whereas method of Charnes-cooper always succeeds. It is shown by Vanita Verma and Puri [53] that the method of Wolf [22] is equally preferable. The proposed ratio algorithm also finds the optimal solution successfully.

Identification of bad-points and singular points as defined by Martos [35], is applicable to almost all practical situations. As stated by Vanita Verma and Puri [53], the method of Charnes-Cooper obtains bad point (or singular point) which corresponds to the limiting end of a feasible ray emanating from the vertex,
where unboundedness (or optimality) is declared. The limiting end is defined as a point

\[ Z = \lim_{\phi \to \infty} (Z + \phi (Z - Z')) \]

where \( Z = (Y, t) \) is a point on the ray and \( Z' = (Y', t') \), is that point at which unboundedness is declared, \( \phi \) being an arbitrary scalar. Martos' algorithm is able to obtain a bad point only if it is at the vertex of \( F \). In this chapter the method of finding the singular and bad points using the ratio algorithm is illustrated.

### 6.2 Optimal Solution over an Unbounded Region

Like the methods of Charnes-cooper and Wolf, the proposed ratio algorithm is also capable of finding the optimal solution of the fractional programming problem when region \( F \) is unbounded. The optimality criterion used earlier in chapter 3 itself is sufficient to find the optimal solution whether the region is bounded or unbounded. This can be illustrated by the following example.

Maximize

\[
Z = \frac{-24x - 7}{5x + x + 1}
\]

subject to

\[
\begin{align*}
-x + x & \leq 1 \\
1 & 2 \\
-x - x & \leq 1 \\
1 & 2 \\
x, x & \geq 0 \\
1 & 2
\end{align*}
\]
The optimal solution of Charnes-Cooper's equivalent problem is

\[(y, y, t, x, x) = (0, \frac{4}{7}, \frac{4}{7}, 0, 1)\]

where \(x = y/t\)

Therefore \(x = (0, 1)\) is the optimal solution and the optimum value of \(Z\) is \(-7/2\).

6.2.1 SOLUTION USING THE RATIO ALGORITHM

The starting tableau for the above problem is given below.

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>Z</th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z - c</td>
<td>1</td>
<td>0</td>
<td>24</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-7</td>
</tr>
<tr>
<td>1j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z - d</td>
<td>0</td>
<td>1</td>
<td>-5</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z - c</td>
<td>-</td>
<td>-</td>
<td>-4.8</td>
<td>-0</td>
<td>0</td>
<td>0</td>
<td>-7</td>
</tr>
<tr>
<td>1j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z - d</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2j</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Iteration 1:

Step 1: Selection of Entering Variable

The entering variable is selected based on the ratios \( \frac{Z - c_{1j}}{Z - d_{2j}} \). In this problem \( Z \) is less than zero and the ratios are negative when \( Z - d_{2j} \)'s are negative. Therefore, there would be improvement only if the maximum negative ratio is greater than the objective function value at that point. The maximum negative ratio corresponds to \( x \) and that value is greater than the \( \frac{Z}{2} \) objective function value. Therefore \( x \) is the entering variable.

Step 2: Selection of Leaving Variable

The leaving variable is selected by computing the \( \theta \) value using the entering variable column and the solution vector is as given below.

\[
\theta = \min_i \left[ \begin{array}{c}
-1 \\
(B_{p_{1i}}) \\
-o_{1i} \\
1 \\
-1 \\
(B_{p_{2j}}) \\
-j_i \\
-1 \\
(B_{p_{2j}}) \\
-j_i \\
1 \\
\end{array} \right] ; \ (B_{p_{2j}}) > 0
\]

\[
S = S_{12} = \min \left[ \begin{array}{c}
1 \\
2 \\
1 \\
\end{array} \right]
\]

\[
\theta = 1 \text{ which corresponds to the variable } S
\]
Step 3

Taking the intersecting element as the pivot and performing the simplex method operation the following table is obtained at the end of iteration 1.

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>1</th>
<th>2</th>
<th>x</th>
<th>1</th>
<th>2</th>
<th>S</th>
<th>1</th>
<th>2</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z - c</td>
<td>1</td>
<td>0</td>
<td>24</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z - d</td>
<td>0</td>
<td>1</td>
<td>-6</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z - c</td>
<td>-</td>
<td>-</td>
<td>-4</td>
<td>0</td>
<td>0</td>
<td>-</td>
<td>-3.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Z - d</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
value or the positive ratio is less than the objective function value. Since neither situation arises for the given problem, there is no more promising variable and hence the optimal solution is reached.

The solution is

\[
x_1 = 0
\]

\[
x_2 = 1
\]

\[
Z = -3.5
\]

6.3 IDENTIFICATION OF SINGULAR POINTS

According to Martos [35] a singular point is defined as a point \(X^F\) at which \(n(x) = d(x) = 0\) provided no bad-point exists in \(F\).

If singular point exists in \(F\), then

(a) Method of Charnes-Cooper[10] obtains it as limiting end of at least one feasible ray emerging from the point at which optimality is declared.

(b) Martos' approach[35] always finds the singular point in one pivot operation from optimal simplex tableau.

(c) Optimal solution corresponds to very first subinterval \([u, v]\) (say) in decomposition of \(\triangle\) in Wolf's method, where \(u\) corresponds to the singular point and \(v\) corresponds to the optimal point.
(d) In the proposed ratio algorithm the singular point is obtained from one pivot operation from the optimal simplex tableau.

This can be illustrated by the following example.

\[
\begin{align*}
\text{Maximize } \quad & Z = \frac{-x - 3x + 9}{1} \\
\text{subject to } & -x + 3x \leq 3 \\
& 2x - x \leq 4 \\
& x, \; x \geq 0.
\end{align*}
\]

Method of Charnes-Cooper obtains the singular-point \((x = 3, \; x = 2)\) which corresponds to the limiting end \((2/7 + 3\phi /2, \; 0 + 2\phi, \; 1/7 + \phi, \; \phi \to \infty)\) of feasible ray through optimal solution \((y = 2/7, \; y = 0, \; t = 1/7)\) of the equivalent problem [53].

Martos' method derives the singular point \((3, \; 2)\) from the optimal point \((2, \; 0)\) in one Simplex iteration. On solving by Wolf's method,

\[
\Delta = \{0, \; 9\} = \{0, \; 6\} \cup \{6, \; 7\} \cup \{7, \; 9\}
\]

Optimal solution corresponds to \(\delta = 6\in\{0, \; 6\}\) and singular point corresponds to \(\delta = 0\).
6.3.1 SINGULAR POINT USING THE RATIO ALGORITHM

The starting tableau for the above problem is given below.

\[
\begin{array}{cccccc|c}
 & Z & x & x & S & S & \text{RHS} \\
1 & 1 & 2 & 1 & 2 & 1 & 2 \\
Z - c & 1j & j & 1 & 0 & 0 & -1 \\
Z - d & 2j & j & 0 & 1 & 3 & 0 & 0 & 9 \\
\hline
Z - c & 1j & j & - & - & -1 & 1/3 & - & - & -1/9 \\
\hline
\hline
S & 1 & 0 & 0 & -1 & 3 & 1 & 0 & 3 \\
S & 2 & 0 & 0 & 2 & -1 & 0 & 1 & 4 \\
\end{array}
\]

Iteration 1:

Step 1: Selection of entering variable

Now \( Z < 0 \).

\[
\frac{Z - c}{1j} < 0 \quad \text{and} \quad \frac{Z - c}{1j} < 0 \quad \text{for} \quad x
\]

and there would be improvement since this ratio is less than the objective function value, \(-1/9\).

For \( x \) the ratio is positive and since \( Z \) is less than zero \( Z \) there won't be any improvement if this variable enters. Hence the entering variable is \( x \).
Step 2: Selection of leaving variable

The leaving variable is determined by finding $\theta$

$$\theta = \min_i \left[ \frac{-1}{(B \ p)_{0i}} \right]$$

$$\theta = \min_i \left[ \frac{-1}{(B \ p)_{0i}} ; (B \ p)_{0i} > 0 \right]$$

$$\theta = \min_i \left[ \frac{-1}{(B \ p)_{0i}} \right]$$

$S$ leaves the basis.

Step 3: Taking the intersecting element as pivot, the table obtained after performing the simplex iteration is as follows.

<table>
<thead>
<tr>
<th></th>
<th>$Z$</th>
<th>$Z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z - c_{1j}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>1/2</td>
<td>1</td>
</tr>
<tr>
<td>$Z - d_{2j}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>7/2</td>
<td>0</td>
<td>-1/2</td>
<td>7</td>
</tr>
<tr>
<td>$Z - c_{1j}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1/7</td>
<td>-</td>
<td>-1</td>
<td>1/7</td>
</tr>
<tr>
<td>$Z - d_{2j}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1/7</td>
<td>-</td>
<td>-1</td>
<td>1/7</td>
</tr>
<tr>
<td>$S_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5/2</td>
<td>1</td>
<td>1/2</td>
<td>5</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1/2</td>
<td>0</td>
<td>1/2</td>
<td>2</td>
</tr>
</tbody>
</table>
Iteration 2

Step 1: \( Z > 0 \)

\[
\begin{align*}
 & Z - c \\
 & 1j \\
\text{for } x : & \quad \text{----} > 0 \\
 & Z - d \\
 & 2j
\end{align*}
\]

for \( S : \quad \text{----} < 0 \)

\[
\begin{align*}
 & Z - c \\
 & 1j \\
\text{This would improve the solution if this ratio} & \quad \text{and } Z - d \text{ is -ve} \\
 & Z - d \\
 & 2j
\end{align*}
\]

is greater than the objective function value.

There will not be any improvement in the objective function value by entering this variable.

However since the ratio is equal to the objective function value that variable may be selected to enter the basis. i.e., \( x \) enters the basis.

Step 2: The leaving variable by finding the ratio \( \theta \) would be \( S \)

\[
\begin{align*}
S_{1} & \quad x_{1} \\
\theta &= \text{min} \{ 2 - 1 \} = 2
\end{align*}
\]

Step 3: The tableau obtained after the simplex iteration is

| \( Z - c \) | 1 | 0 | 0 | 0 | -1/5 | 2/5 | 0 |
| \( 1j \) | \( j \) |
| \( Z - d \) | 0 | 1 | 0 | 0 | -7/5 | -12/10 | 0 |
| \( 2j \) | \( j \) |

\[
\begin{align*}
 & Z - c \\
 & 1j \\
 & \quad \text{----} \\
& Z - d \\
 & 2j
\end{align*}
\]

\[
\begin{align*}
x_{2} & \quad 0 | 0 | 0 | 1 | 2/5 | 1/5 | 2 \\
x_{1} & \quad 0 | 0 | 1 | 0 | 1/5 | 11/10 | 3
\end{align*}
\]
This corresponds to the singular point. Therefore the ratio method also finds the singular point through one simplex iteration of the optimal table.

6.4 IDENTIFICATION OF BAD POINTS :

A point $x$ in $F$ is called bad-point if

$$\lim_{x \to x} \frac{n(x)}{d(x)} = \infty$$

If a bad point exists in $F$, then

a) Method of Charnes-cooper [10] obtains it as a point corresponding to the limiting-end of a feasible ray emanating from the vertex where unboundedness is declared.

b) Martos [35] obtains a bad-point only if it exists at a vertex of $F$ and $d(x) > 0$ for all $x \in F$.

c) Wolf’s procedure [23] always succeeds in finding out a bad point.

d) In the ratio algorithm by treating the $Z - d_j$ row also as a constraint row while determining the leaving variable a bad point can be obtained. This is illustrated by the following example

Example:

Maximize $Z = \begin{bmatrix} -2x - x - 2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} -5x + 2x + 10 \\ 1 \end{bmatrix}$$
subject to \[-x + \frac{x}{2} \leq 1 \]
\[x - \frac{x}{2} \leq 3 \]
\[x + 2x \leq 5 \]
\[x, x \geq 0 \]

By Charnes method [10] the bad point is found as \((2, 0)\) But Martos' approach does not identify the bad point but declares optimality at \((0, 0)\) [35].

Wolf's approach [23] also gives the bad point at \((2, 0)\)

6.4.1 IDENTIFICATION OF BAD POINT USING RATIO ALGORITHM

The starting tableau of the above problem is as given below.

<table>
<thead>
<tr>
<th></th>
<th>Z</th>
<th>x</th>
<th>x</th>
<th>S</th>
<th>S</th>
<th>S</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z - c 1</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z - d 2</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Z - c 1</td>
<td>0</td>
<td>-2/5</td>
<td>1/2</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2/10</td>
</tr>
<tr>
<td>Z - d 2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>S 1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>S 2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>S 3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Iteration 1

Step 1: Selection of entering variable

\[ \begin{align*}
Z - c \\
1j & \quad j \\
\text{Z is positive, } & \quad \text{is negative} \\
1 \\
Z - d \\
2j & \quad j
\end{align*} \]

and \( Z - c \) is negative for \( x \) where as \( 1j \) \( j \) \( 1 \)

\[ \begin{align*}
Z - c \\
1j & \quad j \\
\text{is positive} \quad \text{and } Z - c \text{ is negative for } x \\
1j & \quad j \\
Z - d \\
2j & \quad j
\end{align*} \]

Therefore \( x \) enters the basis if the ratio is less than the \( 1 \) objective function value where as for \( x \) the ratio should be \( 2 \) greater than the objective function value. Here both \( x \) and \( x \) \( 1 \) \( 2 \) qualify for the same. Let us take \( x \) as the entering variable.

Step 2: Selection of leaving variable

This is selected by finding the ratio \( \theta \),

\[ \theta = \min \left[ \min_{j} \left( \frac{-1}{(B^p)_{0i} -1} ; (B^p_{ji}) > 0 \right) \right] \]

\[ \begin{bmatrix}
T & d & B^p \\
B^o & \text{Z - d} \\
2j & j
\end{bmatrix} \]

\[ \begin{bmatrix}
S & S & S & Z - d \\
1 & 2 & 3 & 2j & j
\end{bmatrix} \]

\[ \begin{bmatrix}
\min ( \begin{array}{c}
-3 \\
5 \end{array} ) \\
2
\end{bmatrix} \]
\[ S \begin{bmatrix} Z - d \\ \end{bmatrix} \begin{bmatrix} z \\ 2j \\ j \end{bmatrix} = \min \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 2 \]

i.e., \( Z - d \) is the leaving variable

\[ \begin{align*}
\text{Step 3:} & \text{ By considering the pivot element and performing the} \\
& \text{simplex iteration the table obtained is given below.} \\
\end{align*} \]

\[
\begin{array}{cccc|ccc}
 & x & x & S & S & S & \text{RHS} \\
1 & 0 & 0 & -9/5 & 0 & 0 & 0 \\
2 & 0 & 0 & -2/5 & 0 & 0 & 0 \\
\hline
Z - c_{1j} & - & - & - & - & - & \_ \\
\hline
Z - d_{2j} & - & - & - & - & - & \_ \\
\hline
S & 0 & 0 & 3/5 & 1 & 0 & 0 \\
S & 0 & 0 & -3/5 & 0 & 1 & 0 \\
S & 0 & 0 & 12/5 & 0 & 0 & 1 \\
\end{array}
\]

Thus the bad point (2, 0) is obtained.
6.5 CONCLUSION:

The proposed ratio algorithm which selects the entering variable based on the ratio of the contribution coefficients of the variables both in the numerator and the denominator can be applied for identifying the singular points, and bad points and also for solving the unbounded region problems. This method compares well with the methods of Charnes-Cooper[10], Martos[35] and Wolf[23].

** -- ** -- **