Chapter 10

Chaos in Fractional Ordered Liu System

This chapter is based on the following paper:

10.1 Introduction

Study of chaotic systems is an important aspect of dynamical systems that finds applications in different areas ranging from engineering to ecology. Although more than three decades have passed since the existence of “chaotic solutions” was demonstrated, still we do not have a theory of chaos from which the existence of chaotic solutions can be predicted. Theorems such as Poincare-Bendixon have proven useful in some cases, albeit basically numerical simulations is the only tool on which we have to rely on while studying chaos and related phenomena. Extensive numerical work has been carried out to understand chaos in dynamical systems. In a seminal paper Grigorenko and Grigorenko [3] demonstrated existence of chaotic solutions in fractional ordered Lorenz dynamical system. Several fractional ordered dynamical systems have been investigated and are being investigated.

One of the important question being addressed herewith is: what is the minimum effective dimension in a fractional order dynamical system for which the system remains chaotic? The effective dimension being defined as the sum of orders of all involved derivatives. Minimum effective dimension has been numerically calculated for various systems including fractional order Lorenz system [3], fractional order Chua system [4], fractional order Rossler system [5], fractional order Newton-Leipnik system [10] and so on. It should be mentioned that some results have discrepancies/errors due to computational methods. As a pursuance to this analytical conditions necessary for a system to exhibit chaotic behaviour have been presented in the literature [8, 13].

In the present chapter we study fractional version of Liu system [6] and observe existence of chaos. We study commensurate and incommensurate ordered systems and find lowest order at which chaos exist by numerical experiments. In case of commensurate orders the lowest order turns out to be 2.76, whereas in incommensurate case the lowest order is 2.60. We use the analytical conditions [8, 13] to check stability and the Lyapunov exponents [9] for the existence of chaos.
10.2 Liu system

In [6] Liu et al studied dynamical behavior of the system

\[
\begin{align*}
\dot{x} &= -ax - ey^2 \\
\dot{y} &= by - kxz \\
\dot{z} &= -cz + mxy
\end{align*}
\]  

(10.1)

where \(a = e = 1, b = 2.5, k = m = 4, c = 5\) and initial conditions \((0.2, 0, 0.5)\) yield chaotic trajectory. In this chapter we study corresponding fractional order system

\[
\begin{align*}
D^{\alpha_1}x &= -ax - ey^2 \\
D^{\alpha_2}y &= by - kxz \\
D^{\alpha_3}z &= -cz + mxy
\end{align*}
\]  

(10.2)

where \(\alpha_i \in (0, 1)\). If \(\alpha_1 = \alpha_2 = \alpha_3 = \alpha\) then the system (10.2) is called commensurate otherwise incommensurate.

The system (10.1) has five equilibrium points out of which two are complex \(v\acute{\iota}\). \((0.8839, \pm 0.9402i, \pm 0.6648i)\). The real equilibrium points and the eigenvalues of corresponding Jacobian matrix

\[
J(x, y, z) = \begin{pmatrix}
-a & -2ey & 0 \\
kz & b & kx \\
-my & -mx & -c
\end{pmatrix}
\]  

(10.3)

are given in Table 10.1. An equilibrium point \(p\) of the system (10.1) is called as saddle point if the Jacobian matrix at \(p\) has at least one eigenvalue with negative real part (stable) and one eigenvalue with non-negative real part (unstable). A saddle point is said to have index one \((/two)\) if there is exactly one \((/two)\) unstable eigenvalue/s. It is established in the literature [1, 2, 7, 11, 12] that, scrolls are generated only around the saddle points of index two. Saddle points of index one are responsible only for connecting scrolls.

It is clear from the Table 10.1 that the equilibrium points \(E_1\) and \(E_2\) are saddle points of index two, hence there exists a two-scroll attractor [12], in the system (10.2).
### Table 10.1: Equilibrium points and corresponding eigenvalues

<table>
<thead>
<tr>
<th>Equilibrium point</th>
<th>Eigenvalues</th>
<th>Nature</th>
</tr>
</thead>
<tbody>
<tr>
<td>O(0, 0, 0)</td>
<td>-5, 2.5, -1</td>
<td>Saddle point</td>
</tr>
<tr>
<td>E₁(-0.8839, 0.9402, -0.6648)</td>
<td>-4.3878, 0.4439 ± 3.3464i</td>
<td>Saddle point</td>
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<td>Saddle point</td>
</tr>
</tbody>
</table>

#### 10.2.1 Commensurate ordered system

Consider the system (10.2) with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ (commensurate order). In this case a system shows regular behavior if it satisfies the condition (9.12), Chapter 9,

$$\alpha < \frac{2}{\pi} \min_i |\arg(\lambda_i)| \approx 0.91.$$  \hspace{1cm} (10.4)

Also, by observing Fig. 10.1(a) it is clear that the Lyapunov exponent for the commensurate order system (10.2) is positive only if $\alpha > 0.91$. Thus the system does not show chaotic behaviour for $\alpha < 0.91$. This result is supported by numerical experiments. Fig. 10.1(b) shows phase portrait in xy-plane for $\alpha = 0.91$. It is observed using numerical experiments that the system shows chaotic behaviour for $\alpha \geq 0.92$. For $\alpha = 0.92$ phase portraits are shown in Fig. 10.1(c) for xy-plane and in Fig. 10.1(d) for yz-plane. Figs. 10.1(e), (f) and (g) shows solutions $x(t), y(t)$ and $z(t)$ respectively for $\alpha = 0.92$. We have used Adams-Bashforth predictor-corrector method (Chapter 7) for numerical computations, where the step size is taken to be 0.01.
Fig. 10.1(a): Lyapunov exponent

Fig. 10.1(b): Phase portrait $\alpha = 0.91$

Fig. 10.1(c): Phase portrait $\alpha = 0.92$

Fig. 10.1(d): Phase portrait $\alpha = 0.92$
In this subsection we show that the condition (9.18) (Chapter 9) i.e. IMFOS > 0 is not sufficient for existence of chaos. Consider the fractional order system (10.2). Lyapunov exponent is positive for $\alpha_1 \geq 0.79$, $\alpha_2 = \alpha_3 = 1$ (cf. Fig. 10.2), for $\alpha_2 \geq 0.66$, $\alpha_1 = \alpha_3 = 1$ (cf. Fig. 10.3) and for $\alpha_3 \geq 0.56$, $\alpha_1 = \alpha_2 = 1$ (cf. Fig. 10.4). Now consider the following cases.
\* \( \alpha_1 = 39/50, \alpha_2 = \alpha_3 = 1. \) Therefore \( M = LCM(50, 1, 1) = 50. \) Since \( \Delta(\lambda) = \text{diag}(\lambda^{39} \lambda^{50} \lambda^{50}) - J(E_1), \)

\[
det(\Delta(\lambda)) = \lambda^{129} + \lambda^{100} + 2.5\lambda^{89} + 7.5\lambda^{50} - 1.7764 \times 10^{-15}\lambda^{39} + 50. 
\] (10.5)

The IMFOS of the system is

\[
\frac{\pi}{100} - 0.0 = 0.03141 > 0. \quad (10.6)
\]

In this case IMFOS > 0 but chaos is absent in this case (cf. Fig. 10.5). This shows that the condition IMFOS > 0 is not sufficient for existence of chaos.

\* \( \alpha_1 = 4/5, \alpha_2 = \alpha_3 = 1. \) Therefore \( M = LCM(5, 1, 1) = 5. \) Since \( \Delta(\lambda) = \text{diag}(\lambda^{4} \lambda^{5} \lambda^{5}) - J(E_1), \)

\[
det(\Delta(\lambda)) = \lambda^{14} + \lambda^{10} + 2.5\lambda^{9} + 7.5\lambda^{5} - 1.7764 \times 10^{-15}\lambda^{4} + 50. 
\] (10.7)

The IMFOS of the system is

\[
\frac{\pi}{10} - 0.3131 = 0.0012 > 0. \quad (10.8)
\]

It is observed from numerical experiments that the system exhibits chaos in this case (cf. Fig. 10.6). Note that this is the lowest dimension (2.60) for which the system shows chaos.

\* Consider \( \alpha_1 = 1, \alpha_2 = 13/20, \alpha_3 = 1, \) so \( M = LCM(20, 1, 1) = 20. \)

\[
det(\Delta(\lambda)) = \lambda^{53} - 2.5\lambda^{40} + 6\lambda^{33} + 2.5\lambda^{20} + 5\lambda^{13} + 50. 
\] (10.9)

The IMFOS of the system is

\[
\frac{\pi}{40} - 0.0786 = -0.0001 < 0. \quad (10.10)
\]

Eq. (10.10) implies that the system does not exhibit chaos. This is confirmed numerically in Fig. 10.7.

\* Consider \( \alpha_1 = 1, \alpha_2 = 7/10, \alpha_3 = 1, \) so \( M = LCM(10, 1, 1) = 10. \)

\[
det(\Delta(\lambda)) = \lambda^{27} - 2.5\lambda^{20} + 6\lambda^{17} + 2.5\lambda^{10} + 5\lambda^{7} + 50. 
\] (10.11)
The IMFOS of the system is

$$\frac{\pi}{20} - 0.1559 = 0.0012 > 0. \quad (10.12)$$

Numerical results in Fig. 8 demonstrate that the system is chaotic in this case.

- Consider $\alpha_1 = \alpha_2 = 1, \alpha_3 = 11/20$, so $M = \text{LCM}(20, 1, 1) = 20$.

$$\det(\Delta(\lambda)) = \lambda^{51} + 5\lambda^{49} - 1.5\lambda^{31} + 5\lambda^{20} + 2.5\lambda^{11} + 50. \quad (10.13)$$

The IMFOS of the system is

$$\frac{\pi}{40} - 0.0785 = 0.000077 > 0, \quad (10.14)$$

Though IMFOS > 0 the system shows regular behaviour (cf. Fig. 10.9).

- Consider $\alpha_1 = \alpha_2 = 1, \alpha_3 = 3/5$, so $M = \text{LCM}(5, 1, 1) = 5$.

$$\det(\Delta(\lambda)) = \lambda^{13} + 5\lambda^{10} - 1.5\lambda^{8} + 5\lambda^{5} + 2.5\lambda^{3} + 50. \quad (10.15)$$

The IMFOS of the system is

$$\frac{\pi}{10} - 0.3116 = 0.0025 > 0. \quad (10.16)$$

Fig. 10.10 shows that the system is chaotic.

Note that the step size used for the numerical experiments in this section is 0.01.
Fig. 10.5: $\alpha_1 = 39/50, \alpha_2 = \alpha_3 = 1$

Fig. 10.6: $\alpha_1 = 4/5, \alpha_2 = \alpha_3 = 1$

Fig. 10.7: $\alpha_1 = \alpha_3 = 1, \alpha_2 = 13/20$

Fig. 10.8: $\alpha_1 = \alpha_3 = 1, \alpha_2 = 7/10$
Fig. 10.9: $\alpha_1 = \alpha_2 = 1, \alpha_3 = 11/20$

Fig. 10.10: $\alpha_1 = \alpha_2 = 1, \alpha_3 = 3/5$
Bibliography


