CHAPTER-6

HEAT TRANSFER IN A VISCO-ELASTIC FLUID PAST A
STRETCHING SHEET WITH VISCOUS DISSIPATION AND
INTERNAL HEAT GENERATION

6.1 FLOW ANALYSIS

Consider the problem of a flat, possibly porous surface issuing from a very
thin slit at x=0, y=0 and subsequently being stretched, as in a polymer extrusion
process (Fig.6.1). It is assumed that the speed of a point on the surface is proportional
to its distance from the slit, and the boundary layer approximations are applicable.

6.11 Momentum transfer

The velocity boundary layer equations for the steady, two-dimensional,
incompressible visco-elastic (Walters' liquid B') fluid flow are

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  ...(6.11.1)

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - k \left( u \frac{\partial^3 u}{\partial x \partial y^2} + v \frac{\partial^3 u}{\partial x^2 \partial y} + \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial y \partial x} \right) \]  ...(6.11.2)

Where \( u \) and \( v \) are the velocity components in x and y directions respectively,
and \( \nu \) is the fluid kinematic viscosity. The suitable boundary conditions are:

\[ u = bx; \quad v = v_a \quad \text{at} \quad y = 0 \ (b > 0). \]

\[ u = 0; \quad \frac{\partial u}{\partial y} = 0 \quad \text{as} \quad y \rightarrow \infty \]  ...(6.11.3)

Equations (6.11.1) and (6.11.2) admit a self-similar solution

\[ u = bx f'(\eta), \quad v = -\sqrt{\nu} f(\eta). \]  ...(6.11.4)

\[ \eta = (b/\nu)^{1/2} y \]  ...(6.11.5)
lib(6.1) Boundary layer on a stretching flat, porous sheet

Fig.(6.1) Boundary layer on a stretching flat, porous sheet
Where \( \eta \) is the similarity variable and \( f(\eta) \) is the dimensionless stream function. Clearly continuity equation (6.11.1) is satisfied by equation (6.11.4) by the considered velocity components \( u \) and \( v \). Substituting (6.11.4) and (6.11.5) in momentum equation (6.11.2) and in (6.11.3) we get,

\[
f'^{2} - f'' = f''' - k_{1}\left\{ 2f'f'' - f'f'' - f''^{2}\right\} \quad \ldots (6.11.6)
\]

where

\[ k_{1} = \frac{k_{b}b}{v} = \text{visco-elastic parameter.} \]

and

\[ f'(0)=1, \quad f(0) = \frac{v_{w}}{\sqrt{b}v} = v_{0}, \quad f' (\infty) = 0, \quad f'' (\infty) = 0 \quad \ldots (6.11.7) \]

Where prime denotes differentiation with respect to \( \eta \).

The exact solution to the differential equation (6.11.6) satisfying the boundary conditions (6.11.7) is obtained as.

\[
f(\eta) = \left[ \frac{1-e^{-\eta\alpha}}{\alpha}\right] - v_{0} \quad \ldots (6.11.8)
\]

Where \( \alpha \) is the positive root of the cubic equation

\[
\alpha^{3} + R\alpha^{2} - W\alpha - Z = 0 \quad \ldots (6.11.9)
\]

and \( \alpha \) is calculated by Grafe’s square root method where

\[
R = 1 - k_{4}/k_{1} v_{0}, \quad W = 1/k_{1}, \quad Z = 1/k_{1} v_{0}
\]

and within the range \( 0<\alpha<1 \) corresponding to blowing \( (v_{w}>0) \) and \( \alpha>1 \) corresponding to suction \( (v_{w}<0) \). In the case when \( \alpha \) is unity, the stretching sheet is impermeable.

The velocity components thus become,
6.2 SKIN FRICTION

Skin friction for Visco-elastic fluid flow is given by,

\[ \tau_0 = [-\mu \frac{\partial u}{\partial y} - \nu k_0 \frac{\partial u}{\partial y^2}] \bigg|_{y=0} \]

and leads to

\[ \tau_0 = u_0 \left[ \mu b (v)^{1/2} + \nu_0 k_0 (b)^{1/2} \right] \]

where \( u_0 = \frac{b^{1/2} \alpha}{\nu} \) \( \ldots (6.2.1) \)

6.3 HEAT TRANSFER ANALYSIS

The boundary layer energy equation with viscous dissipation (frictional heating) and internal heat generation or absorption is,

\[ \rho c_p \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) - k \left( \frac{\partial^2 T}{\partial y^2} + \mu \left( \frac{\partial u}{\partial y} \right)^2 \right) + Q(T - T_\infty) \] \( \ldots (6.3.1) \)

Where \( k \) is the thermal conductivity, \( c_p \) is the specific heat and \( Q \) is the volumetric rate of heat generation.

The thermal boundary conditions depend on the type of heating process under consideration. We consider two different heating processes, namely (i) wall with prescribed surface temperature (PST-Case) and (ii) wall with prescribed heat flux (PHF-Case). Both the cases are dealt in the following sections 6.31 and 6.32.

6.31 Prescribed power law surface temperature (PST)

For this case, the boundary conditions are

\[ T = T_w \left[ = T_{w0} + A (x/ \ell)^2 \right] \] \( \text{at } y = 0 \)

\[ T \to T_\infty \] \( \text{as } y \to \infty \) \( \ldots (6.31.1) \)
Where \( \ell \) is the characteristic length. Defining the dimensionless temperature by,

\[
\theta(\eta) = \frac{(T - T_c)}{(T_w - T_c)} \quad \ldots (6.31.2)
\]

and using the relations (6.11.4) and (6.11.5), the energy equation (6.3.1) and the boundary conditions (6.3.1) become,

\[
\theta'' + \sigma \left( f + \frac{v_n}{\sqrt{v}} \right) \theta' - \Pr (2f' - \beta) \theta = -\Pr Ec (f'')^2 \quad \ldots (6.31.3)
\]

with \( \theta = 1 \) at \( \eta = 0 \)

\[ \theta \to 0 \quad \text{as} \quad \eta \to \infty \quad \ldots (6.31.4) \]

where \( \Pr = \mu C_p / \kappa \), the Prandtl number

\( \beta = Q / \rho \alpha C_p \), the heat source/sink parameter

\( Ec = a^2 \ell^2 / \alpha C_p A \), the Eckert number,

and prime denotes differentiation with respect to \( \eta \).

Introducing a new dimensionless variable

\[
\zeta = (Pr/\alpha^2)e^{-\alpha \eta} \quad \ldots (6.31.5)
\]

and substituting the solution for \( f \) into the dimensionless energy equation (6.31.3) and the boundary conditions (6.31.4) yields,

\[
\zeta 0'' + [1 - a_0 - \zeta] 0' + [2 + \beta (Pr/\alpha^2)/\zeta] 0 = -\Pr Ec (\alpha^2/Pr)^2 \zeta \quad \ldots (6.31.6)
\]

\[ \theta (-Pr/\alpha^2) = 1 \quad \text{and} \quad \theta (0, \zeta) = 0 \quad \ldots (6.31.7) \]

Where \( a_0 = [Pr/\alpha^2 - \nu_0/\alpha] \)

and prime denotes differentiation w.r.t. \( \zeta \)

To solve the above differential equation (6.31.6) we assume a solution of the type
\[ \theta (\zeta) = \theta_c (\zeta) + \theta_p (\zeta) \]

For the particular solution, trying \( \theta_p = C\zeta^2 \) gives,

\[ \theta_p (\zeta) = -\text{PrEc} \left[ 4 - 2a_o + \beta \left( \frac{\text{Pr}}{\alpha^2} \right) \right] \left( \frac{\alpha^2}{\text{Pr}} \right) \zeta^2 \]  \hspace{1cm} \text{(6.31.9)}

For the homogeneous solution, solving equation (6.3.16) by series method, assuming \( \theta(\zeta) = \sum_{r=0}^{\infty} a_r \zeta^{r+s} \), the solution can be presented in terms of Kummer's functions as

\[ \theta_c (\zeta) = C_1 \zeta^{1a_o \cdot s+1/2} M \left[ \frac{a_o + s - 4}{2}, 1 + s, \zeta \right] \]  \hspace{1cm} \text{(6.3.10)}

where \( M(a,c;\zeta) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n z^n}{(c)_n n!} \)

and \( a_0 = [\text{Pr}/\alpha^2 - \nu_0/\alpha], \hspace{1cm} s = [a_0^2 - 4\beta(\text{Pr}/\alpha^2)]^{1/2} \)

Substituting equation (6.3.9) and (6.3.10) into equation (6.3.8), the solution of equation (6.3.6) satisfying the boundary conditions (6.3.7) is obtained as,

\[ \theta (\zeta) = \frac{1 + \text{PrEc} \left[ 4 - 2a_o + \beta \left( \frac{\text{Pr}}{\alpha^2} \right) \right]^{-1}}{M \left[ \frac{a_o + s - 4}{2}, 1 + s, \frac{-\zeta}{\text{Pr}/\alpha^2} \right]} M \left[ \frac{a_o + s - 4}{2}, 1 + s, \zeta \right] \]

\[ - \frac{\text{PrEc}}{\left( 4 - 2a_o + \beta \frac{\text{Pr}}{\alpha^2} \right)^2} \left( \frac{\zeta}{\text{Pr}/\alpha^2} \right)^{2} \]  \hspace{1cm} \text{(6.3.11)}

Above solution can be written in terms of \( \eta \) as,

\[ \theta(\eta) = \frac{1 + \text{PrEc} / (4 - 2a_o + \beta \frac{\text{Pr}}{\alpha^2})}{M \left[ \frac{a_o + s - 4}{2}, 1 + s, \frac{-\text{Pr}}{\alpha^2} e^{-a_o \eta} \right]} e^{-a_o \eta} M \left[ \frac{a_o + s - 4}{2}, 1 + s, \frac{-\text{Pr}}{\alpha^2} e^{-a_o \eta} \right] \]

\[ \left( \frac{\text{PrEc}}{4 - 2a_o + \beta \frac{\text{Pr}}{\alpha^2}} \right) e^{-2a_o \eta} \]  \hspace{1cm} \text{(6.3.12)}
The dimensionless temperature gradient derived from (6.31.12) is,

\[
\theta'(0) = \frac{1 + \text{Pr} \left[ 4 - 2a_o + \beta \left( \frac{\text{Pr}}{\alpha^2} \right) \right]}{M \left( \frac{a_o + s - 4}{2} \right) - \alpha \left( \frac{a_o + s}{2} \right) M \left( \frac{a_o + s - 4}{2} \right) - \alpha^2 \left( \frac{a_o + s}{2} \right) M \left( \frac{a_o + s - 4}{2} \right) - \alpha^2 \left( \frac{a_o + s}{2} \right)} + \frac{\text{Pr} \left( \frac{a_o + s - 4}{2} \right) M \left( \frac{a_o + s - 2}{2} \right) - \alpha^2 \left( \frac{a_o + s}{2} \right) M \left( \frac{a_o + s - 2}{2} \right) - \alpha^2 \left( \frac{a_o + s}{2} \right)}{4 - 2a_o + \beta \left( \frac{\text{Pr}}{\alpha^2} \right)} \]

and the local wall heat flux can be expressed as,

\[
q_w = -k \left( \frac{\partial T}{\partial y} \right)_{\omega} = -k \sqrt{\frac{b}{\nu}} \theta'(0)(x/\nu)^2 \]

for several sets of values of the dimensionless parameters \( \alpha, \beta, \text{Ec} \) and \( \text{Pr} \) the temperature profiles are plotted in figs. (6.3a) and (6.3b).

6.32 Prescribed power law wall heat flux (PHF)

For this case, the boundary conditions are,

\[
-k \left( \frac{\partial T}{\partial y} \right) = q_w = D(x/\nu)^2 \quad \text{at} \quad y=0
\]

\[
T \rightarrow T_x \quad \text{as} \quad y \rightarrow \infty
\]

Defining \( T - T_x = D/k (v/\nu)^{1/2} (x/\nu)^2 g(\eta) \)

and substituting the relations (6.11.4),(6.11.5) into (6.3.1) and (6.32.1), they transform to

\[
g'' + \text{Pr} g' - \text{Pr} (2T' - \beta) g = -\text{Pr} \text{Ec} (T'')^2 \]

\[
g'(0) = -1 \quad \text{and} \quad g(\infty) = 0
\]

Where,

\( \text{Pr} = \mu C_p/\kappa \), the Prandtl number

\( \beta = Q/\kappa C_p \), the heat source/sink parameter

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$Ec = \frac{kb^2 \xi}{\sqrt{b\nu}}$, the Eckert number,

and prime denotes differentiation w.r.t. $\eta$.

Substituting the solution for $f$ and using transformation (6.31.5), equations (6.32.3) and (6.32.4) transform to.

\[
\zeta g'' + (1-a_0-\zeta) g' + \left[2 + \beta \left(\frac{Pr}{\alpha^2}\right)^{\zeta-1}\right] g = -PrEc \left(\frac{\alpha^2}{Pr}\right)^2 \zeta \quad \ldots \ (6.32.5)
\]

\[g(-Pr/\alpha^2) = -\alpha/Pr \quad \text{and} \quad g(\alpha) = 0 \quad \ldots \ (6.32.6)
\]

The equation (6.32.5) is of the same form as the energy equation (6.31.6) for the PST-case. Therefore the solution method is parallel to the method used for solving the equation (6.31.6). The solution of (6.32.5) satisfying the boundary conditions (6.32.6) is obtained as.

\[
g(\zeta) = \left(\frac{1}{\alpha} + \frac{2PrEc}{\sqrt{4-2a_0 + \beta Pr}} \alpha \right) \left(\frac{a_0 + s - \frac{4}{\alpha^2}}{M} \left(\frac{a_0 + s - \frac{4}{\alpha^2}}{2} \cdot \frac{Pr}{\alpha^2} \right) - \left(\frac{a_0 + s - \frac{4}{\alpha^2}}{2} \cdot \frac{Pr}{\alpha^2} \right) \right) \ldots \ (6.32.7)
\]

\[\left(\frac{-\zeta}{Pr/\alpha^2}\right)^{a_0^2 + s^2} \cdot \left(\frac{a_0 + s - \frac{4}{\alpha^2}}{2} \cdot \frac{Pr}{\alpha^2} \right) - \left(\frac{PrEc}{Pr} \right) \left(\frac{\alpha^2}{\sqrt{4-2a_0 + \beta Pr}} \alpha \right) \ldots \ (6.32.7)
\]

Where $a_0 = |Pr/\alpha^2 - v_0/\alpha|$, $s = |a_0^2 - 4\beta(Pr/\alpha^2)|^{1/2}$.
The solution (6.32.7) can be recast into
terms of \( q \) as,

\[
g(q) = \left( 1 + \frac{2PrEc}{\alpha} \frac{Pr}{4 - 2a_o + \beta \frac{Pr}{\alpha^2}} \right) e^{-\frac{a_o + s - \frac{4Pr}{\alpha^2}}{2} \left( 1 + s, -\frac{Pr}{\alpha^2} \right)}
\]

The wall temperature can now be determined from equation (6.32.2) as

\[
T = T_o + \frac{D}{k} \left( \frac{v}{b} \right)^4 g(0) \left( \frac{x}{\ell} \right)^2
\]

Numerical values of \( g(0) \) for several sets of values of the dimensionless parameters \( \alpha, \beta, Pr \), and \( Ec \) are obtained and presented graphically in fig. (6.8).

### 6.4 RESULTS AND DISCUSSION

In fig. (6.2) a graph of \( f'(\eta) \) versus \( \eta \) is drawn for various values of viscoelastic parameter \( k_1 = 0.2, 0.4, 0.6 \). From the figure it is seen that the transverse component of velocity increases with increasing values of \( k_1 \).

Fig. (6.3a) displays several dimensionless temperature profiles \( \theta(\eta) \) versus space variable \( \eta \), for \( \alpha, \beta \), and \( Ec \) when \( Pr = 10 \). Fig. (6.3b) presents the similar temperature profiles when \( Pr = 20 \). Comparing the curves in Fig. (6.3a) with the corresponding ones in Fig. (6.3b) it is observed that the temperature at a given point (fixed) decreases with an increase in the Prandtl number \( Pr \). This is in agreement with the physical fact that the thermal boundary layer thickness decreases with increasing
Pr. From Fig. (6.3b) it is also seen that the temperature increases with an increase in the heat source/ sink parameter $\beta$. The same trend occurs in the increase in the frictional heating parameter $E_c$. Further, for fixed values of $Pr$, $\beta$ and $E_c$, the smaller the value of the suction/ blowing parameter $\alpha$, the larger is the temperature at a given point. This phenomena implies that the thermal boundary layer thickness is thinner than that in the blowing ($0<\alpha<1$) case.

In fig. (6.4) a graph of dimensionless wall temperature gradient $\theta'(0)$ as a function of Prandtl number $Pr$ is drawn for several sets of values of the dimensionless parameters $\alpha$ and $\beta$ when $E_c = 0.01$. For fixed values of $\alpha$ and $\beta$ the magnitude of the wall temperature gradient decreases with increasing $Pr$. It is also evident from fig. (6.4) that as $\alpha$ increases, the magnitude of $\theta'(0)$ decreases. It is exactly opposite treatment in Vajravelu’s [46] case. Due to the effect of visco-elasticity, our case is different from their case. Further more the negative values of $\theta'(0)$ for all values of $Pr$ are indicative of the physical fact that the heat flows from the ambient fluid to the surface.

In figures (6.5a) and (6.5b) several temperature profiles are drawn in both PST and PHF cases respectively. The effect of Prandtl number on heat transfer may be analysed from these figures. Both the graphs implicate that the increase of Prandtl number results in the decrease of temperature distribution at a particular point. This is due to the fact that there would be a decrease of thermal boundary layer thickness with the increasing values of Prandtl number. Temperature distribution in both the situations asymptotically approaches to zero in the free stream region.
Figs. (6.6a) and (6.6b) are the graphs for temperature distribution \( \theta(\eta) \) and \( g(\eta) \) Vs distance \( \eta \) from the sheet for different values of visco-elastic parameter \( k_1 \) in both PST and PHF cases respectively. The effect of increasing values of \( k_1 \) is seen to increase the temperature distribution in the flow region. This is in conformity with the fact that the increase of non-Newtonian visco-elastic parameter leads to the increase of thermal boundary layer thickness. The results of PHF cases are qualitatively similar to that of PST case but quantitatively in reduced magnitude.

The graphs for temperature distribution from the sheet for different values of Eckert number are plotted in figures (6.7a) and (6.7b) in PST and PHF cases respectively. The analysis of the graphs reveal the fact that effect of increasing values of Eckert number is to increase the temperature distribution in the flow region in both the cases of PST and PHF. This is due to the fact that heat energy is stored in the fluid due to the frictional heating. From these graphs it is noticed that the combined effect of suction parameter \( v_w \) and increasing values of Eckert number \( Ec \) is to reduce the temperature distribution significantly.

For PHF case, in fig. (6.8) a graph of \( g(0) \) versus Pr with changing values in \( \alpha \), \( \beta \) when \( Ec = 0.01 \) is drawn. It is evident from the figure that the wall temperature decreases as Pr increases. The decrease is rapid for small Prandtl numbers and lower for high Prandtl numbers. Further more, it is evident from the figure that for fixed Pr, the larger the suction/ blowing parameter \( \alpha \), the smaller is the wall temperature.

In fig. (6.9) a graph of skin friction versus visco-elastic parameter \( k_1 \) is drawn. It is observed from the figure that the skin friction decreases with increasing values of
the visco-elastic parameter $k_1$. For industrial applications it is of some importance, since the power expenditure in stretching the sheet decreases with increasing values of $k_1$. The same idea has been already investigated by Raja Gopal et al. [98] in case of viscous fluid and also by Subhas Abel and Veena P [106] in case of unsteady visco-elastic flow.

6.5 CONCLUSIONS

A mathematical analysis about the heat transfer in a visco-elastic fluid past a stretching sheet has been carried out for momentum and heat transfer with viscous dissipation and internal heat generation. An analysis is carried out for two different cases of heating processes namely; (i) prescribed surface temperature (PST) and (ii) prescribed wall heat flux (PIIF) to get the effect of visco-elastic parameter $k_1$ for various situations. Some important findings of the study are listed below.

1) The increasing effect of visco-elastic parameter is to decrease the skin friction on the sheet and also the transverse component of velocity increases as $k_1$ increases.

2) The effect of Prandtl number on temperature distribution is seen to reduce the thermal boundary layer thickness.

3) The combined effect of suction/blowing parameter ($v_w < 0$) and heat source or sink parameter $\beta$ implied that the thermal boundary layer thickness is thinner than that in the blowing ($0 < \alpha < 1$) case.

4) Magnitude of the dimensionless temperature gradient is noticed to be decreased with increasing values of Prandtl number which is exactly opposite treatment in case of Vajravelu [46] in viscous flow.
Fig. (6.2) Variation of velocity component $f'(\eta)$ vs $\eta$ for different values of visco-elastic parameter $k_1$. 
Fig. (6.3a) Dimensionless Temperature profiles $\theta (\eta)$ Vs $\eta$ for $Pr = 10$ (PST case)
Fig (6.3b) Dimensionless temperature profiles for Pr = 20 (PST case)

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</tr>
</tbody>
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Fig (6.4) Dimensionless temperature gradient (PST case) for Ec = 0.01
Fig (6.5a) Dimensionless temperature profiles $\theta(\eta)$ for various values of $Pr$ with $Ec=.05$, $k_1=0.2$ and $\beta=-0.1$ in PST case.
Fig (6.5b) Temperature profiles $g(\eta)$ for various values of $Pr$ with $Ec=0.1, k_1=0.2$ and $\beta=-1.0$ in PHF case.
Fig. (6.6a) Temperature profile $\theta(\eta)$ for various values of $k_1$ with $Ec=0.5$, $\beta=-0.1$ and $Pr=1.0$ in PST case
Fig. (6.6b) Temperature profile $g(\eta)$ for various values of $k_1$ with $Ec=0.5$, $\beta=0.1$ and $Pr=1.0$ in PHF case
Fig (6.7a) Temperature profile $\theta(\eta)$ for various values of Ec with $Pr=1.0, k_1=0.2$ and $\beta=-0.5$ in PST case
Fig (6.7b) Temperature profile $g(\eta)$ for various values of Ec with $Pr=1.0$, $k_i=0.2$, and $\beta=0.1$ in PHF case.
Fig (6.8) Dimensionless wall temperature (PHF case) for Ec = 0.01
Fig. (6.9) Graph of skin friction versus visco-elastic parameter $k_i$