Chapter 12

Synchronization of Nonidentical Fractional Order Chaotic Systems

This chapter is based on the following paper:
12.1 Introduction

Synchronization of chaotic systems [5, 18, 29, 30] has been focus of attention in recent literature owing to its applications in secure communications of analog and digital signals [14] and for developing safe and reliable cryptographic systems [13]. A variety of approaches which include nonlinear feed-back control [16], adaptive control [22, 28, 34, 39], back-stepping design [4, 42] and active control/feedback linearization [1, 2, 40] have been proposed for synchronization of chaotic systems.

Study of chaos in fractional order dynamical systems and related phenomena is receiving growing attention [17, 26]. Grigorenko and Grigorenko investigated fractional ordered Lorenz system and observed that below a threshold order the chaos disappears [12]. Further many systems such as Chen [19], Lü [24], Rossler [20], Liu [6] were investigated in this regard. Effect of delay on chaotic solutions in fractional order dynamical system is investigated by the present authors [3]. It is demonstrated that the chaotic systems can be transformed into limit cycles or stable orbits with appropriate choice of delay parameter.

Synchronization of fractional order chaotic systems was first studied by Deng and Li [7] who carried out synchronization in case of the fractional Lü system. Further they have investigated synchronization of fractional Chen system [8]. The theory for synchronization problems in an $\omega$-symmetrically coupled fractional differential systems have been studied by Zhou and Li [43]. Since then many fractional systems have been investigated by various researchers. A few examples in this regards are Li et. al [21] (Chua system), Wang et. al [35] (Chen system), Wang and Zhang [36] (unified system), Xingyuan and Yijie [38] (unified system), Yu and Li [41] (Rossler hyperchaos system), Tavazoei and Haeri [31] (Lü system and Chen system). Of late Matouk [27] has synchronized fractional Lü system with fractional Chen system and fractional Chen system with fractional Lorenz system. Hu et. al [15] have synchronized fractional Lorenz and fractional Chen systems.

In the present chapter we synchronize the following fractional systems using active control method: (i) Lorenz with Liu, (ii) Liu with Chen, (iii) Lü with Liu.

We have employed analytical results presented by Matignon [26] in our analysis rather than the Laplace transform method used by most of the researchers. Secondly, examples of the synchronization of two different systems are few in the literature. Further we study
the inter-relation between fractional order and synchronization.

### 12.2 System description

The fractional order Lorenz system [12, 37] is described by

\[
\begin{align*}
D^\alpha x &= \sigma (y - x), \\
D^\alpha y &= rx - y - xz, \\
D^\alpha z &= xy - \mu z,
\end{align*}
\]

where \( \sigma = 10 \) is the Prandtl number, \( r = 28 \) is the Rayleigh number over the critical Rayleigh number and \( \mu = 8/3 \) gives the size of the region approximated by the system. The minimum effective dimension for this system is 2.97 [37].

A fractional version of the chaotic system by Liu et al. [23] is studied in [6]. The system is described by

\[
\begin{align*}
D^\alpha x &= -ax - ey^2, \\
D^\alpha y &= by - kxz, \\
D^\alpha z &= -cz + mxy,
\end{align*}
\]

where \( a = 1, e = 1, b = 2.5, k = 4, c = 5, m = 4 \). The lowest value of \( \alpha \) for which the system exhibits chaos is given by 0.92 [6].

Li and Peng [19] studied chaos in fractional order Chen system

\[
\begin{align*}
D^\alpha x &= a_1 (y - x), \\
D^\alpha y &= (c_1 - a_1) x - xz + c_1 y, \\
D^\alpha z &= xy - b_1 z,
\end{align*}
\]

where \( a_1 = 35, b_1 = 3, c_1 = 27 \). The minimum effective dimension reported is 2.92 [19].

Fractional order Lü system is the lowest-order chaotic system among all the chaotic systems reported in the literature [24]. The minimum effective dimension reported is 0.30.
The system is given by
\[
\begin{align*}
D^\alpha x &= a_2 (y - x), \\
D^\alpha y &= c_2 y - xz, \\
D^\alpha z &= xy - b_2 z,
\end{align*}
\]
where \(a_2 = 35, b_2 = 3, c_2 = 28\).

### 12.3 Chaos synchronization between fractional order Lorenz and Liu system

In this section we study the synchronization between Lorenz and Liu systems. Assuming that the Lorenz system drives the Liu system, we define the drive (master) and response (slave) systems as follows
\[
\begin{align*}
D^\alpha x_1 &= \sigma (y_1 - x_1), \\
D^\alpha y_1 &= r x_1 - y_1 - x_1 z_1, \\
D^\alpha z_1 &= x_1 y_1 - \mu z_1, \\
\end{align*}
\]
and
\[
\begin{align*}
D^\alpha x_2 &= -ax_2 - ey_2^2 + u_1(t), \\
D^\alpha y_2 &= by_2 - kx_2 z_2 + u_2(t), \\
D^\alpha z_2 &= -cz_2 + mx_2 y_2 + u_3(t).
\end{align*}
\]
The unknown terms \(u_1, u_2, u_3\) in (12.6) are active control functions to be determined. Define the error functions as
\[
e_1 = x_2 - x_1, \quad e_2 = y_2 - y_1, \quad e_3 = z_2 - z_1.
\]
Equation (12.7) together with (12.5) and (12.6) yields the error system
\[
\begin{align*}
D^\alpha e_1 &= -ae_1 - ax_1 - ee_2^2 - 2ee_2 y_1 - ey_1^2 - \sigma (y_1 - x_1) + u_1(t), \\
D^\alpha e_2 &= be_2 + by_1 - ke_1 (e_3 - z_1) - kx_1 (z_1 + e_3) + y_1 + x_1 (z_1 - r) + u_2(t), \\
D^\alpha e_3 &= -ce_3 - cz_1 + me_1 (e_2 + y_1) + mx_1 (y_1 + e_2) + \mu z_1 - x_1 y_1 + u_3(t).
\end{align*}
\]
We define active control functions \( u_i(t) \) as
\[
\begin{align*}
  u_1(t) &= V_1(t) + ax_1 + ee_2^2 + 2ee_2y_1 + ey_1^2 + \sigma (y_1 - x_1), \\
  u_2(t) &= V_2(t) - by_1 + ke_1 (e_3 - z_1) + kx_1 (z_1 + e_3) - y_1 - x_1 (z_1 - r), \\
  u_3(t) &= V_3(t) + cz_1 - me_1 (e_2 + y_1) - mx_1 (y_1 + e_2) - \mu z_1 + x_1 y_1. 
\end{align*}
\] (12.9)

The terms \( V_i(t) \) are linear functions of the error terms \( e_i(t) \). With the choice of \( u_i(t) \) given by (12.9) the error system (12.9) becomes
\[
\begin{align*}
  D^a e_1 &= -ae_1 + V_1(t), \\
  D^a e_2 &= be_2 + V_2(t), \\
  D^a e_3 &= -ce_3 + V_3(t). 
\end{align*}
\] (12.10)

The control terms \( V_i(t) \) are chosen so that the system (12.10) becomes stable. There is not a unique choice for such functions. We choose
\[
\begin{pmatrix}
  V_1 \\
  V_2 \\
  V_3 
\end{pmatrix} = A \begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 
\end{pmatrix} 
\] (12.11)

where \( A \) is a 3 \times 3 real matrix, chosen so that for all eigenvalues \( \lambda_i \) of the system (12.10) the condition
\[
|\text{arg}(\lambda_i)| > \alpha \pi / 2. 
\] (12.12)

is satisfied. (The stability condition (12.12) is discussed in the literature [26, 32, 33]). If we choose
\[
A = \begin{pmatrix}
  a - 1 & 0 & 0 \\
  0 & -1 - b & 0 \\
  0 & 0 & c - 1 
\end{pmatrix} 
\] (12.13)
then the eigenvalues of the linear system (12.10) are \(-1, -1\) and \(-1\). Hence the condition (12.12) is satisfied for \( \alpha < 2 \). Since we consider only the values \( \alpha \leq 1 \) we get the required synchronization.
12.3.1 Simulation and results

Parameters of the Lorenz system are taken as $\sigma = 10$, $r = 28$, $\mu = 8/3$ and Liu system as $a = 1$, $e = 1$, $b = 2.5$, $k = 4$, $c = 5$, $m = 4$. The fractional order $\alpha$ is taken to be 0.99 for which both the systems are chaotic. The initial conditions for drive and response system are $x_1(0) = 10$, $y_1(0) = 5$, $z_1(0) = 10$ and $x_2(0) = 0.2$, $y_2(0) = 0$, $z_2(0) = 0.5$ respectively. Initial conditions for the error system are thus $e_1(0) = -9.8$, $e_2(0) = -5$, $e_3(0) = -9.5$.

Figs. 12.1(a), 12.1(b) and 12.1(c) show the synchronization between Lorenz and Liu system, the response system is given by dashed line. The errors $e_i(t)$ for the drive and response system are shown in Fig. 12.1(d).
12.4 Chaos synchronization between Liu and Chen systems of fractional order

Assuming that Chen system is synchronized with Liu system, define the drive system as:

\[
\begin{align*}
D^\alpha x_1 &= -ax_1 - ey_1^2, \\
D^\alpha y_1 &= by_1 - kx_1z_1, \\
D^\alpha z_1 &= -cz_1 + mx_1y_1
\end{align*}
\] (12.14)

and the response system as

\[
\begin{align*}
D^\alpha x_2 &= a_1(y_2 - x_2) + u_4, \\
D^\alpha y_2 &= (c_1 - a_1)x_2 - x_2z_2 + c_1y_2 + u_5, \\
D^\alpha z_2 &= x_2y_2 - b_1z_2 + u_6.
\end{align*}
\] (12.15)

Let \( e_1 = x_2 - x_1, e_2 = y_2 - y_1, e_3 = z_2 - z_1 \) be error functions. For synchronization it is essential that the errors \( e_i \to 0 \) as \( t \to \infty \). Note that

\[
\begin{align*}
D^\alpha e_1 &= a_1(e_2 - e_1) + (a - a_1)x_1 + a_1y_1 + ey_1^2 + u_4(t), \\
D^\alpha e_2 &= (c_1 - a_1)e_1 + c_1e_2 - z_1e_1 - x_1e_3 - e_1e_3 \\
&\quad + (c_1 - a_1)x_1 + (c_1 - b)y_1 + (k - 1)x_1z_1 + u_5(t), \\
D^\alpha e_3 &= -b_1e_3 + y_1e_1 + x_1e_2 + e_1e_2 + (c - b)z_1 + (1 - m)x_1y_1 + u_6(t). \quad (12.16)
\end{align*}
\]

The control functions are chosen as

\[
\begin{align*}
u_4 &= V_4 - (a - a_1)x_1 - a_1y_1 - ey_1^2, \\
u_5 &= V_5 + z_1e_1 + x_1e_3 + e_1e_3 - (c_1 - a_1)x_1 - (c_1 - b)y_1 - (k - 1)x_1z_1, \\
u_6 &= V_6 - y_1e_1 - x_1e_2 - e_1e_2 - (c - b)z_1 - (1 - m)x_1y_1. \quad (12.17)
\end{align*}
\]

The linear functions \( V_4, V_5, V_6 \) are given by

\[
\begin{align*}
V_4 &= (a_1 - 1)e_1 - a_1e_2, \\
V_5 &= -(1 + c_1)e_2, \\
V_6 &= (b_1 - 1)e_3. \quad (12.18)
\end{align*}
\]
With the values given in (12.17) and (12.18), the error system (12.16) becomes

\[
\begin{pmatrix}
D^\alpha e_1 \\
D^\alpha e_2 \\
D^\alpha e_3
\end{pmatrix} =
\begin{pmatrix}
-1 & 0 & 0 \\
0 & c_1 - a_1 & -1 \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}.
\]

(12.19)

It can be observe that the coefficient matrix of the error system (12.19) has eigenvalues $-1, -1, -1$. So the system is stable and synchronization is achieved.

### 12.4.1 Simulations and results

We take parameters for fractional order Chen system as $a_1 = 35$, $b_1 = 3$, $c_1 = 27$. Parameters for the Liu system are same as given in Section 12.3.1. Experiments are done for fixed value of fractional order $\alpha = 0.95$, which is same for drive and response system (12.14) and (12.15). The initial conditions for the systems (12.14) and (12.15) are $x_1(0) = 0.2$, $y_1(0) = 0$, $z_1(0) = 0.5$ and $x_2(0) = 10$, $y_2(0) = 25$, $z_2(0) = 36$ respectively. For the error system (12.19) the initial conditions turns out to be $e_1(0) = 9.8$, $e_2(0) = 25$, $e_3(0) = 35.5$. The simulation results are summerised in Figs. 12.2. Synchronization between fractional Liu and Chen system is shown in Figs. 12.2(a) (signals $x_1$, $x_2$), 12.2(b) (signals $y_1$, $y_2$) and 12.2(c) (signals $z_1$, $z_2$). Note that the drive systems are shown by solid lines whereas response systems are shown by dashed lines. The errors $e_1(t)$ (solid line), $e_2(t)$ (dashed line) and $e_3(t)$ (dot-dashed line) in the synchronization are shown in Fig. 12.2(d).

![Fig. 12.2(a): Signals $x_1$, $x_2$](image1)

![Fig. 12.2(b): Signals $y_1$, $y_2$](image2)
12.5 Chaos synchronization between fractional Lü and Liu system

In this case consider Lü system as the drive system

\[
\begin{align*}
D^\alpha & x_1 = a_2 (y_1 - x_1), \\
D^\alpha & y_1 = c_2 y_1 - x_1 z_1, \\
D^\alpha & z_1 = x_1 y_1 - b_2 z_1, \\
\end{align*}
\]

(12.20)

and the response system as the Liu system

\[
\begin{align*}
D^\alpha & x_2 = -a x_2 - e y_2^2 + u_7, \\
D^\alpha & y_2 = b y_2 - k x_2 z_2 + u_8, \\
D^\alpha & z_2 = -c z_2 + m x_2 y_2 + u_9 \\
\end{align*}
\]

(12.21)

Let \( e_1 = x_2 - x_1, e_2 = y_2 - y_1, e_3 = z_2 - z_1 \) be error functions. For synchronization it is essential that the errors \( e_i \to 0 \) as \( t \to \infty \). To achieve this one should choose the control terms \( u_7, u_8, u_9 \) properly. The error system thus becomes

\[
\begin{align*}
D^\alpha & e_1 = -a e_1 - a x_1 - e e_2 (e_2 + 2 y_1) - y_1 (e y_1 + a_2) + a_2 x_1 + u_7(t), \\
D^\alpha & e_2 = b e_2 + b y_1 - k e_1 e_3 + (1 - k) x_1 z_1 - k (e_1 z_1 + e_3 x_1) - c_2 y_1 + u_8(t), \\
D^\alpha & e_3 = -c e_3 - c z_1 + m e_1 (e_2 + y_1) + (m - 1) x_1 y_1 + m x_1 e_2 + b_2 z_1 + u_9(t). \\
\end{align*}
\]

(12.22)
The control functions are chosen as

\[
\begin{align*}
    u_7 &= V_7 + ax_1 + ee_2 (e_2 + 2y_1) + y_1 (ey_1 + a_2) - a_2x_1, \\
    u_8 &= V_8 - by_1 + ke_1e_3 - (1 - k) x_1z_1 + k (e_1z_1 + e_3x_1) + c_2y_1, \\
    u_9 &= V_9 + cz_1 - me_1 (e_2 + y_1) - (m - 1) x_1y_1 - m x_1 e_2 - b_2z_1. 
\end{align*}
\] (12.23)

The linear functions \( V_7, V_8, V_9 \) are given by

\[
\begin{align*}
    V_7 &= (a - 1) e_1, \\
    V_8 &= - (1 + b) e_2, \\
    V_9 &= (c - 1) e_3. 
\end{align*}
\] (12.24)

With the values given in (12.23) and (12.24), the error system (12.22) becomes

\[
\begin{pmatrix}
    D^\alpha e_1 \\
    D^\alpha e_2 \\
    D^\alpha e_3
\end{pmatrix} =
\begin{pmatrix}
    -1 & 0 & 0 \\
    0 & -1 & 0 \\
    0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
    e_1 \\
    e_2 \\
    e_3
\end{pmatrix}.
\] (12.25)

It can be observe that the coefficient matrix of the error system (12.25) has eigenvalues \(-1, -1, -1\). So the system is stable and synchronization is achieved.

### 12.5.1 Simulations and results

Parameters for the Lü system are \( a_2 = 35, b_2 = 3, c_2 = 28 \), whereas parameters for Liu system are unaltered. The initial conditions for drive system are \( x_1(0) = 0.2, y_1(0) = 0, z_1(0) = 0.5 \) whereas the initial conditions for response system are \( x_2(0) = 10, y_2(0) = 25, z_2(0) = 36 \). Hence the initial conditions for the error system (12.25) are \( e_1(0) = 9.8, e_2(0) = 25, e_3(0) = 35.5 \). We perform the numerical simulations for the two cases of fractiona order \( \alpha \) viz. 0.95 and 0.84 of the drive system (12.20) and response system (12.21). Figs. 12.3(a, b, c) and Figs. 12.3(e, f, g) shows synchronization between fractional Lü and Liu system for \( \alpha = 0.95 \) and \( \alpha = 0.84 \) respectively. Fig. 12.3(d) and 12.3(h) shows the errors \( e_1(t) \) (solid line), \( e_2(t) \) (dashed line) and \( e_3(t) \) (dot-dashed line) in the synchronization for \( \alpha = 0.95 \) and \( \alpha = 0.84 \) respectively.
Fig. 12.3(a): $\alpha = 0.95$, Signals $x_1$, $x_2$

Fig. 12.3(b): $\alpha = 0.95$, Signals $y_1$, $y_2$

Fig. 12.3(c): $\alpha = 0.95$, Signals $z_1$, $z_2$

Fig. 12.3(d): $\alpha = 0.95$, Error system

Fig. 12.3(e): $\alpha = 0.84$, Signals $x_1$, $x_2$

Fig. 12.3(f): $\alpha = 0.84$, Signals $y_1$, $y_2$
12.6 Effect of order $\alpha$ on synchronization

It is well known that the order $\alpha$ of the derivative affects the chaotic behaviour of fractional chaotic dynamical systems. In this section we focus on the effect of the order $\alpha$ on synchronization. As a representative we consider the coupled system (12.20)–(12.21) i.e. synchronization between Lü and Liu system. In the Table 1, we have summarized some observations about the error functions for different values of $\alpha$ at prescribed time values.
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Table 12.1.

It is clear from the Table 12.1 that the error in synchronization decreases as the order $\alpha$ is increased. In other words, for larger value of $\alpha$ the synchronization starts earlier.
Bibliography


