Chapter 3

Applications of the New Iterative Method

This chapter is based on the following papers:
3.1 Introduction

Many physical processes can be modeled by nonlinear differential equations. Since very few of the nonlinear problems have exact analytical solutions, we have to resort to numerical/iterative methods.

In the present chapter we use the new iterative method (NIM) to solve some nonlinear equations. In Section 3.2 we solve some diffusion-wave equations of fractional order. Section 3.3 deals with evolution equations containing integer order derivatives and the Section 3.4 various partial differential equations (PDEs) of fractional order. We compare the results obtained with existing methods and exact solutions.

3.2 Fractional diffusion-wave equations

A space-time fractional diffusion-wave equation is obtained from classical diffusion equation by replacing the second order space derivative by fractional derivative of order \( \beta \) (1 < \( \beta \) \leq 2), and first order time derivative by fractional derivative of order \( \alpha \) (0 < \( \alpha \) \leq 1), [11].

Similar generalizations of classical wave equation have been discussed in the literature ([4, 9, 12, 13, 22] etc.). Diffusion-wave equations involving Riemann-Liouville derivative [4, 17], Caputo derivative [1, 3, 9, 15, 22] and Grünwald-Letnikov derivative [20] have been discussed by various researchers. Fujita [4] has presented the existence and uniqueness of the solution of the following equation

\[
D_t^\alpha u = D_x^\beta u, \quad 0 < \alpha \leq 1, \ 0 < \beta \leq 2.
\]  

(3.1)

Schneider and Wyss [22] have shown that the time-fractional diffusion equation (\( \beta = 2 \)) represents sub-diffusion for 0 < \( \alpha \) < 1. It has further been observed that 1 < \( \alpha \) < 2 represents enhanced diffusion only in one-dimension. The solutions need not remain non-negative and can not represent physical diffusion of any kind [9, 22] in higher dimensions for \( \alpha > 1 \).

Fractional diffusion-wave equation has been used widely in many branches of Science and Engineering. These equations represent propagation of mechanical waves in visco-elastic media [12–14], a non-Markovian diffusion process with memory [15], charge transport in amorphous semiconductors [21] and many more. Mainardi et al. [12–14]

We consider the following fractional initial value problem (IVP), for $\bar{x} \in \mathbb{R}^n$:

$$D_t^\alpha u(\bar{x}, t) = \sum_{i=1}^{n} a_i D_{x_i}^{\beta_i} u(\bar{x}, t) + A(u(\bar{x}, t)), \quad t > 0, \quad m - 1 < \alpha \leq m,$$

(3.2)

$$\frac{\partial^j u}{\partial t^j}(\bar{x}, 0) = h_j(\bar{x}), \quad 0 \leq j \leq m - 1, \quad m = 1, 2, \quad 1 < \beta_i \leq 2,$$

(3.3)

where $a_i$ are constants, $A(u)$ is non-linear function of $u$ and $h_k$ are functions of $\bar{x}$. Applying $I_t^\alpha$ on both sides of (3.2) and using (3.3) we get

$$u(\bar{x}, t) = \sum_{j=0}^{m-1} h_j(\bar{x}) \frac{t^j}{j!} + I_t^\alpha \left( \sum_{i=1}^{n} a_i D_{x_i}^{\beta_i} u(\bar{x}, t) \right) + I_t^\alpha A(u).$$

(3.4)

Comparing equation (3.4) with standard form

$$u = f + L(u) + N(u)$$

we get $f = \sum_{j=0}^{m-1} h_j(\bar{x}) \frac{t^j}{j!}$, $L(u) = I_t^\alpha \left( \sum_{i=1}^{n} a_i D_{x_i}^{\beta_i} u \right)$ and $N(u) = I_t^\alpha A(u)$. Thus (3.4) can be solved using NIM.

### 3.2.1 Illustrative examples

Some illustrative examples are presented below.

**Example 3.1** Consider the time-fractional diffusion equation

$$D_t^\alpha u(x, t) = u_{xx}(x, t), \quad t > 0, \quad x \in \mathbb{R}, \quad 0 < \alpha \leq 1,$$

(3.5)

$$u(x, 0) = \sin(x).$$

(3.6)
System (3.5)–(3.6) is equivalent to
\[ u = \sin(x) + I_t^\alpha u_{xx}. \]  
(3.7)

Using the NIM algorithm, we get the recurrence relation
\[ u_0 = \sin(x), \ u_1 = -\sin(x) \frac{\mathcal{I}^\alpha}{\Gamma(\alpha + 1)}, \ldots \]

In general \( u_j = (-1)^j \sin(x) \frac{\mathcal{I}^\alpha}{\Gamma(j\alpha + 1)}, \ j = 0, 1, 2, \ldots \) The solution of (3.5)–(3.6) is thus
\[ u(x, t) = \sum_{j=0}^{\infty} u_j(x, t) = \sin(x) \sum_{j=0}^{\infty} \frac{(-t^\alpha)^j}{\Gamma(j\alpha + 1)} \]
\[ = \sin(x)E_\alpha(-t^\alpha). \]

**Example 3.2** Consider the time-fractional wave equation
\[ D_t^\alpha u(x, t) = k \cdot u_{xx}(x, t), \quad t > 0, \quad x \in \mathbb{R}, \quad 1 < \alpha \leq 2, \]  
(3.8)
\[ u(x, 0) = x^2, \ u_t(x, 0) = 0. \]  
(3.9)

We get the equivalent integral equation of initial value problem (3.8)–(3.9) as
\[ u = x^2 + k \cdot I_t^\alpha u_{xx}. \]  
(3.10)

Applying the NIM, we get \( u_0 = x^2, \ u_1 = 2k \cdot \frac{\mathcal{I}^\alpha}{\Gamma(\alpha + 1)}, \ u_2 = 0, \ldots \) The solution of (3.8)–(3.9) is
\[ u(x, t) = \sum_{i=0}^{\infty} u_i = x^2 + 2k \cdot \frac{t^\alpha}{\Gamma(\alpha + 1)}. \]  
(3.11)
Example 3.3 Consider the space-fractional diffusion equation

\[ u_t(x, t) = k \cdot D_x^\beta u(x, t), \quad t > 0, \quad x \in \mathbb{R}, \quad 1 < \beta \leq 2, \quad (3.12) \]

\[ u(x, 0) = \frac{2x^\beta}{\Gamma(1 + \beta)}. \quad (3.13) \]

Integrating (3.12) and using (3.13) we get

\[ u(x, t) = \frac{2x^\beta}{\Gamma(1 + \beta)} + k \int_0^t \left( D_x^\beta u(x, t) \right) dt. \quad (3.14) \]

Applying the NIM, we get

\[ u_0 = \frac{2x^\beta}{\Gamma(1 + \beta)}, \quad u_1 = 2kt, \quad u_2 = 0, \ldots \]

The solution of (3.12)–(3.13) turns out to be

\[ u(x, t) = \frac{2x^\beta}{\Gamma(1 + \beta)} + 2kt. \quad (3.15) \]

Example 3.4 Now we consider the space and time fractional diffusion equation

\[ D_t^\alpha u(x, t) = k \cdot D_x^\beta u(x, t), \quad t > 0, \quad x \in \mathbb{R}, \quad (3.16) \]

\[ u(x, 0) = \frac{3x^\beta}{\Gamma(1 + \beta)} \quad 0 < \alpha \leq 1, \quad 1 < \beta \leq 2. \quad (3.17) \]

Applying \( I_t^\alpha \) on both sides of (3.16) and using condition (3.17), we get

\[ u(x, t) = \frac{3x^\beta}{\Gamma(1 + \beta)} + I_t^\alpha(D_x^\beta u(x, t)). \quad (3.18) \]

Using the algorithm of NIM we get

\[ u_0 = \frac{3x^\beta}{\Gamma(1 + \beta)}, \quad u_1 = 3k \frac{t^\alpha}{\Gamma(\alpha + 1)}, \quad u_2 = 0, \ldots \quad (3.19) \]

Thus \( u(x, t) = \frac{3x^\beta}{\Gamma(1 + \beta)} + 3k \frac{t^\alpha}{\Gamma(\alpha + 1)} \) is solution of (3.16)–(3.17).
Example 3.5 Consider the two-dimensional time fractional wave equation

\[ D_t^\alpha u(\bar{x}, t) = 2 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) u(\bar{x}, t), \quad t > 0, \quad \bar{x} \in \mathbb{R}^2, \]  
\[ u(\bar{x}, 0) = \sin(x_1) \cdot \sin(x_2), \quad u_t(\bar{x}, 0) = 0, \quad 1 < \alpha \leq 2. \]  

The problem (3.20)–(3.21) is equivalent to

\[ u = \sin(x_1) \cdot \sin(x_2) + 2I_t^\alpha \left( \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right). \]  

In the view of the NIM,

\[ u_j = \sin(x_1) \cdot \sin(x_2) \frac{(-4t^\alpha)^j}{\Gamma(j\alpha + 1)}, \quad j = 0, 1, 2, \ldots \]  

Hence,

\[ u(\bar{x}, t) = \sum_{j=0}^{\infty} u_j = \sin(x_1) \sin(x_2) \sum_{j=0}^{\infty} \frac{(-4t^\alpha)^j}{\Gamma(j\alpha + 1)} \]  
\[ = \sin(x_1) \sin(x_2) E_\alpha (-4t^\alpha) \]  

is solution of (3.20)– (3.21).

Comment. In two-dimensions, since \( \alpha > 1 \), the solution is not necessarily positive hence does not represent diffusion (cf. Fig. 3.5) of any kind \([9, 22]\).

Example 3.6 Consider the nonlinear time fractional diffusion equation

\[ D_t^\alpha u(x, t) = u_{xx}(x, t) + 2u(x, t)^2, \quad t > 0, \quad x \in \mathbb{R}, \quad 0 < \alpha \leq 1, \]  
\[ u(x, 0) = e^{-x}. \]
The equivalent integral equation of the initial value problem (3.25)–(3.26) is
\[ u = e^{-x} + I_{\alpha}^0 (u_{xx}) + 2I_{\alpha}^0 \left( u^2 \right). \] (3.27)
Equation (3.27) has the required form with \( f = e^{-x} \), \( L(u) = I_{\alpha}^0 (u_{xx}) \), and \( N(u) = 2I_{\alpha}^0 \left( u^2 \right) \).

The algorithm of NIM gives
\[
\begin{align*}
  u_0 &= e^{-x}, \\
  u_1 &= e^{-x} (1 + 2e^{-x}) \frac{t^\alpha}{\Gamma(\alpha + 1)}, \\
  u_2 &= e^{-x} (1 + 8e^{-x}) \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + 2 \left( e^{-x} + 2e^{-2x} \right)^2 \frac{t^{3\alpha} \Gamma(2\alpha + 1)}{\Gamma(3\alpha + 1) \Gamma(\alpha + 1)^2} \\
  &\quad + 4e^{-2x} (1 + 2e^{-x}) \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)},
\end{align*}
\]
and so on. Three term solution of (3.25)–(3.26) is \( u = u_0 + u_1 + u_2 \).

\[ \text{Fig. 3.5: (Ex. 3.5, } x_1 = \pi/4, \alpha = 1.5) \quad \text{Fig. 3.6: (Ex. 3.6, } \alpha = 0.5) \]

### 3.3 Evolution equations

**Example 3.7** Consider the regularized long-wave (RLW) equation [5]
\[
\begin{align*}
  &u_t - u_{xxx} + \left( \frac{u^2}{2} \right)_x = 0, \quad -\infty < x < \infty, \ t > 0, \\
  &u(x, 0) = x.
\end{align*}
\] (3.28) (3.29)

The exact solution is given by \( \frac{x}{1-t} \). Integrating (3.28) and using (3.29) we get
\[
\begin{align*}
  u &= x + \int_0^t \left( u_{xx} - \frac{u^2}{2} \right)_x \ dt = f + L(u) + N(u)
\end{align*}
\] (3.30)
where \( f = x \), \( L(u) = \int_0^t u_{xx} \, dt \) and \( N(u) = -\int_0^t \left( \frac{u^2}{2} \right)_x \, dt \). Applying NIM we get the following iterative scheme

\[
\begin{align*}
    u_0 &= x, \\
    u_1 &= -xt, \\
    u_2 &= -\frac{t^2}{3}x(t-3), \\
    \ldots.
\end{align*}
\]

The six-term solution of (3.28)–(3.29) is

\[
\begin{align*}
    u(x,t) &= x \left(1 - t + t^2 - t^3 + t^4 - t^5 + \frac{43t^6}{45} - \frac{13t^7}{15} + \frac{943t^8}{1260} - \frac{3497t^9}{5670} + \frac{27523t^{10}}{56700} \right. \\
    &\quad \left. - \frac{1477t^{11}}{4050} + \frac{17779t^{12}}{68040} - \frac{13141t^{13}}{73710} + \frac{1019t^{14}}{8820} - \frac{63283t^{15}}{893025} + \frac{43363t^{16}}{1058400} \right) \\
    &\quad + \frac{48580560}{1080013t^{17}} + \frac{2588t^{18}}{229635} - \frac{162179t^{19}}{30541455} + \frac{16511t^{20}}{7144200} - \frac{207509t^{21}}{225042300} \\
    &\quad + \frac{1666980}{55722} - \frac{22504230}{2447t^{23}} + \frac{16927t^{24}}{5309t^{25}} - \frac{675126900}{109876902975} + \frac{595350}{6751269} - \frac{2t^{27}}{13t^{28}} \\
    &\quad + \frac{3544416225}{135059220} - \frac{315294415}{236294415} - \frac{109876902975}{109876902975}.
\end{align*}
\]

which is exactly equal to the solution obtained by variational iteration method (VIM) [5]. The solution (3.31) has been presented in Fig. 3.8(a). Approximate solution (dashed line) is compared with exact solution (solid line) in Fig. 3.8(b) for \( t = 1 \).

\[\text{Fig. 3.8(a): solution given by (3.31)} \quad \text{Fig. 3.8(b): } t = 1\]

**Example 3.8** Consider the following equation [5]

\[
\begin{align*}
    u_t + u_{xxxx} &= 0, \quad -\infty < x < \infty, \quad t > 0, \\
    u(x,0) &= \sin(x).
\end{align*}
\]

(3.32) (3.33)
Equivalent integral equation of (3.32)–(3.33) is

\[ u = \sin(x) - \int_{0}^{t} u_{xxxx} \, dt. \]  

(3.34)

Applying algorithm of NIM we get

\[ u_0 = \sin(x), \]
\[ u_1 = -t \sin(x), \]
\[ u_2 = \frac{t^2}{2} \sin(x), \]
\[ u_3 = \frac{-t^3}{6} \sin(x), \]
\[ u_4 = \frac{t^4}{24} \sin(x). \]

Five term solution is

\[ u(x, t) = \frac{\sin(x)}{24} \left( 24 - 24t + 12t^2 - 4t^3 + t^4 \right) \]  

(3.35)

which is in high agreement with the exact solution \( e^{-t} \sin(x) \) and matches with the VIM solution [5]. Fig. 3.10(a) represents solution (3.35). Approximate solution (dashed line) is compared with exact solution (solid line) in Figs. 3.10(b) and 3.10(c) for \( t = 0.9 \) and \( t = 1.3 \) respectively.

![Fig. 3.10(a) solution given by (3.35)](image1)

![Fig. 3.10(b) t = 0.9](image2)

![Fig. 3.10(c) t = 1.3](image3)

**Example 3.9** Consider the following equation

\[ u_t + \frac{1}{2} u u_{xx} = u_{xxx}, \quad -\infty < x < \infty, \quad t > 0, \]  

(3.36)

\[ u(x, 0) = x^2. \]  

(3.37)

Integrating with respect to \( t \), we get

\[ u = x^2 + \int_{0}^{t} \left( u_{xxx} - \frac{1}{2} u u_{xx} \right) \, dt. \]  

(3.38)
In view of NIM algorithm

\[ u_0 = x^2, \]
\[ u_1 = -tx^2, \]
\[ u_2 = \frac{-t^2x^2}{3}(t - 3), \]
\[ u_3 = \frac{-t^3x^2}{63} \left( 42 - 42t + 21t^2 - 7t^3 + t^4 \right). \]

Fig. 3.11(a) represents the six-term solution of (3.36)–(3.37). The graph matches with the graph of exact solution of (3.36)–(3.37), \( u(x, t) = \frac{x^2}{1+4t} \). Approximate solution (dashed line) is compared with exact solution (solid line) in Figs. 3.11(b) and 3.11(c) for \( t = 0.3 \) and \( t = 0.9 \) respectively.

**Example 3.10**  Consider the evolution equation

\[ u_t + e^t u u_x + u u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \quad (3.39) \]
\[ u(x, 0) = x. \quad (3.40) \]

Eq. (3.39)–(3.40) are equivalent to the integral equation

\[ u = x - \int_0^t (e^t u u_x + u u_{xx}) \, dt. \quad (3.41) \]

We set up the following iterative scheme, in view of NIM

\[ u_0 = x, \]
\[ u_1 = x(1 - e^t), \]
\[ u_2 = \frac{-x}{3} (e^t - 4)(e^t - 1)^2, \]
\[ u_3 = \frac{-x}{63} \left( 113 - 109e^t + 48e^{2t} - 11e^{3t} + e^{4t} \right) (e^t - 1)^3. \]
Fig. 3.12 represents six-term solution of (3.41).

**Example 3.11** Consider the cubic Burgers’ equation [8]

\[ u_t + u^2u_x = (0.001)u_{xx}, \quad 0 \leq x \leq 1, \quad t \geq 0, \]  

(3.42)

subject to the periodic boundary conditions \( u(0, t) = u(1, t) \) and the initial Gaussian condition

\[ u(x, 0) = \exp(-100(x - 0.5)^2) \]  

(3.43)

centered at 0.5.

Four term approximate solution using NIM is shown in Fig. 3.13 for \( t = 0.01, 0.02, 0.03, 0.04 \) and 0.05. Solution matches with the solution by Adomian decomposition method (ADM) [8].

![Fig. 3.12: (Ex. 3.10: six-term solution) Fig. 3.13:(Ex. 3.11: 4-term solutions)](image)

**Example 3.12** Consider the nonlinear partial differential equation [2, 23]

\[ u_{xx} - uu_t = 1 - \frac{x^2 + t^2}{2}, \]  

(3.44)

\[ u(0, t) = \frac{t^2}{2}, \quad u_x(0, t) = 0. \]  

(3.45)

Equivalent integral equation of system (3.44)–(3.45) is

\[ u = \frac{t^2}{2} + \frac{x^2}{2} - \frac{x^2t^2}{4} - \frac{x^4}{24} + \int_0^x \int_0^x u \cdot u_t dx dx. \]  

(3.46)
Using NIM algorithm we get

\[ u_0 = \frac{t^2}{2} + \frac{x^2}{2} - \frac{x^4}{4} - \frac{x^6}{24}, \]

\[ u_1 = \frac{t^2 x^2}{4} + \frac{x^4}{24} - \frac{x^6}{24} - \frac{7x^6}{720} + \frac{x^8}{240} + \frac{x^8}{2688}, \]

\[ u_2 = \frac{x^4 t^2}{24} + \frac{7x^6}{720} - \frac{t^2 x^6}{144} - \frac{13x^8}{10080} + \frac{t^2 x^8}{6720} + \frac{61x^{10}}{1209600} + \frac{t^2 x^{10}}{25920} + \frac{7x^{12}}{1140480} - \frac{t^2 x^{12}}{190080} - \frac{271x^{14}}{440294400} + \frac{t^2 x^{14}}{5241600} + \frac{x^{16}}{77414400}. \]

In Fig. 3.14 we compare five-term solution of (3.44)–(3.45) using NIM, ADM and exact solution \( \frac{\partial^2 u}{\partial t^2} \) at time \( t = 0.5 \).

![Fig. 3.14: (Ex. 3.12: Solution of (3.44)–(3.45) at \( t = 0.5 \))](image)

**Example 3.13** Consider the Boussinesq equation

\[ u_{tt} = u_{xx} + u_{xxxx} + \left( u^2 \right)_{xx}, \quad (3.47) \]

\[ u(x, 0) = -\frac{3}{8} sech^2 \left( \frac{x}{4} \right), \quad u_t(x, 0) = \frac{3 \sqrt{5}}{32} sech^2 \left( \frac{x}{4} \right) tanh \left( \frac{x}{4} \right). \quad (3.48) \]

Applying NIM we get

\[ u_0 = -0.375 sech^2(0.25x) (1 - 0.559017t \ tanh(0.25x)), \]

\[ u_1 = t^2 \left( -0.0292969 sech^6(0.25x) + 0.134766 sech^4(0.25x) \ tanh^4(0.25x) \right) + t^3 \left( 0.0414895 sech^6(0.25x) \ tanh(0.25x) \right) \]

\[ -0.0117188 sech^2(0.25x) \ tanh^4(0.25x) + t^4 \left( 0.00457764 sech^8(0.25x) - 0.00549316 sech^6(0.25x) \ tanh^2(0.25x) \right) \]

\[ + t^5 \left( 0.00366211 sech^8(0.25x) \ tanh^4(0.25x) \right) \].

44
The four-term approximate solution of (3.47)–(3.48) is compared with the solution by VIM and HPM [6] in the Table 3.1.

Table 3.1: Comparison of solutions for Eq. (3.47)–(3.48)

<table>
<thead>
<tr>
<th>x</th>
<th>t = 0.1</th>
<th></th>
<th>t = 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>HPM</td>
<td>VIM</td>
<td>NIM</td>
</tr>
<tr>
<td>-2</td>
<td>-0.2949</td>
<td>-0.2949</td>
<td>-0.3025</td>
</tr>
<tr>
<td>-1.5</td>
<td>-0.3267</td>
<td>-0.3267</td>
<td>-0.3335</td>
</tr>
<tr>
<td>-1</td>
<td>-0.3524</td>
<td>-0.3524</td>
<td>-0.3575</td>
</tr>
<tr>
<td>-0.5</td>
<td>-0.369</td>
<td>-0.369</td>
<td>-0.372</td>
</tr>
<tr>
<td>0</td>
<td>-0.3748</td>
<td>-0.3748</td>
<td>-0.3753</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.369</td>
<td>-0.369</td>
<td>-0.3667</td>
</tr>
<tr>
<td>1</td>
<td>-0.3524</td>
<td>-0.3524</td>
<td>-0.3478</td>
</tr>
<tr>
<td>1.5</td>
<td>-0.3267</td>
<td>-0.3267</td>
<td>-0.3203</td>
</tr>
<tr>
<td>2</td>
<td>-0.2949</td>
<td>-0.2949</td>
<td>-0.2873</td>
</tr>
</tbody>
</table>

3.4 Fractional partial differential equations

Hemeda [10] used variational iteration method (VIM) to solve PDEs of integer order. PDEs of fractional order are solved by Momani and Odibat using VIM and ADM in [18]. Homotopy perturbation method (HPM) has further been used by Momani and Odibat in [16] which provides the solutions of PDE. In the present section we apply the NIM to solve PDEs of integer as well as those of fractional order. The results are compared with those obtained by existing methods such as VIM, ADM, HPM.
3.4.1 Solving PDEs using NIM

Consider the partial differential equation of arbitrary order:

\[ D^\alpha_t u(x, t) = A(u, \partial u) + B(x, t), \quad m - 1 < \alpha \leq m, m \in \mathbb{N}, \]  
\[ \frac{\partial^k u}{\partial t^k}(x, 0) = h_k(x), \quad k = 0, 1, \ldots, m - 1, \]  

where \( A \) is a nonlinear function of \( u \) and \( \partial u \) (partial derivatives of \( u \) with respect to \( x \) and \( t \)), and \( B \) is the source function. The initial value problem (3.49)-(3.50) is equivalent to the following integral equation:

\[ u(x, t) = \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + I^\alpha_t B + I^\alpha_t A = f + N(u), \]  

where \( f = \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + I^\alpha_t B \) and \( N(u) = I^\alpha_t A \). We get the solution of (3.51) by employing the NIM.

3.4.2 Numerical examples

In this section we illustrate NIM with numerical examples and compare the results with those obtained by other methods.

Example 3.14 Consider nonlinear PDE [10]

\[ u_{tt} + u_{xx} + u_x^2 = 2x + t^4, \]
\[ u(x, 0) = 0, \quad u_t(x, 0) = a, \quad u(0, t) = at, \quad u_x(0, t) = t^2. \]  

The boundary value problem (3.52) is equivalent to the following integral equation

\[ u = at + \frac{t^6}{30} + xt^2 - I^2_t (u_{xx} + u_x^2). \]

Let \( N(u) = -I^2_t (u_{xx} + u_x^2) \). In view of NIM,

\[ u_0 = at + \frac{t^6}{30} + xt^2, \quad u_1 = N(u_0) = -\frac{t^6}{30}, \]
\[ u_2 = 0, \quad u_3 = 0, \ldots. \]

Thus \( \sum_{i=0}^{\infty} u_i = u(x, t) = at + xt^2 \) which in fact is the exact solution.
Example 3.15 Consider the linear PDE [10]

\[ u_{tt} + u_{xx} + u_{xt} = 2(x + t), \]
\[ u(x, 0) = ax, \; u_t(x, 0) = x^2, \; u(0, t) = 0, \; u_x(0, t) = a. \]  
(3.53)

Eq. (3.53) is equivalent to the integral equation:

\[ u = ax + xt^2 + x^2t + \frac{t^3}{3} - \int_t^2 (u_{xx} + u_{xt}). \]  
(3.54)

In view of the NIM algorithm

\[ u_0 = ax + xt^2 + x^2t + \frac{t^3}{3}, \; u_1 = -\frac{2t^3}{3} - xt^2, \]
\[ u_2 = \frac{t^3}{3}, \; u_3 = 0, \ldots. \]

Thus \( u(x, t) = u_0 + u_1 + u_2 = ax + x^2t \) which is exact solution of (3.53).

Example 3.16 Consider the nonlinear time-fractional Fisher’s equation [16, 18]

\[ D_t^\alpha u(x,t) = u_{xx}(x,t) + 6u(x,t)(1 - u(x,t)), \]
\[ x \in \mathbb{R}, \; 0 < \alpha \leq 1, \]
\[ u(x, 0) = \frac{1}{(1 + e^{x-5})^2}. \]  
(3.55)

The exact solution for \( \alpha = 1 \) case is \( u(x, t) = \left(1 + e^{x-5t}\right)^{-2}. \)

We have plotted the three-term solution in Fig. 3.15 for the case \( \alpha = 1, \; t = 0.4, \) and it is compared with corresponding solution given by exact and ADM solution [18]. It is remarkable to note that the graphs of VIM, HPM and NIM coincide and in Fig. 3.16 for the case \( \alpha = 1, \; t = 0.2, \) the exact solution and NIM solution coincides.
Fig. 3.15: (Ex. 3.16, \( \alpha = 1, t = 0.4 \))

Fig. 3.16: (Ex. 3.16, \( \alpha = 1, t = 0.2 \))
Bibliography


