CHAPTER 3

HEAT TRANSFER IN A FLUID OVER A LINEARLY STRETCHING SHEET WITH VARIABLE THERMAL CONDUCTIVITY AND INTERNAL HEAT GENERATION

3.1 FLOW ANALYSIS

The study considers the problem of two-dimensional flow of a porous sheet issuing from a thin slit at \( x=0, y=0 \) into an otherwise quiescent fluid and the flow is influenced by suction/blowing. The sheet is then stretched so that the speed at any point \( x \) on the sheet is proportional to its distance from the origin.

The basic boundary layer equations for the steady two-dimensional flow are

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.1.1}
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \tag{3.1.2}
\]

\[
\rho C_p \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + Q(T - T_w) \tag{3.1.3}
\]

where \( u \) and \( v \) are the flow velocities in \( x \) and \( y \) directions respectively. \( \nu \) is the kinematic viscosity, \( \rho \) is the fluid density, \( Q \) is the heat source/sink, according as \( Q > 0 \) or \( Q < 0 \). \( T_\infty \) is the free stream temperature, \( T_w \) is the wall temperature, \( C_p \) is the specific heat capacity and \( k \) is the thermal conductivity of the fluid and is a constant.

The appropriate boundary conditions are

\[
u = bx \quad v = v_w \quad \text{at} \quad y = 0 \quad \text{at} \quad y \to \infty \tag{3.1.4}
\]

where \( b \) is a constant and \( v_w \) is the blowing or suction velocity at the sheet.

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3.2 Momentum Transfer

Equations (3.1.1) and (3.1.2) admit self similar solution of the form

\[ u = bx f_\eta(\eta) \quad \nu = -(bv)^{1/2} f(\eta) \quad \eta = \frac{b}{\nu} y \]  

...(3.2.1)

where \( f \) is a dimensionless stream function and \( \eta \) is the similarity variable. The equation (3.1.1) is automatically satisfied. With the substitution of (3.2.1) equation of momentum (3.1.2) reduces to the similarity form

\[ f_{\eta\eta\eta}(\eta) + f(\eta) \cdot f_{\eta\eta}(\eta) - f^2(\eta) = 0 \]  

...(3.2.2)

with boundary conditions

\[ f(0) = -\frac{\nu_{*}}{\sqrt{bv}} \quad f_{\eta}(0) = 1 \quad f_{\eta}(\infty) = 0 \]  

...(3.2.3)

The exact solution of equations (3.2.2) and (3.2.3) have been shown to be (see Gupta and Gupta 4)

\[ f(\eta) = \gamma - \frac{1}{\gamma} e^{-\gamma} \]  

...(3.2.4)

where \( \gamma \) is the mass transfer parameter which is related to \( v_w \) by

\[ v_w = \sqrt{bv} \left( \frac{1}{\gamma} - \gamma \right) \]  

...(3.2.5)

It should be noted that \( \gamma < 1 \) corresponds to blowing where as \( \gamma > 1 \) corresponds to suction and when \( \gamma = 1 \) the sheet is impermeable.

3.3 Heat Transfer

In the heat transfer two dimensional energy equation (3.1.3), \( k \) which is the thermal conductivity assumed to be a variable keeping in view that thermal conductivity \( k \) varies approximately linearly with temperature which is true in some
of the polymer solutions, thermal conductivity of this problem is considered in the form

\[ k = k_\infty \left(1 + \varepsilon \theta \right), \quad \theta = \frac{T - T_\infty}{T_w - T_\infty}, \quad \varepsilon = \frac{k_w - k_\infty}{k_\infty} \quad \ldots (3.3.1) \]

where \( T_\infty \) is the constant temperature of the fluid far away from the sheet and \( T_w \) is the sheet (wall) temperature. \( \varepsilon \) is a small parameter which depends on the nature of the fluid and \( k_\infty \) is the thermal conductivity of the fluid far away from the sheet.

The thermal boundary conditions depend on the type of heating process through the wall surface under consideration. Here it is considered two general cases of non-isothermal boundary conditions.

(i) **PST Case:** Surface with prescribed power law temperature.

(ii) **PHF Case:** Surface with prescribed power law heat flux.

### 3.3A PST Case

The prescribed power law surface temperature is assumed to be a quadratic function of \( x \) and it is given by

\[ T = T_w = T_\infty + A \left( \frac{x}{\ell} \right)^2 \quad \text{at} \quad y = 0 \quad \text{and} \]

\[ T = T_\infty \quad \text{as} \quad y \to \infty \quad \ldots (3.3.2) \]

where \( T_w \) is the variable wall temperature, \( A \) is a constant and \( \ell \) is the characteristic length (Chen and Char, 1988).

Energy equation (3.1.3) is non-dimensionalised with dimensionless temperature variables given by (3.3.1).
Considering equations (3.2.1), (3.2.4) and (3.3.1), equation (3.1.3) takes the form 

\[(1 + e \theta(\eta)) \theta_{\eta\eta}(\eta) + 2 e \theta^2_{\eta}(\eta) + Pr \left( \frac{1}{\gamma} c^{m} \right) \theta_{\eta}(\eta) - Pr \left( 2 e^{m} - \beta \right) \theta(\eta) = 0 \quad \ldots (3.3.3)\]

where \( Pr = \frac{\mu C_p}{k_n} \) denotes the Prandtl number.

The corresponding boundary conditions are

\[\theta(0) = 1 \quad ; \quad 0(\infty) = 0 \quad \ldots (3.3.4)\]

### 3.3B PHF CASE

The power law heat flux on the wall surface is considered to vary quadratically with the distance and it is given by

\[-k \frac{\partial T}{\partial y} = D \left( \frac{x}{r} \right) \qquad \text{at} \quad y = 0\]

\[T - T_w = T_w \quad \text{as} \quad y \to \infty \quad \ldots (3.3.5)\]

where \( D \) is another constant and \( r \) is the characteristic length.

Defining

\[g(\eta) = \frac{T - T_w}{T_w - T_s}\]

\[T - T_s = \frac{D}{k} \left( \frac{x}{r} \right)^2 \sqrt{\frac{y}{b}} g(\eta) \quad \ldots (3.3.6)\]

where \( k = k_r (1 + eg) \)

Substituting (3.2.1), (3.2.4) and (3.3.6) in the energy equation (3.1.3) leads to a non-linear ordinary differential equation of the form

\[(1 + eg(\eta)) \theta_{w\eta}(\eta) + eg^2(\eta) + Pr \left( \frac{1}{\gamma} c^{m} \right) \theta_{w}(\eta) - Pr \left( 2 e^{m} - \beta \right) \theta(\eta) = 0 \quad \ldots (3.3.7)\]
subjected to the boundary conditions

\[ g_{\eta}(0) = -1 \quad ; \quad g(\infty) = 0 \] \quad ...(3.3.8)

In the next section, equations (3.3.3) and (3.3.7) are being solved.

Perturbation Solution

Perturbation technique is employed to solve the non-linear equation (3.3.3) and (3.3.7), and so assume

\[ \theta = \theta_0 + \varepsilon \theta_1 + \varepsilon^2 \theta_2 + \ldots \] \quad ...(3.3.9)

\[ g = g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \ldots \] \quad ...(3.3.10)

substituting (3.3.9) and (3.3.10) in to equations (3.3.3), (3.3.4), (3.3.7) and (3.3.8) and equating the terms with the like powers of \( \varepsilon \), the following sequence of boundary value problems for \( \theta_0, \theta_1, \theta_2, \theta_3 \) and \( g_0, g_1, g_2, g_3 \)

PST Case

\[ \theta_{0\eta\eta}(\eta) + \text{Pr} \left( \gamma - \frac{1}{\gamma} e^{-\gamma \eta} \right) \theta_{0\eta}(\eta) - \text{Pr} \left( 2e^{-\gamma m} - \beta \right) \theta_0(\eta) = 0 \] \quad ...(3.3.11)

\[ \theta_0(0) = 1 \quad ; \quad \theta_0(\eta_\infty) = 0 \] \quad ...(3.3.12)

\[ \theta_{1\eta\eta}(\eta) + \text{Pr} \left( \gamma - \frac{1}{\gamma} e^{-\gamma \eta} \right) \theta_{1\eta}(\eta) - \text{Pr} \left( 2e^{-\gamma m} - \beta \right) \theta_1(\eta) = \theta_0(\eta) \theta_{0\eta\eta}(\eta) - \theta_0^2(\eta) \] \quad ...(3.3.13)

\[ \theta_1(0) = 0 \quad ; \quad \theta_1(\eta_\infty) = 0 \] \quad ...(3.3.14)

\[ \theta_{2\eta\eta}(\eta) + \text{Pr} \left( \gamma - \frac{1}{\gamma} e^{-\gamma \eta} \right) \theta_{2\eta}(\eta) - \text{Pr} \left( 2e^{-\gamma m} - \beta \right) \theta_2(\eta) = \theta_0(\eta) \theta_{1\eta\eta}(\eta) - \theta_1(\eta) \theta_{0\eta\eta}(\eta) - 2 \theta_{0\eta}(\eta) \theta_{1\eta}(\eta) \] \quad ...(3.3.15)

\[ \theta_2(0) = 0 \quad ; \quad \theta_2(\infty) = 0 \]
\[ \theta_{3\eta}(\eta) + \Pr\left(\gamma - \frac{1}{\gamma} e^{-\eta}\right) \theta_{3\eta}(\eta) - \Pr(2e^{-\eta} - \beta) \theta_1(\eta) = 0 \]

\[ = -\theta_0(\eta) \theta_{2\eta}(\eta) - 2\theta_{0\eta}(\eta) \theta_{2\eta}(\eta) - \theta_{2\eta}(\eta) - \theta_{1\eta}(\eta) - \theta_{3\eta}(\eta) - \theta_1(\eta) \theta_{1\eta}(\eta) \]

\[ \theta_3(0) = 0 ; \quad \theta_3(\infty) = 0 \]  

\textbf{PHF Case}

\[ g_{0\eta}(\eta) + \Pr\left(\gamma - \frac{1}{\gamma} e^{-\eta}\right) g_{0\eta}(\eta) - \Pr(2e^{-\eta} - \beta) g_0(\eta) = 0 \]

\[ g_{0\eta}(0) = -1 ; \quad g_0(\eta) = 0 \]

\[ g_{1\eta}(\eta) + \Pr\left(\gamma - \frac{1}{\gamma} e^{-\eta}\right) g_{1\eta}(\eta) - \Pr(2e^{-\eta} - \beta) g_1(\eta) = -g_0(\eta) g_{0\eta}(\eta) - g_{1\eta}(\eta) \]

\[ g_{1\eta}(0) = 0 ; \quad g_1(\eta) = 0 \]

\[ g_{2\eta}(\eta) + \Pr\left(\gamma - \frac{1}{\gamma} e^{-\eta}\right) g_{2\eta}(\eta) - \Pr(2e^{-\eta} - \beta) g_2(\eta) = -g_0(\eta) g_{1\eta}(\eta) - g_1(\eta) g_{0\eta}(\eta) - 2g_0(\eta) g_{1\eta}(\eta) \]

\[ g_{2\eta}(0) = 0 ; \quad g_2(\infty) = 0 \]

\[ g_{3\eta}(\eta) + \Pr\left(\gamma - \frac{1}{\gamma} e^{-\eta}\right) g_{3\eta}(\eta) - \Pr(2e^{-\eta} - \beta) g_3(\eta) = -g_0(\eta) g_{2\eta}(\eta) - g_0(\eta) g_{2\eta}(\eta) - g_2(\eta) - g_0(\eta) g_{2\eta}(\eta) - g_2(\eta) - g_1(\eta) g_{1\eta}(\eta) \]

\[ g_{3\eta}(0) = 0 ; \quad g_3(\infty) = 0 \]

Higher order equations can be similarly obtained.

The zeroth order equation has been considered by Gupta and Gupta [4] and Chaim [58]. The former expressed the solution of equation (3.3.11) as

\[ \theta_0(\eta) = \frac{\int_{0}^{u_{\eta}^{1-n}} e^{-u} \, du}{\int_{0}^{u_{\eta}^{1-n}} u^{n-1} e^{-u} \, du} \]

\[ \ldots (3.3.23) \]
where as in the latter work it was obtained in the form

\[ 0_{\xi}(\eta) = \frac{\int_{\xi}^{\eta} \exp \left[ -\frac{Pr}{\gamma} \left( \frac{\gamma^{2} \xi + 1}{\gamma} e^{-\xi} \right) \right] d\xi}{\int_{0}^{\eta} \exp \left[ -\frac{Pr}{\gamma} \left( \frac{\gamma^{2} \xi + 1}{\gamma} e^{-\xi} \right) \right] d\xi} \]  

...(3.3.24)

The solution can be easily computed by using standard numerical integration algorithms such as Simpson’s rule and this has been done by Chaim [58].

3.4 SOLUTION METHODS

To solve equations (3.3.11) and (3.3.17) defining a new variable

\[ \xi = -\frac{Pr}{\gamma^{2}} e^{-\eta} \]  

...(3.4.1)

with this substitution equations (3.3.11) and (3.3.17) transform to

\[ \xi \theta_{\xi} \theta_{\xi} + (1 - \Pr - \xi) \theta_{\xi} \theta_{\xi} + \left( 2 + \frac{Pr \beta}{\gamma^{2}} \xi^{-1} \right) \theta_{\xi} = 0 \]  

...(3.4.2)

\[ \theta_{\xi} \left( -\frac{Pr}{\gamma^{2}} \right) = 1 \quad ; \quad \theta_{\xi}(\infty) = 0 \]  

...(3.4.3)

\[ \xi \xi_{\xi} \xi_{\xi} + (1 - \Pr - \xi) \xi_{\xi} \xi_{\xi} + \left( 2 + \frac{Pr \beta}{\gamma^{2}} \xi^{-1} \right) \xi_{\xi} = 0 \]  

...(3.4.4)

\[ \xi \xi_{\xi} \left( -\frac{Pr}{\gamma^{2}} \right) = -1 \quad ; \quad \xi_{\xi}(\eta) = 0 \]  

...(3.4.5)

The solutions of equations (3.4.2) and (3.4.4) subjected to the boundary conditions (3.4.3) and (3.4.5) in terms of \( \xi \) are obtained respectively as

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\[ \theta_0(\xi) = \xi^{\frac{a_0 + b_0}{2} - \frac{1}{2}} \frac{M\left[\frac{a_0 + b_0}{2} - 2, 1 + b_0, -\xi\right]}{M\left[\frac{a_0 + b_0}{2} - 2, 1 + b_0, -\frac{Pr}{\gamma^2}\right]} \quad \text{(3.4.6)} \]

\[ g_0(\xi) = C_1 \xi^{\frac{a_0 + b_0}{2} - \frac{1}{2}} \frac{M\left[\frac{a_0 + b_0}{2} - 2, 1 + b_0, -\frac{Pr}{\gamma^2}e^{-\eta}\right]}{M\left[\frac{a_0 + b_0}{2} - 2, 1 + b_0, -\frac{Pr}{\gamma^2}\right]} \quad \text{(3.4.7)} \]

Solutions (3.4.6) and (3.4.7) can be recast in terms of \( \eta \) as

\[ \theta_0(\eta) = e^{-\frac{(a_0 + b_0)}{2}} \frac{M\left[\frac{a_0 + b_0}{2} - 2, 1 + b_0, -\frac{Pr}{\gamma^2}e^{-\eta}\right]}{M\left[\frac{a_0 + b_0}{2} - 2, 1 + b_0, -\frac{Pr}{\gamma^2}\right]} \quad \text{(3.4.8)} \]

and

\[ g_0(\eta) = C_1 e^{-\frac{(a_0 + b_0)}{2}} \frac{M\left[\frac{a_0 + b_0}{2} - 2, 1 + b_0, -\frac{Pr}{\gamma^2}e^{-\eta}\right]}{M\left[\frac{a_0 + b_0}{2} - 2, 1 + b_0, -\frac{Pr}{\gamma^2}\right]} \quad \text{(3.4.9)} \]

where

\[ C_1 = \left\{ e^{-\frac{(a_0 + b_0)}{2}} M\left[\frac{a_0 + b_0}{2} - 2, 1 + b_0, -\frac{Pr}{\gamma^2}\right] \right\} \frac{Pr\left(\frac{a_0 + b_0 - 4}{2(1 + b_0)}\right)}{\eta^4} \quad \text{(3.4.9a)} \]

where \( a_0 = Pr \), \( b_0 = \left( a_0^2 - 4\frac{Pr}{\gamma^2}\right)^{1/2} \)

dimensionless temperature gradient and dimensionless wall temperature in PST and PHF cases can be derived as

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\[
\theta_{\alpha_1}(0) = C_2 \left[ \frac{\Pr \left( \frac{a_0 + b_0 - 4}{2(l + b_0)} \right)}{\gamma} \right] \left[ \frac{a_0 + b_0}{2} - 1, 2 + b_0, -\frac{\Pr}{\gamma^2} \right]

- \gamma \left( \frac{a_0 + b_0}{2} \right) M \left[ \frac{a_0 + b_0}{2} - 2, 1 + b_0, -\frac{\Pr}{\gamma^2} \right] \right]
\]

...(3.4.10)

where

\[
C_2 = \frac{1}{M \left[ \frac{a_0 + b_0}{2} - 2, 1 + b_0, -\frac{\Pr}{\gamma^2} \right]}
\]

and

\[
g_0(0) = C_1 M \left[ \frac{a_0 + b_0}{2} - 2, 1 + b_0, -\frac{\Pr}{\gamma^2} \right]
\]

...(3.4.11)

and \(C_1\) is as given in (3.4.9a) and \(M(a,b,z)\) is the Kummer's function [128] and is defined as

\[
M(a,b,z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n z^n}{(b)_n n!}
\]

...(3.4.12)

and

\[
(a)_n = a(a+1)(a+2) \quad (a+n-1)
\]

\[
(b)_n = b(b+1)(b+2) \quad (b+n-1).
\]

Heat transfer analysis would be carried out by analysing the terms of local heat flux in both PST and PHF cases as

\[
q_w = -k \left( \frac{\partial T}{\partial y} \right)_w = k \sqrt{\frac{b}{\nu}} (T_w - T_v) \left[ -\theta_{\alpha_1}(0) \right]
\]

...(3.4.13)

\[
T_w = T_v + \frac{Dx^2}{k} \sqrt{\frac{b}{\nu}} g_0(0)
\]

...(3.4.14)
3.5 RESULTS AND DISCUSSION

In Fig.(3.1a) a graph of $\theta_0(\eta)$ Vs $\eta$ for a small value of Prandtl number $\text{Pr}=0.023$ (PST Case) is plotted and it is observed from the figure that the temperature function $\theta_0(\eta)$ is a very slowly varying function. Physically it means that small Prandtl number leads to thick thermal boundary layers and the effects of stronger blowing should be felt at larger distance from the belt.

Fig.(3.1b) is a graph of $g_0(\eta)$ Vs $\eta$ for $\text{Pr}=0.023$ (PHF Case). It is noticed from the figure that temperature function is very slowly varying function. It is also observed that as the small parameter $\varepsilon$ goes on increasing, temperature is also go on increasing.

Figs. (3.2a), (3.2b) and (3.2c) represent the graphs of $\theta_{0q}(0)$ Vs $\varepsilon$ for different values of Prandtl number i.e., (i) $\text{Pr}=0.024$  (ii) $\text{Pr}=0.1$ (iii) $\text{Pr}=1.0$. It can observed from the figures that the magnitude of wall temperature gradient increases with increasing values of the mass transfer parameter $\gamma$. This is also true for increase in the values of Prandtl number. However, its variation with respect to the small parameter $\varepsilon$ is more complicated with the exception of one case ($\gamma=0.5$, Pr=1.0). The general trend that can be observed from the figures (3.2a), (3.2b) and (3.2c) is that $-\theta_{0q}(0)$ decreases as $\varepsilon$ increases. The decrease is sharper for higher values of the mass transfer parameter $\gamma$.

Various temperature profiles for $g_0(0)$ Vs $\varepsilon$ for $\text{Pr}=0.023$ and $\text{Pr}=1.0$ in PHF Case are drawn in figures (3.3a) and (3.3b). It is noticed from the figures that $g_0(0)$
decreases as \( \varepsilon \) increases. The decrease is sharper for higher values of \( \gamma \). But when \( \gamma = 0.5 \) and \( Pr = 1.0 \), the trend is reversed. It reveals the fact that, infact when \( \gamma \) is less than about 0.6, \(-g_0(0)\) actually increases to a maximum and then decreases for higher values of \( \varepsilon \).

3.6 CONCLUSIONS

The important findings of present study are as follows:

1) Temperature functions are the slowly varying functions for \( Pr = 0.023 \) (mercury) in both PST and PHF Cases.

2) Dimensionless temperature gradient increases with increase in the values of suction parameter and also it increases with increasing values of Prandtl number. However, temperature gradient is decreasing for increasing values of small parameter \( \varepsilon \) and this drop in temperature is sharper for higher values of \( \gamma \) in both the PST and PHF Cases.
Fig (3.1a) Plots of $\Theta_\eta$ versus $\eta$ for $Pr = 0.023$ in PST case
Fig (3.2a) Dimensionless temperature gradient $-\theta_n'(0)$ versus $\epsilon$ for $Pr = 0.024$ in PST case.
Fig (3.2b) Plots of wall temperature gradient \(-\theta_{w}(0)\) vs \(\varepsilon\) for \(Pr = 0.1\) in PST case
Fig 3.2c: Plots of wall temperature gradient $-\theta_0(0)$ Versus $\varepsilon$ for $Pr = 1.0$ in PST case
Fig. (3.3a) Plots of dimensionless wall temperature $g_0(0)$ Versus $\varepsilon$ in PHF case for $Pr = 1.0$
Fig (3.3b) Plots of dimensionless wall temperature $g_0(0)$ versus $\varepsilon$ for various values of Prandtl number for $\gamma=0.5$ in PHF case.