ABSTRACT

The present thesis deals with some investigations in nonlinear systems of fractional order. We discuss linear and nonlinear ordinary/partial differential equations of fractional order using some decomposition methods such as Adomian Decomposition Method (ADM) and New Iterative Method (NIM) (Chapter 2 to Chapter 5) proposed by Daftardar-Gejji and Jafari. We further tackle delay differential equations of fractional order (Chapter 6 to Chapter 8) and interesting phenomena such as chaos and synchronization in fractional order dynamical systems (Chapter 9 to Chapter 12).

Chapter 1 deals with historical development and basic framework of Fractional Calculus [22, 29–32].

In Chapter 2, we discuss decomposition methods such as ADM [1, 9] and NIM [10] proposed by Daftardar-Gejji and Jafari. These decomposition methods are useful in solving nonlinear functional equations. Further we prove the convergence of the NIM [3].

In Chapter 3 we employ NIM to solve some fractional order nonlinear equations such as:


\[ D_t^\alpha u(x, t) = A(u, \partial u) + B(x, t), \quad m - 1 < \alpha \leq m, m \in N, \]

\[ \frac{\partial^k u}{\partial \tau^k}(x, 0) = h_k(x), \quad k = 0, 1, \cdots, m - 1, \]

where \( A \) is a nonlinear function of \( u \) and \( \partial u \) (partial derivatives of \( u \) with respect to \( x \) and \( t \)), and \( B \) is the source function.


\[ D_t^\alpha u(\bar{x}, t) = \sum_{i=1}^{n} a_i D_{x_i}^{\beta_i} u(\bar{x}, t) + A(u(\bar{x}, t)), \quad 1 < \beta_i \leq 2, \]

where \( \bar{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n \), \( a_i \) are constants, \( A(u) \) is non-linear function of \( u \), \( D_t^\alpha \) and \( D_{x_i}^{\beta_i} \) denote Caputo partial fractional derivative with respect to \( t \) and with respect to \( x_i \), respectively.

3. Some evolution equations [5] (with integer order derivative) such as regularized long-wave (RLW) equation, cubic Burgers’ equation.
Fractional order boundary value problems (FBVP) \[12\]

\[ D_t^\alpha u = u_{xx} + A(u), \quad t > 0, \quad 0 < x < l, \]

where \( m - 1 < \alpha < m, m = 1, 2 \) and \( A(u) \) a given nonlinear function of \( u \) together with the Dirichlet boundary conditions

\[ \frac{\partial^k u}{\partial t^k}(x, 0) = p_k(x), \quad k = 0, \cdots, (m - 1), \]
\[ u(0, t) = f_0(t), \quad u(l, t) = f_l(t) \]

are solved in Chapter 4 using NIM.

The classical separation of variables method is used in Chapter 5 for solving multi-term linear fractional order differential equations (FDE). The following equation is solved explicitly.

\[ P(D)u(x, t) = k \frac{\partial^2 u(x, t)}{\partial x^2} + q(t), \quad 0 < x < \pi, \quad t > 0, \]

where \( P(D) = D_t^\mu - \sum_{i=1}^{r-1} \lambda_i D_t^{\mu_i}, \quad 0 < \mu_{r-1} < \mu_{r-2} < \cdots < \mu_1 < \mu \leq 2. \)

We [14] also solve the following multi-term non-linear FDE using ADM

\[ P(D)u(\bar{x}, t) = \sum_{i=1}^{n} N_i \frac{\partial^2 u}{\partial x_i^2} + \phi(\bar{x}, t)u^m(\bar{x}, t) \]

where \( P(D) \equiv D_t^{s_1} - \sum_{j=2}^{r} \lambda_j D_t^{s_j}, \)

\( r \geq 2, \quad r \in \mathbb{N}, \quad 0 < s_r < s_{r-1} < \cdots < s_2 < s_1 < 2, \quad m = 0, 1, 2, \cdots, N_i(\bar{x}, t) \in C_\alpha. \) \( D_t^{s_i} \)
denote Caputo fractional derivatives.

In this context we have made an observation [13] that, unlike in the one term case, solution of multi-term fractional diffusion-wave equation is not necessarily non-negative, and hence does not represent anomalous diffusion in general.

Our work on multi-term fractional diffusion equation is further exploited by Srivastava and Rai [33] in studying the model of oxygen delivery through capillary to tissues. They have used NIM and ADM to solve the corresponding differential equation.

Delay differential equation (DDE) is a differential equation in which the derivative of the function at any time depends on the solution at previous time. Introduction of delay in the model enriches its dynamics and allows a precise description of the real life phenomena. DDEs are proved useful in control systems [19], lasers, traffic models [16].
metal cutting, epidemiology, neuroscience, population dynamics [23], chemical kinetics [18] etc. Chapter 6 deals with the delay differential equations.

In Chapter 7, we have carried out a modification in fractional order Adams-Bashforth method [17] to solve fractional delay differential equations. We employ this algorithm to solve some problems in biology such as life cycle of a population of lemmings and enzyme kinetics. It is observed that one dimensional delayed systems of fractional order show chaotic behavior in some cases. We have studied the effect of fractional order on the phase portrait.

Fractional order generalization of the Bloch equation has been undertaken by several scientists to account for the anomalous relaxation and anomalous diffusion observed in NMR studies of complex materials. We propose fractional order Bloch equation involving delay [6]

\[
\begin{align*}
D^\alpha M_x(t) &= \omega_0 M_y(t - \tau) - \frac{M_x(t - \tau)}{T_2}, \\
D^\alpha M_y(t) &= -\omega_0 M_x(t - \tau) - \frac{M_y(t - \tau)}{T_2}, \\
D^\alpha M_z(t) &= \frac{M_0 - M_z(t - \tau)}{T_1}, \\
M_x(t) &= 0, \quad M_y(t) = 100, \quad M_z(t) = 0, \quad \text{for } t \leq 0,
\end{align*}
\]

and study its properties in Chapter 8.

In 1963, Lorenz [26] discovered chaotic solutions in a system of three autonomous ordinary differential equations. To this end a plenty of examples of dynamical systems exhibiting chaos have been presented in various branches of Science [2]. Chaotic solutions have been obtained in fluid dynamics, electronic circuits, low-energy lasers. We discuss chaos and stability conditions for fractional ordered chaotic dynamical systems

\[
D^\alpha y_i(t) = f_i(y_1(t), y_2(t), \ldots, y_n(t)), \quad 1 \leq i \leq n
\]

in Chapter 9.

Study of dynamical systems of fractional order is receiving increasing attention in the recent years. Financial systems in economics displaying fractional order dynamics are known [24]. Furthermore Lorenz, Chen, Lü, Rossler [20, 25, 27, 28] systems of fractional order have been studied widely in the literature. One of the important problems that
concerns here is to find the minimum effective dimension (MED) in a fractional order dynamical system for which the system remains chaotic. In Chapter 10, we propose a fractional version of Liu system [15]

\[
\begin{align*}
D^{\alpha_1} x &= -ax - ey^2 \\
D^{\alpha_2} y &= by - kxz \\
D^{\alpha_3} z &= -cz + mxy
\end{align*}
\]

where \( a = e = 1, b = 2.5, k = m = 4, c = 5, \alpha_i \in (0, 1) \) and initial conditions \((0.2, 0, 0.5)\). If \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha \) then the system is called commensurate otherwise incommensurate. We have studied commensurate and incommensurate ordered systems. In case of commensurate orders the MED turns out to be 2.76, whereas in incommensurate case the MED is 2.60.

It is interesting to observe effect of delay parameter on chaotic behavior of fractional order chaotic dynamical systems. It was first time observed by us that the chaotic system becomes periodic with some suitable values of delay. In particular we have studied Liu system [7]

\[
\begin{align*}
D^{\alpha_1} x(t) &= a(y(t) - x(t - \tau)) , \\
D^{\alpha_2} y(t) &= bx(t - \tau) - kx(t) z(t) , \\
D^{\alpha_3} z(t) &= -cz(t - \tau) + hx^2(t) , \\
x(t) &= 2.2, y(t) = 2.4, z(t) = 38, t \in [-\tau, 0],
\end{align*}
\]

of fractional order. All this discussion is covered in Chapter 11.

Synchronization in chaotic dynamical systems is very useful and interesting phenomenon. It is used in secure communication [21]. The trajectories of oscillatory system with distinct initial positions evolve together after sufficient time when synchronized. We have proposed the synchronization of fractional order chaotic systems using active control [8] in Chapter 12.
Bibliography


