CHAPTER-10

UNSTEADY THREE-DIMENSIONAL MHD BOUNDARY LAYER FLOW DUE TO THE IMPULSIVE MOTION WITH HEAT AND MASS TRANSFER OF A STRETCHING SURFACE IN A SATURATED POROUS MEDIUM

10.1 MATHEMATICAL FORMULATION

In this present paper it is considered the unsteady laminar viscous boundary layer flow of an electrically conducting fluid induced by the impulsive stretching of the flat surface in two lateral directions in an otherwise quiescent fluid embedded in a porous medium with heat and mass transfer.

![Diagram](image)

Fig.(10.1) Physical model and coordinate system.

The problem is formulated in such a manner that for a small time it reduces to that of the Rayleigh type of flow and for large time it reduces to that of the Wang type flow. Th
steady state results without magnetic field and porous medium are compared with those of Wang [35] and are found to be in excellent agreement.

We consider the unsteady laminar incompressible flow of an electrically conducting fluid over the flat surface in two lateral directions in an otherwise quiescent fluid in a porous medium with heat and mass transfer. At the same time, the wall temperature is raised from \( T_w \) to \( T_w' \) (\( T_w' > T_w \)) and wall concentration is raised from \( C_w \) to \( C_w' \) (\( C_w' > C_w \)). The magnetic field is applied in the z-direction. It is assumed that the magnetic Reynolds number is small, i.e., \( Rm = \frac{\mu_0 V L}{\mu} \ll 1 \), where \( \mu_0 \) is the magnetic permeability, \( \mu \) is the electrical conductivity, and \( V, L \) are the characteristic velocity and length, respectively. Under these conditions we can neglect the effect of the induced magnetic field in comparison to the applied magnetic field. The electrical current flowing in the fluid gives rise to an induced magnetic field if the fluid were in electrical insulator, but here we have taken the fluid to be electrically conducting and embedded in a porous medium. Hence the applied magnetic field \( B_0 \) plays a role which gives rise to magnetic forces \( F_x = \sigma B_x u, F_y = \sigma B_y v \) and \( F_z = \sigma B_z \rho \) and applied porous medium gives rise to \( G_x = \frac{V}{k} u \) and \( G_y = \frac{V}{k} v \) in \( x \) and \( y \) directions, respectively. The effects of viscous dissipation, Ohmic heating and Hall current are neglected. The wall and ambient temperatures and concentrations are considered to be constant. Under these above assumptions, the boundary layer equations governing the unsteady laminar flow due to an impulsive motion are given by [59], [107] as follows.
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma \partial^2 u}{\rho k'} \quad \ldots (10.1.1)
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = \nu \frac{\partial^2 v}{\partial y^2} - \frac{\sigma \partial^2 v}{\rho k'} \quad \ldots (10.1.2)
\]

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = \nu \frac{\partial^2 w}{\partial y^2} - \frac{\sigma \partial^2 w}{\rho k'} \quad \ldots (10.1.3)
\]

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial y^2} \quad \ldots (10.1.4)
\]

\[
\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} = D \frac{\partial^2 C}{\partial y^2} \quad \ldots (10.1.5)
\]

The initial conditions are given by

\[ u(x,y,z,0) = v(x,y,z,0) = w(x,y,z,0) = 0 \]

\[ T(x,y,0,0) = T, \quad \text{for} \quad t < 0 \]

\[ C(x,y,0,0) = C, \quad \text{for} \quad t < 0 \quad \ldots (10.1.6) \]

The boundary conditions for \( t \geq 0 \) given by

\[ u(x,y,0,t) = u_0, : \quad v(x,y,0,t) = v_0 \]

\[ w(x,y,0,t) = 0 \quad : \quad T(x,y,0,t) = T_n, \quad C(x,y,0,t) = C_n \]

\[ u(x,y,Z,t) = u(x,y,Z,t) = 0 \]

\[ T(x,y,Z,t) = T, \quad : \quad C(x,y,Z,t) = C \quad \ldots (10.1.7) \]

Here \( x, y \) and \( z \) are the longitudinal, the transverse and the normal directions, respectively; \( t \) is the time, \( T \) is the temperature, \( C \) is the concentration, \( B \) is the magnetic field applied, \( k' \) the porous parameter in \( x \)-direction. \( u_n \) and \( v_n \) are the surface velocities.
in x and y directions, respectively: the subscripts w and \( \infty \) denote conditions at the wall and in the ambient fluid, respectively.

We make use of the scales

\[
R = \frac{Z}{\sqrt{\nu t}} \quad t^* = \frac{u_* t}{x} \quad \text{or} \quad \eta = \sqrt{\frac{u_*}{\nu x}} z^* = \frac{u_* t}{x}
\]

which are valid for small and large times, respectively. Hence, one has to find such scales where both small and large time solutions fit in properly. Defining

\[
\eta = \sqrt{\frac{u}{v z}} \quad \xi = 1 - \exp(-t^*) \quad t^* = a_* t, \ a > 0
\]

\[u(x, y, z, t) = a_{in} f^* (\xi, \eta) \quad v(x, y, z, t) = a_{in} y S^* (\xi, \eta)\]

\[w(x, y, z, t) = -\sqrt{a_* v} \sqrt{\xi} (f + s), \ u_* = a_{in} x.\]

\[v_* = b, y = b, \ g(t) = 0 \quad T(x, y, z, t) = T + (T_* - T_0) g(\xi, \eta)\]

\[C(x, y, z, t) = C_* + (C_* - C_0) h(\xi, \eta) \quad M_* = \frac{\sigma B^*}{\rho a_*}\]

\[
k = \frac{v}{k' a_*} \quad C = \frac{b}{a_*} \geq 0 \quad Pr = \frac{v}{\alpha} \quad Sc = \frac{v}{D} \quad \ldots (10.1.8)
\]

Substituting relations (10.1.8) in equations (10.1.1) to (10.1.5) we find that equation (10.1.1) is identically satisfied and equations (10.1.2) - (10.1.5) reduce to

\[
\gamma_{\eta\eta}(\eta) + \frac{1}{\xi} \eta(1 - \xi) \gamma_{\xi\xi}(\eta) - \xi (f + s) \gamma_{\eta\eta}(\eta) - \xi \gamma_{\xi\xi}(\eta)
\]

\[= (M_* + k_*) \xi \gamma_{\eta\eta}(\eta) = \xi (1 - \xi) \gamma_{\xi\xi}(\eta) \quad \ldots (10.1.9)\]

\[
S_{\eta\eta}(\eta) + \frac{1}{\xi} \eta(1 - \xi) S_{\xi\xi}(\eta) + \xi (f + s) S_{\eta\eta}(\eta) - \xi \gamma_{\xi\xi}(\eta)
\]

\[= (M_* + k_*) \xi S_{\eta\eta}(\eta) = \xi (1 - \xi) S_{\xi\xi}(\eta) \quad \ldots (10.1.10)\]
Pr \frac{1}{2} \eta \left(1 - \xi \right) f_\eta(\eta) + \xi \left( f + S \right) f_\eta(\eta) = \xi \left(1 - \xi \right) \frac{\partial \xi}{\partial \xi} \quad \ldots \quad (10.1.11)

Sc \frac{1}{2} \eta \left(1 - \xi \right) h_\eta(\eta) + \xi \left( f + S \right) h_\eta(\eta) = \xi \left(1 - \xi \right) \frac{\partial \xi}{\partial \xi} \quad \ldots \quad (10.1.12)

The corresponding boundary conditions are given by

\begin{align*}
& f(\xi, 0) = 0, \quad f_\eta(\xi, 0) = 1, \quad S(\xi, 0) = 0, \quad S_\eta(\xi, 0) = C_\eta, \\
& g(\xi, 0) = 1, \quad h_\eta(\xi, 0) = 1, \\
& f_\eta(\xi, \infty) = S_\eta(\xi, \infty) = g(\xi, \infty) = h(\xi, \infty) = 0 \quad \ldots \quad (10.1.13)
\end{align*}

Here \( \eta \) is the dimensionless transformed similarity variable. \( \tau^* \) and \( \xi \) are dimensionless times. \( f_\eta \) and \( S_\eta \) are the dimensionless velocity components along the \( x \) and \( y \) directions, respectively; \( g \) is the dimensionless temperature and \( h \) the dimensionless concentration. \( C_\eta \) is the ratio of the surface velocity gradients along the \( y \) and \( x \) directions; \( \text{Pr} \) is the Prandtl number. \( \text{Sc} \) is the Schmidt number and suffix denotes the partial derivative with respect to \( \eta \).

It is justified that equations (10.1.9) – (10.1.12) are parabolic partial differential equations, but for \( \xi \to 0 \) (\( \tau^* \to 0 \)) and \( \xi \to 1 \) (\( \tau^* \to \infty \)) they reduce to ordinary differential equations. For

\begin{align*}
& \xi = 0, \quad \text{equations (10.1.9)} \quad \text{(10.1.12) reduce to} \\
& f_{\eta \eta}(\eta) + \frac{1}{2} \eta f_\eta(\eta) = 0 \quad \ldots \quad (10.1.14) \\
& S_{\eta \eta}(\eta) + \frac{1}{2} \eta S_\eta(\eta) = 0 \quad \ldots \quad (10.1.15) \\
& \frac{1}{\text{Pr}} g_{\eta \eta}(\eta) + \frac{1}{2} \eta g_\eta(\eta) = 0 \quad \ldots \quad (10.1.16)
\end{align*}
where \( \text{erfc}(\eta) = 1 - \text{erf}(\eta) \)

\[
\text{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\eta} \exp(-x^2) \, dx
\]

Equations (10.1.18) (10.1.21) for \( C_o=0 \) (two-dimensional case) admit closed form solutions.

For \( C_o \neq 0 \) and \( S \neq 0 \) equation (10.1.19) is not required.

For \( C_o = 0 \), the solution of equation (10.1.18) under the boundary conditions (10.1.22) is given by

\[
f(\eta) = \left[ 1 - \exp(-\gamma\eta) \right], \quad \gamma = \sqrt{1 + \frac{M}{\eta} + k_z}
\]  
...(10.2.2)

The solution of equations (10.1.20) and (10.1.21) using the solution (10.2.2) iterms of Kummer's functions are given by

where

\[
g(\eta) = e^{-\eta} \left[ M \cdot \frac{Pr}{\gamma} \cdot \frac{1}{\gamma - \frac{Pr}{\gamma}} \cdot \frac{1}{\gamma - \frac{Pr}{\gamma}} \right]
\]  
...(10.2.3)

\[
h(\eta) = e^{-\eta} \left[ M \cdot \frac{Sc}{\gamma} \cdot \frac{1}{\gamma - \frac{Sc}{\gamma}} \cdot \frac{1}{\gamma - \frac{Sc}{\gamma}} \right]
\]  
...(10.2.4)

The dimensionless temperature gradient and concentration gradient at the wall are expressed as
where \( M \) is the Kummer's function. Also for \( C_w=0, \gamma=1, (M_n=0, k_n=0) \) and \( Pr=Sc=1 \), the solutions \( g \) and \( h \) can be expressed in the simple form, and given by

\[
\begin{align*}
g &= e^{\frac{\gamma}{e - 1} \left[ 1 - \exp(-\frac{\gamma}{e}) \right]}
g &= e^{\frac{\gamma}{e - 1} \left[ 1 - \exp(-\frac{\gamma}{e}) \right]} \quad \ldots (10.2.7)
\end{align*}
\]

\[
\begin{align*}
h &= e^{\frac{\gamma}{e - 1} \left[ 1 - \exp(-\frac{\gamma}{e}) \right]}
h &= e^{\frac{\gamma}{e - 1} \left[ 1 - \exp(-\frac{\gamma}{e}) \right]} \quad \ldots (10.2.7)
\end{align*}
\]

10.3 ASYMPTOTIC SOLUTION

For large \( \eta \) when \( \xi = 1, t \to 0, S \to 0, g \to 0, h \to 0 \). Hence asymptotic behaviour of \( g \) and \( h \) are verified as

\[
\begin{align*}
\lim_{\eta \to \infty} \eta (\delta - 1) & \to 0, \\
\lim_{\eta \to \infty} (\delta - S) & \to 0 \\
\end{align*}
\]

\ldots (10.3.1)

Hence for large \( \eta (\eta \to \infty) \) we get

\[
\begin{align*}
f \delta_1 \cdot F, & \quad S \delta_2 \cdot S, & \quad g \cdot G, & \quad h \cdot H, & \quad \delta : \delta_1 \cdot \delta_2
\end{align*}
\]

where \( F, S, G \) and \( H \) are small. Linearizing the equations (10.1.18) to (10.1.21) we get

\[
\begin{align*}
(1_0 - (\eta)) \cdot \delta_1 \cdot I_{\infty}(\eta) \cdot (M_n \cdot k_n) F(\eta) & = 0 \\
S_{\infty}(\eta) \cdot \delta_1 \cdot S_{\infty}(\eta) \cdot (M_n \cdot k_n) S(\eta) & = 0 \\
\end{align*}
\]

\ldots (10.3.2)
\[ G_{\eta\eta}(\eta) + Pr \delta_1 G_\eta(\eta) = 0 \]  
\[ H_{\eta\eta}(\eta) + \text{Sc} \delta_1 H_\eta(\eta) = 0 \]  
\[ \delta_1 \begin{bmatrix} \delta \cdot (\delta - 4(Mn - k)) \end{bmatrix} \]  

The boundary conditions as \( \eta \to \infty \) are given by

\[
F_\eta, S_\eta, G, H = 0
\]  

Equations (10.3.3) (10.3.5) are linear differential equations with constant coefficients. Thus their solutions satisfying the boundary conditions (10.3.6) are obtained as

\[
F_\eta, S_\eta = A_1 \exp(-\delta_1 \eta)
\]
\[
G = A_2 \exp\left[-(\text{Pr} \delta_1 \eta)\right]
\]
\[
H = A_3 \exp\left[-(\text{Sc} \delta_1 \eta)\right]
\]

where \( \delta_1 \begin{bmatrix} \delta \cdot (\delta - 4(Mn - k)) \end{bmatrix} \) Here \( \delta_1, \delta_2, \delta_3, \delta_4 \) are constants and \( A_1, A_2 \) and \( A_3 \) are some arbitrary constants. It is noticed that equations (10.3.7) - (10.3.9) that \( F_\eta, S_\eta, G \) and \( H \) (hence, \( f, S, g, h \)) tend to zero in an exponential manner as \( \eta \to \infty \). if \( \delta_1 < 0 \)

10.4 RESULTS AND DISCUSSION

Figures (10.2) and (10.3) show the variation of the surface shear stresses in \( x \) and \( y \) directions \( [f(x, z), S(x, z)] \) for different values of permeability parameter \( k_2 \) and \( Mn \) when \( C_1, 0.5 \) and \( Pr, 0.7 \). From the figures we observe that at the start of the motion \( (\xi, 0) \), the surface shear stresses are independent of \( k_2 \). The surface shear stresses increase with the permeability \( k_2 \) due to the enhanced Lorentz force which imparts
additional momentum into the boundary layer. This reduces the boundary layer thickness which, in turn, increases the surface shear stress.

The variation of surface shear stresses in x and y directions with dimensionless time \( \dot{\xi} \) for several values of the stretching parameter \( C_{1} \) \((0 \leq C_{1} \leq 1) \) when \( k_{2} = 1.0 \) and \( \text{Mn} = 1.0 \) and \( \text{Pr} = 0.7 \) are presented in figures (10.4) and (10.5) when \( C_{1} = 0, \; S_{\text{yp}}(\xi, 0) = 0 \), as the problem reduces to the two dimensional case. The surface shear stresses in x and y directions \( -f_{x} (\xi, 0), \; -S_{y} (\xi, 0) \) for \( 0 \leq C_{1} \leq 1 \) increase with \( \dot{\xi} \) almost linearly. The effect is significantly pronounced on the surface shear stress in the y direction \( (-S_{\text{yp}}(\xi, 0)) \). For \( \text{Mn} = 1.0, \; \text{Pr} = 0.7 \) the surface shear stresses in the y and x directions increase as \( C_{1} \) increases from 0.0 to 0.75.

In Fig. (10.6a) and (10.6b) we have plotted the graphs for surface heat transfer \( g_{a} (\xi, 0) \) for \( 0 \leq \xi \leq 1 \) with \( C = 0.5, \; \text{Pr} = 0.7 \) for various values of permeability parameter \( k_{2} \) and magnetic parameter \( \text{Mn} \) respectively. It is noticed from the figures that the surface heat transfer decreases with increasing values of \( k_{2} \) and \( \text{Mn} \). This is due to the reduction in the functions \( f \) and \( S \) with increasing \( k_{2} \) and \( \text{Mn} \) which increases the thermal boundary layer thickness. This results in the reduction of the surface heat transfer \( (-g_{a}(\xi, 0)) \) as \( k_{2} \) and \( \text{Mn} \) increase in the respective figures. As \( k_{2} \) increases from 0 to 3, \( -g_{a}(\xi, 0) \) decreases by about 33%. Also as \( \text{Mn} \) increases from zero to 5, \( -g_{a}(\xi, 0) \) decreases by about 36%. Also for all \( k_{2} \) and \( \text{Mn} \) there is a smooth transition from the short-time solution to the long-time solution.
additional momentum into the boundary layer. This reduces the boundary layer thickness which, in turn, increases the surface shear stress.

The variation of surface shear stresses in x and y directions with dimensionless time $\xi$, for several values of the stretching parameter $C_1 (0 \leq C_1 \leq 1)$ when $k_2 = 1.0$ and $Mn = 1.0$ and $Pr = 0.7$ are presented in figures (10.4) and (10.5) when $C_1 = 0, S_{\eta\eta} (\xi, 0) = 0$. As the problem reduces to the two dimensional case. The surface shear stresses in x and y directions ($-f_{\eta\eta} (\xi, 0), -S_{\eta\eta} (\xi, 0)$ for $0 \leq C_1 \leq 1$) increase with $\xi$ almost linearly. The effect is significantly pronounced on the surface shear stress in the y direction ($-S_{\eta\eta} (\xi, 0)$). For $Mn = k_2 = 1.0, Pr = 0.7$ the surface shear stresses in the y and x directions increase as $C_1$ increases from 0.0 to 0.75.

In Fig (10.6a) and (10.6b) we have plotted the graphs for surface heat transfer $g_{\alpha} (\xi, 0)$ for $0 \leq \xi \leq 1$ with $C_1 = 0.5, Pr = 0.7$ for various values of permeability parameter $k_2$ and magnetic parameter $Mn$ respectively. It is noticed from the figures that the surface heat transfer decreases with increasing values of $k_2$ and $Mn$. This is due to the reduction in the functions $f$ and $S$ with increasing $k_2$ and $Mn$ which increases the thermal boundary layer thickness. This results in the reduction of the surface heat transfer ($-g_{\alpha} (\xi, 0))$ as $k_2$ and $Mn$ increase in the respective figures. As $k_2$ increases from 0 to 3, $-g_{\alpha} (\xi, 0)$ decreases by about 33%. Also as $Mn$ increases from zero to 5, $-g_{\alpha} (\xi, 0)$ decreases by about 36%. Also for all $k_2$ and $Mn$ there is a smooth transition from the short-time solution to the long-time solution.
The variation of the surface mass transfer $-h_d(\xi,0)$ for various values of permeability parameter $k_2$, magnetic parameter $M_n$ with Schmidt number $Sc=3.0$, $C_1=0.5$ is shown in the figures (10.7a) and (10.7b) respectively. It is observed from the figures that the surface mass transfer decreases with increasing values of $k_2$ and $M_n$. On the other hand it can be explained as the surface mass transfer $-h_d(\xi,0)$ increases with $\xi$ for all $k_2$ (Fig 10.7a) and for all $M_n$ (Fig 10.7b) except $-h_d(\xi,0)$ for $\xi<\xi_1$, which depends on $k_2$ and $M_n$ ($\xi_1=0.42$ when $k_2=0$, $M_n=0$, $\xi=0.8$ for $k_2=M_n=1$).

Effect of stretching parameter $C_1$ on the surface mass transfer is plotted in Fig (10.8a) for $k_2=M_n=1.0$ and $Sc=3.0$. It is found from the figure that the surface mass transfer $-h_d(\xi,0)$ increases with $\xi$ for $\xi<\xi_2=0.8$ when $k_2=M_n=1$, $C_1=0.25$. Beyond this value, it decreases. The surface mass transfer also increases with increasing values of stretching parameter $C_1$, because increasing values of $C_1$ implies higher surface velocity which accelerates the fluid in the boundary layer.

In Fig (10.8b) a graph of surface mass transfer $-h_d(\xi,0)$ for various values of Schmidt number is plotted and it is noticed from the figure that the surface mass transfer increases with increasing values of Schmidt number $Sc$.

### 10.5 Conclusions

The surface shear stresses, the surface heat transfer and the surface mass transfer increase with the stretching parameter $C_1$, permeability parameter $k_2$ and magnetic parameter $M_n$ and there is a smooth transition from the short-time solution to the long-time solution. The surface shear stresses increase with time $\xi$, but the surface heat
transfer and mass transfer increases up to a certain instant of time, but beyond this time they decrease. The effects of the stretching parameter $C_1$, permeability parameter $k_2$, magnetic parameter $Mn$ and Schmidt number $Sc$ are most pronounced on the surface shear stress and surface mass transfer.
Fig 10.2: Effect of permeability parameter $k$ on the surface shear stress in x-direction for $0 < \xi < 1$ and $M_n = 1.0$
Fig (10.3) Effect of permeability parameter $k_2$ on the surface shear stress in Y-direction ie. $S_{y} = \tau \cdot 10$ for $0 \leq \xi \leq 1$ and for $Mn=1.0$
Fig (10.4) Effect of the stretching parameter $C_1$ on the surface shear stress in $x$-direction. $f_{xx}(\xi, 0)$ for $0 \leq \xi \leq 1$ for $k_2 = 1.0$ and $Mn = 1.0$.
Fig (10.5) Effect of stretching parameter $C_1$ on the surface shear stress in $Y$-direction $-\sigma_{xy}(\xi, 0)$ for $0 \leq \xi \leq 1$ and $k_d=1.0$ and $Mn=1.0$
Fig (10.6b) Effect of magnetic parameter $Mn$ on the surface heat transfer $-\mathbf{g}_{n}(\xi, \theta)$ for $0 \leq \xi \leq 1$ with $C_{1} = 0.5$ and $Pr = 0.7$ and $k_{c} = 1$
Fig (10.7a) Effect of permeability parameter $K_2$ on the surface heat transfer $-h_1(\xi, 0)$ for $0 \leq \xi \leq 1$ with $C_1=0.5$, $Sc=3$ and $Mr=1.0$. 
Fig (10.7b) Effect of magnetic parameter $Mn$ on the surface mass transfer $-h_{x}(\xi , 0)$ for $0 \leq \xi \leq 1$ with $C_{f}=0.5$ Sc=3 and $K_{r}=1.0$
Fig (10.8a) Effect of stretching parameter $C_1$ on the surface mass transfer $-h_1(\xi, 0)$ for $0 \leq \xi \leq 1$ with $Mn=1.0$, $Sc=3.0$, $k_0=1.0$
Fig (10.8b) Effect of Schmidt number on the surface mass transfer $-h_f(\xi, 0)$ for $0 \leq \xi \leq 1$ with $M_n=1.0, k_f=1.0, c_l=0.5$. 