CHAPTER 4

QUASI-STATIC MODELING FOR MR DAMPERS

This chapter presents two quasi-static models namely axisymmetric & parallel-plate model, based on the Navier-Stokes equation are established for MR damper behaviour. The Herschel-Bulkley visco-plasticity model is used to define the MR fluid field-dependent features and shear thinning/thickening effects. Simple equations created on these damper models are given which can be used in the initial design phase. Effects of geometry on Controllable force, dynamic range and MR damper performance are also discussed.

4.1 Introduction

Magneto rheological fluids (or simply “MR” fluids) be a member of a class of controllable fluids which react to an applied magnetic field with a dramatic change in their rheological behaviour. Generally, this change is expressed by a very big change of the dampeing force in which MR fluid is used. A small-scale MR fluid damper has been designed and constructed to show the scalability of MR fluid technology to devices of applicable size for engineering uses. A schematic diagram of the small-scale MR fluid damper is shown in Fig. 4.1. The damper uses a particularly simple geometry in which the outer cylindrical housing is part of the magnetic circuit. The operative fluid orifice is the whole annular space between the piston outside diameter and the inside of the damper cylinder housing. A detail explanation of this damper and a summary of nominal design parameters are specified in chapter 5.

![Linear MR damper with its basic assemblies](image)

FIGURE 4.1 Schematic of small scale MR fluid damper
Introduction

Fig. 4.1 shows the fluids flow in the annular gap between the piston and the cylinder housing during motion of the MR damper piston.

Following assumptions are made for quasi-static analysis of MR fluid dampers:

1) MR fluid flow is fully developed
2) MR dampers move at a constant velocity; and
3) The Herschel-Bulkley visco-plasticity model is applied to describe MR fluid field-dependent characteristics and shear thinning/thickening properties.

The whole shear stress in the Herchel-Bulkley model is specified by:

\[
\tau = (\tau_0(H) + K\dot{\gamma}^{\frac{1}{m}}) \text{sgn} (\dot{\gamma}) \tag{4.1}
\]

Where

\( \tau_0 \) = yield stress due to the applied magnetic field

\( H \) = magnitude of the applied magnetic field

\( \dot{\gamma} \) = shear strain rate and

\( m, K \) = fluid parameters and \( m, K > 0 \).

Note that the Herschel-Bulkley model is reduce to the Bingham visco-plasticity model while the fluid parameter \( m = 1 \).

Many efforts have been made to develop quasi-static models for governable fluid damper investigation by the Bingham model of MR fluids. Phillips (1969) [59] established a set of non-dimensional variables and a corresponding quantic equation to describe the pressure gradient of flow through a parallel duct. This method was utilized by Gavin et. al. (2004) [26] in their studies. An axisymmetric model is necessary to exactly describe MR dampers quasi-static behaviour because of having cylindrical geometry. Gavin et. al. (2004) [26] developed such an axisymmetric model. From all these study, it is assumed that yield stress is constant in the annular gap. To account for the radial field distribution, Gavin et. al. (2004) [26] assumed that the yield stress satisfied an inverse power law.

To reflect MR fluid shear thinning/thickening properties, Wang and Gordaninejad (2001) [17] applied the Herschel-Bulkley model to predict fluid flow in a parallel duct with fixed boundaries. Wang and Gordaninejad also established an axisymmetric model for a circular pipe with constant yield stress. However, to precisely model the small-scale prototype damper given in Fig. 4.1, an annular duct model is necessary.
In the subsequent section, an axisymmetric model is established which is based on the Navier-Stokes equation for the MR flow through an annular duct. To accommodate MR fluid shear thinning/thickening properties, the Herschel-Bulkley visco-plasticity model is applied. The pressure gradient can be explained numerically from the resulting equations, and the damping force can then be calculated. The resulting equations of the Herschel-Bulkley model can be replaced by the Bingham model when the post-yield shear thinning or thickening effect is ignored by assigning the fluid parameter \( m = 1 \). Section 4.3 displays that a considerably simpler parallel-plate model is perfect for a wide range of device parameters and can be applied to examine the damper’s behaviour. Simple equations based on the parallel-plate model are given which can be employed in the initial design phase. Controllable force, dynamic range and effects of geometry on MR damper performance are also discussed.

### 4.2 MR Fluid Flow in an Annular Duct

The pressure gradient along the flow is resisted by the fluid shear stress, which is governed by the Navier-Stokes equation (Constantinescu 1995):

\[
\frac{\partial p}{\partial x} = \rho \frac{\partial}{\partial t} u_x(r) + \frac{\partial}{\partial r} \tau_{xr}(r) + \frac{\tau_{xr}(r)}{r} \quad (4.2)
\]

Where
- \( u_x(r) = \) flow velocity
- \( \tau_{xr}(r) = \) shear stress
- \( r = \) radial co-ordinate
- \( x = \) longitudinal co-ordinate
- \( \rho = \) density of fluid and
- \( \frac{\partial p}{\partial x} = \) pressure gradient.

To analyse the quasi-static motion of the flow inside the damper, the fluid inertial can be ignored. Due to this effect, Eq. (4.2) can be expressed as:

\[
\frac{dp}{dx} = \frac{d}{dr} \tau_{xr}(r) + \frac{\tau_{xr}(r)}{r} \quad (4.3)
\]
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It is important to note that for oscillatory or unsteady flow, the fluid inertia must be taken into account.

The solution of Eq. (4.3) is:

$$
\tau_{rx}(r) = \frac{1}{2} \frac{dp}{dx} r + \frac{D_1}{r}
$$

(4.4)

Where $D_1$ = a constant, can be estimated with boundary conditions.

A typical shear stress diagram along with variation of velocity for MR fluid flow through the annular gap is presented by Fig. 4.2 (Spencer et. al. 1998) [44]. In areas I and II, the shear stress has exceeded the yield stress and fluids flow. In area C, there is no shear flow. It is because of the shear stress is less than the yield stress. This is often referred to as the plug flow area.

**4.2.1 Modelling based on the Herschel-Bulkley model**

To interpretation for the fluid shear thinning or thickening influence, the Herschel-Bulkley visco-plasticity model is employed. In area I, the shear strain rate $\dot{\gamma} = \frac{du_x}{dr} \geq 0$.

![Diagram of MR fluids flow](image)

**FIGURE 4.2**: Velocity and stress profiles of MR fluids through an annular duct [63].

Hence, the shear stress $\tau_{rx}(r)$ given by Eq. (4.1) becomes:

$$
\tau_{rx}(r) = \tau_0(r) + K \left( \frac{du_x(r)}{dr} \right)^{\frac{1}{m}}
$$

(4.5)
This is replaced into Eq. (4.4) and integrated once with respect to \( r \). One gets:

\[
\begin{align*}
  u_x(r) &= \int_{R_1}^{r} \left[ \frac{1}{K} \left( \frac{1}{2} \frac{dp(x)}{dx} r + \frac{D_1}{r} - \tau_0(r) \right) \right]^m dr - v_0 \quad (R_1 \leq r \leq r_1) \\
  u_x(r) &= \int_{R_2}^{r} \left[ \frac{1}{K} \left( \frac{1}{2} \frac{dp(x)}{dx} r + \frac{D_1}{r} + \tau_0(r) \right) \right]^m dr \quad (r_2 \leq r \leq R_2)
\end{align*}
\]  

(4.6)

By imposing the boundary condition that the flow velocity at \( r = R_1 \) is \( u_x(R_1) = -v_0 \). In area II, the shear strain rate \( \gamma = du_x/dr \leq 0 \). Therefore the shear stress is specified by:

\[
\tau_{rx}(r) = -\tau_0(r) - K \left( -\frac{du_x(r)}{dr} \right)^m
\]  

(4.7)

Similarly, proceeding in area II with the boundary condition \( u_x(R_2) = 0 \) at \( r = R_2 \) gives:

\[
\begin{align*}
  u_x(r) &= \int_{R_1}^{R_2} \left[ \frac{1}{K} \left( \frac{1}{2} \frac{dp(x)}{dx} r + \frac{D_1}{r} + \tau_0(r) \right) \right]^m dr \quad (r_2 \leq r \leq R_2)
\end{align*}
\]  

(4.8)

Note that the flow velocity is a constant in the plug flow area because the shear stress is less than the yield stress. Thus, the flow velocity at boundaries of the plug flow area satisfies \( u_x(r_1) = u_x(r_2) \). Combining Eqs. (4.6) and (4.8) yields:

\[
\begin{align*}
  \int_{R_1}^{r_1} \left[ \frac{1}{K} \left( \frac{1}{2} \frac{dp(x)}{dx} r + \frac{D_1}{r} - \tau_0(r) \right) \right]^m dr - \int_{r_2}^{R_2} \left[ \frac{1}{K} \left( \frac{1}{2} \frac{dp(x)}{dx} r + \frac{D_1}{r} + \tau_0(r) \right) \right]^m dr = v_0
\end{align*}
\]  

(4.9)

Also the shear stresses \( \tau_{rx} \) satisfy \( \tau_{rx}(r_1) = \tau_0(r_1) \) and \( \tau_{rx}(r_2) = -\tau_0(r_2) \), therefore, \( D_1 \) can be determined by using Eq. (4.4) as:

\[
D_1 = \frac{r_1 r_2 (\tau_0(r_2) r_1 + \tau_0(r_1) r_2)}{r_2^2 - r_1^2}
\]  

(4.10)

The expression for the volume flow rate \( Q \) is given by:

\[
Q = 2\pi \int_{R_1}^{R_2} ru_x(r) dr
\]  

(4.11)

Because the shear strain rate \( du_x(r)/dr \) is zero in the plug flow region \( r_1 < r < r_2 \), Eq. (4.11) can also be written as:

\[
Q = v_0 A_p = \pi R_1^2 v_0 - \pi \int_{R_1}^{r_1} r^2 \frac{du_x(r)}{dr} dr - \pi \int_{r_2}^{R_2} r^2 \frac{du_x(r)}{dr} dr
\]  

(4.12)
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Where

\( A_p \) = cross section area of the piston head; and

\( v_0 \) = piston head velocity.

Substitution of Eqs. (4.6) and (4.8) into Eq. (4.12) gives:

\[
Q = v_0 A_p = \pi R_1^2 v_0 \\
- \pi \int_{R_1}^{R_2} r^2 \left[ \frac{1}{2} \frac{d(p(x))}{dx} - \frac{D_1}{r} + \tau_0(r) \right]^m dr \\
+ \pi \int_{R_2}^{R_1} r^2 \left[ -\frac{1}{2} \frac{d(p(x))}{dx} + \frac{D_1}{r} + \tau_0(r) \right]^m dr
\]

(4.13)

Fig. 4.3 shows the free body diagram of MR fluids through an annular duct.

![Free body diagram of MR fluids through an annular duct](image)

**FIGURE 4.3:** Free body diagram of MR fluids through an annular duct [63].

The equation of motion of fluid materials bounded by \( r = r_1 \) and \( r = r_2 \) is:

\[
\frac{dp}{dx} \pi (r_2^2 - r_1^2) dx + 2\pi r_2 \tau_0(r_2) dx + 2\pi r_1 \tau_0(r_1) dx = 0
\]

(4.14)

These yields:

\[
\frac{dp(x)}{dx} (r_2^2 - r_1^2) + 2[\tau_0(r_2)r_2 + \tau_0(r_1)r_1] = 0
\]

(4.15)
In summary, the resulting equations which can be solved numerically to determine \( r_1 \), \( r_2 \), and the pressure gradient \( dp/dx \) between the two ends of the cylinder using the Herschel-Bulkley model are given by:

\[
v_0 = \int_{R_1}^{R_2} \left[ \frac{1}{K} \left( \frac{1}{2} \frac{dp(x)}{dx} r + \frac{D_1}{r} - \tau_0(r) \right) \right]^m dr - \int_{r_2}^{R_2} \left[ -\frac{1}{K} \left( \frac{1}{2} \frac{dp(x)}{dx} r + \frac{D_1}{r} + \tau_0(r) \right) \right]^m dr
\]

\[(4.16)\]

\[
Q = v_0 A_p = \pi R_1^2 v_0 - \pi \int_{R_1}^{R_2} r^2 \left[ \frac{1}{K} \left( \frac{1}{2} \frac{dp(x)}{dx} r + \frac{D_1}{r} - \tau_0(r) \right) \right]^m dr + \pi \int_{r_2}^{R_2} r^2 \left[ -\frac{1}{K} \left( \frac{1}{2} \frac{dp(x)}{dx} r + \frac{D_1}{r} + \tau_0(r) \right) \right]^m dr
\]

\[(4.17)\]

\[
\frac{dp(x)}{dx} (r_2^2 - r_1^2) + 2[\tau_0(r_2)r_2 + \tau_0(r_1)r_1] = 0
\]

\[(4.18)\]

Where

\[
D_1 = \frac{r_1 r_2 [\tau_0(r_2)r_1 + \tau_0(r_1)r_2]}{r_2^2 - r_1^2}
\]

\[(4.19)\]

To solve the resulting algebraic equations numerically, a method based on the constrained nonlinear least-squares algorithm [87] is applied in combination with the cubic polynomial interpolation and extrapolation method. The integrals in Eqs. (4.16) and (4.17) are evaluated by the adaptive recursive Newton–Cotes approach [86].

From Eq. (4.18), the thickness of the plug flow region can be achieved by:

\[
r_2 - r_1 = -\frac{2[\tau_0(r_1)r_2 + \tau_0(r_2)r_1]}{\frac{dp(x)}{dx}(r_2 + r_1)}
\]

\[(4.20)\]
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Which varies with the fluid yield stress $\tau_0$. Note that the flow can only be established when $r_2 - r_1 < R_2 - R_1$, which implies that the plug flow needs to be within the gap. Otherwise, there is no flow.

The damper force is then computed as:

$$F = \Delta p A_p$$  \hspace{1cm} (4.21)

Where $\Delta p = P_L - P_0 = -L(dp(x)/dx)$; and $L$ = effective axial pole length. The velocity profile can be determined from Eqs. (4.6) and (4.8) as follows:

$$u_x(r) = \begin{cases} 
\int_{R_1}^{r_1} \left[ \frac{1}{K} \left( \frac{1}{2} \frac{dp(x)}{dx} r + \frac{D_1}{r} \right) - \tau_0(r) \right]^{\frac{1}{m}} dr - v_0 & R_1 \leq r \leq r_1 \\
\int_{r_1}^{r_2} \left[ \frac{1}{K} \left( \frac{1}{2} \frac{dp(x)}{dx} r + \frac{D_1}{r} + \tau_0(r) \right) \right]^{\frac{1}{m}} dr & r_1 < r < r_2 \\
\int_{r_2}^{R_2} \left[ \frac{1}{K} \left( \frac{1}{2} \frac{dp(x)}{dx} r + \frac{D_1}{r} + \tau_0(r) \right) \right]^{\frac{1}{m}} dr & r_2 \leq r \leq R_2 
\end{cases}$$  \hspace{1cm} (4.22)

Further, the shear stress illustration can be achieved from Eq. (4.4).

Note that when the yield stress $\tau_0 = 0$, there is no plug flow area which indicates that $r_2 = r_1$. Hence, Eqs. (4.18) and (4.19) are no longer effective due to the singularity. On the other hand, in this case, the velocity reaches its maximum at $r = r_1$ wherever the shear stress is zero. By using Eq. (4.4), the subsequent equations can be employed to find pressure gradient when yield stress $\tau_0 = 0$:

$$v_0 = \int_{R_1}^{r_1} \left[ \frac{1}{K} \left( \frac{1}{2} \frac{dp(x)}{dx} r + \frac{D_1}{r} \right) \right]^{\frac{1}{m}} dr - \int_{r_1}^{r_2} \left[ \frac{1}{K} \left( \frac{1}{2} \frac{dp(x)}{dx} r + \frac{D_1}{r} \right) \right]^{\frac{1}{m}} dr$$  \hspace{1cm} (4.23)

$$Q = v_0 A_p = \pi R_2^2 v_0 - \pi \int_{R_1}^{r_1} r^2 \left[ \frac{1}{K} \left( \frac{1}{2} \frac{dp(x)}{dx} r + \frac{D_1}{r} \right) \right]^{\frac{1}{m}} dr$$

$$+ \pi \int_{r_1}^{R_2} r^2 \left[ \frac{1}{K} \left( \frac{1}{2} \frac{dp(x)}{dx} r + \frac{D_1}{r} \right) \right]^{\frac{1}{m}} dr$$  \hspace{1cm} (4.24)

$$D_1 = \frac{1}{2} \frac{dp}{dx} r_1^2$$  \hspace{1cm} (4.25)
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Note that the solution of the MR flow in an annular duct does not reduce to that of the pipe flow as \( r_1 \to 0 \). This is for the reason that the annular duct model has a boundary condition at \( r_1 \); but, there is no boundary condition at \( r = 0 \) for the pipe flow.

4.2.2 Modelling based on the Bingham model

The Herschel-Bulkley model reduces to the Bingham model once the MR fluid parameter \( m = 1 \). Using Eq. (4.16) – (4.18), the resultant equations for the Bingham model are (Spencer et. al. 1998) [44]:

\[
\frac{dp(x)}{dx}(R_2^2 - r_2^2 - R_1^2 + r_1^2) + \frac{D_1}{4} \ln \left( \frac{R_2 r_1}{r_2 R_1} \right) + D_2 - \eta \nu_0 = 0 \tag{4.26}
\]

\[
Q = \nu_0 A_p = \pi R_1^2 \nu_0 = \frac{\pi}{8\eta} \left[ \frac{dp(x)}{dp} \left( R_2^4 - r_2^4 - R_1^4 + r_1^4 \right) + 4D_1 \left( R_2^2 - r_2^2 - R_1^2 + r_1^2 \right) + 8D_3 \right] \tag{4.27}
\]

\[
\frac{dp(x)}{dx} (r_2^2 - r_1^2) + 2[\tau_0(r_2) r_2 + \tau_0(r_1) r_1] = 0 \tag{4.28}
\]

Where

\[
D_1 = \frac{r_1 r_2 [\tau_0(r_2) r_1 + \tau_0(r_1) r_2]}{r_2^2 - r_1^2} \tag{4.29}
\]

\[
D_2 = \int_{r_2}^{R_2} \tau_0(r) dr + \int_{r_1}^{R_1} \tau_0(r) dr \tag{4.30}
\]

\[
D_3 = \int_{r_2}^{R_2} \tau_0(r) r^2 dr + \int_{r_1}^{R_1} \tau_0(r) r^2 dr \tag{4.31}
\]

And the velocity profile is given by:

\[
u_x(r) = \begin{cases} 
-\frac{1}{4\eta} \frac{dp}{dx} (R_2^2 - r^2) + \frac{D_1}{\eta} \ln \frac{r}{R_1} - \frac{1}{\eta} \int_{r_1}^{R_1} \tau_0(r) dr - \nu_0 & R_1 \leq r \leq r_1 \\
-\frac{1}{4\eta} \frac{dp}{dx} (R_2^2 - r_2^2) - \frac{D_1}{\eta} \ln \frac{R_2}{r_2} - \frac{1}{\eta} \int_{r_2}^{R_2} \tau_0(r) dr & r_1 < r < r_2 \\
-\frac{1}{4\eta} \frac{dp}{dx} (R_2^2 - r_2^2) - \frac{D_1}{\eta} \ln \frac{R_2}{r} - \frac{1}{\eta} \int_{r_2}^{R_2} \tau_0(r) dr & r_2 \leq r \leq R_2 
\end{cases}
\tag{4.32}
\]
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In the absence of the magnetic field, the yield stress $\tau_0 = 0$. The pressure gradient can be achieved directly from:

$$\frac{dp}{dx} = \frac{8\eta\nu_0}{\pi} \frac{\pi}{2} \frac{\left(2R^2_2 - \frac{R^2_2 - R^2_1}{\ln(R^2_2/R^2_1)}\right)}{R^4_2 - R^4_1} \frac{A_p}{R^4_2 - R^4_1} \left(\frac{R^2_2 - R^2_1}{\ln(R^2_2/R^2_1)}\right)$$

(4.33)

Generally, the yield stress $\tau_0$ in the axisymmetric model will be a function of $r$. But while $R_2 - R_1 \ll R_1$, variation of the yield stress in the gap can be ignored, and Eqs. (4.29) – (4.31) can be further simplified significantly as follows:

$$D_1 = \frac{\tau_0 r_1 r_2}{r_2 - r_1}$$

(4.34)

$$D_2 = \tau_0 (R_2 + R_1 - r_1 - r_2)$$

(4.35)

$$D_3 = \frac{1}{3} \tau_0 (R^3_2 + R^3_1 - r^3_2 - r^3_1)$$

(4.36)

Note that in this case, the thickness of the plug flow area can be calculated by using Eq. (4.20):

$$r_2 - r_1 = -\frac{2\tau_0}{\frac{dp}{dx}}$$

(4.37)

Which is a constant, and only depends on the yield stress and pressure gradient of the flow.

4.3 MR Fluid Flow in a Parallel Duct

Due to the small ratio among the flow gap (between the cylinder housing and the piston) and the diameter of the piston, one might conjecture that the axisymmetric flow create in the damper can be approximated as the flow through a parallel duct as presented in Fig. 4.4. Connecting the parallel- plate model to the axisymmetric model, the parameter $w$ is taken to be the mean circumference of the damper’s annular flow path which equals to $\pi(R_1 + R_2)$, and $h$ is taken to be the gap width, equal to $(R_2 - R_1)$, (Spencer et al 1998) [44].
Fig. 4.5 provides the free body diagram and stress and velocity profiles of MR fluids through a parallel duct.

The governing equation for the flow of the parallel-plate model is:

\[
\frac{dx}{dz} = \frac{dp}{dx} \tag{4.38}
\]

Therefore,

\[
\tau(z) = \frac{dp}{dx} z + D \tag{4.39}
\]
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In which \( D \) is a constant which can be evaluated with boundary conditions.

### 4.3.1 Modelling based on the Herschel-Bulkley model

Like to the axisymmetric model, the shear stress \( \tau \) in area I (Fig. 4.5) is given by:

\[
\tau(z) = \tau_0 + K \left( \frac{du_x}{dz} \right)^{\frac{1}{m}}
\]  \hspace{1cm} (4.40)

Replacement of Eq. (4.40) into Eq. (4.39) and applying the boundary condition \( u_x(0) = 0 \) gives:

\[
u_x(z) = \frac{1}{m+1} \left( -\frac{1}{K} \frac{dp}{dx} \right)^m \{ h_1^{m+1} - (h_1 - z)^{m+1} \} \hspace{1cm} (0 \leq z \leq h_1) \hspace{1cm} (4.41)
\]

In area II, the shear stress is specified by:

\[
\tau = -\tau_0 - K \left( -\frac{du_x}{dz} \right)^{\frac{1}{m}}
\]  \hspace{1cm} (4.42)

By the same procedure as in area I, with boundary condition \( u_x(h) = -v_0 \), gives:

\[
u_x(z) = \frac{1}{m+1} \left( -\frac{1}{K} \frac{dp}{dx} \right)^m \{ (h - h_2)^{m+1} - (h_2 - z)^{m+1} \} - v_0 \hspace{1cm} (h_2 \leq z \leq h) \hspace{1cm} (4.43)
\]

The flow velocity at the boundary of the plug flow area satisfies \( u_x(h_1) = u_x(h_2) \).

Joining Eq. (4.41) and (4.43) yields:

\[
\frac{1}{m+1} \left( -\frac{1}{K} \frac{dp}{dx} \right)^m h_1^{m+1} = \frac{1}{m+1} \left( -\frac{1}{K} \frac{dp}{dx} \right)^m (h - h_2)^{m+1} - v_0
\]  \hspace{1cm} (4.44)

The volume flow rate \( Q \) is specified by:

\[
Q = A_p v_0 = w \int_0^h u_x(z)dz
\]

i.e.

\[
Q = \frac{w}{m+1} \left( -\frac{1}{K} \frac{dp}{dx} \right)^m h_1^{m+1} \left[ h - \frac{1}{m+2} (h + h_1 - h_2) \right] - \frac{w}{m+2} v_0 (h - h_2)
\]  \hspace{1cm} (4.45)
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Where $A_p =$ piston head cross section area

$v_0 =$ velocity of piston head.

With reference to Fig. 4.5, the equation of motion for fluid materials bounded by $h = h_1$ and $h = h_2$ is:

$$\frac{dp}{dx} (h_2 - h_1) \delta x + 2\tau_0 \delta x = 0$$  \hspace{1cm} (4.46)

Which yields

$$h_2 - h_1 = -\frac{2\tau_0}{\frac{dp}{dx}}$$  \hspace{1cm} (4.47)

Hence, the resulting equations for the parallel-plate model using the Herschel-Bulkley model consist of Eq. (4.44), (4.45) and (4.47). The pressure gradient $\frac{dp}{dx}$ can be resolved numerically, and the damper resisting force is then calculated with Eq. (4.21).

Note that $h_2 = h_1$ when the yield stress $\tau_0 = 0$.

The velocity profile, which satisfies the boundary conditions $u_x(0) = 0$ and $u_x(h) = -v_0$, is then specified by:

$$u_x(z) = \begin{cases} \frac{1}{m+1} \left( \frac{1}{K} \frac{dp}{dx} \right)^m [h_1^{m+1} - (h_1 - z)^{m+1}] & 0 \leq z \leq h_1 \\ \frac{1}{m+1} \left( \frac{1}{K} \frac{dp}{dx} \right)^m h_1^{m+1} & h_1 < z < h_2 \\ \frac{1}{m+1} \left( \frac{1}{K} \frac{dp}{dx} \right)^m [(h - h_2)^{m+1} - (z - h_2)^{m+1}] - v_0 & h_2 \leq z \leq h \end{cases}$$  \hspace{1cm} (4.48)

Note that when $v_0 = 0$, the resulting equations are the same as those provided by Lee and Wereley (2000).

4.3.2 Modelling based on the Bingham model (f P V)

Once more for the Bingham model, the fluid parameter is $m = 1$. Replacement of Eq. (4.44) and (4.47) into Eq. (4.45), and describing non-dimensional variables:
MR Fluid Flow in a Parallel Duct

\[ V = -\frac{whv_0}{2Q} = -\frac{wh}{2A_p} \]

\[ P = \frac{wh^3}{12\eta Q} \frac{dp}{dx} = -\frac{wh^3}{12\eta A_pv_0} \frac{dp}{dx} \]

\[ \Gamma = \frac{wh^2\tau_0}{12\eta Q} = \frac{wh^2\tau_0}{12\eta A_p v_0} \]

Results in the non-dimensional quantic equation (Phillips 1969) [59]:

\[ 3(P - 2\Gamma)^2(P^3 - (1 + 3\Gamma - V)P^2 + 4\Gamma^3) + \Gamma V^2P^2 = 0, \quad |V| < 3(P - 2\Gamma)^2/P \]

(4.52)

Note that \( V < 0 \), if the piston motion is in the reverse direction of the fluid flow. The force created by the damper is then specified by:

\[ F = -\frac{dy}{dx} A_p L = \frac{12\eta A_p^2 \tau_0 v_0}{wh^3} P \]

(4.53)

When \( |V| > 3(P - 2\Gamma)^2/P \), the flow is directed by the subsequent dimensionless equations which are independent of the dimensionless yield stress \( \Gamma \) (Phillips 1969) [59]:

\[ P = \frac{4V^3}{27(2\Gamma - 1)^2} \quad 3(P - 2\Gamma)^2/P \leq V \leq 3P \]

(4.54)

\[ P = -\frac{4V^3}{27} \quad -3P \leq V \leq -3(P - 2\Gamma)^2/P \]

(4.55)

\[ P + V = 1 \quad |V| > 3P \]

(4.56)

Eq. (4.54) – (4.56) specify that the pressure gradient be governed by the geometry of the device, and that a controllable yield stress has no effect on the force of the damper.

If the piston head velocity \( v_0 = 0 \), then \( V = 0 \), and Eq. (3.52) turn into a cubic equation for \( P \). This cubic equation has the achievable root at (Gavin et. al. (2004) [26]):
\[ P(\Gamma) = \frac{2}{3} (1 + 3\Gamma) \left[ \cos \left( \frac{1}{3} \cos \left( 1 - 54 \left( \frac{r}{(1+3\Gamma)^3} \right) \right) \right) + \frac{1}{2} \right] \] (4.57)

Normally, there is no analytical solution for Eq. (4.52), but it can be easily solved numerically. An approximate solution can be used to estimate the desired root for the condition \( 0 < \Gamma < 1000 \) and \(-0.5 < V < 0\), which encompasses most practical designs in which the flow is in the opposite direction of the piston velocity (Spencer et. al. 1998) [44]:

\[ P(\Gamma, V) = 1 + 2.07\Gamma - V + \frac{r}{1+0.4\Gamma} \] (4.58)

### 4.4 Simple Geometry Design Considerations

With reference to the parallel-plate Bingham model which was established in prior sections, simple equations that make available the understanding on the influence of various damper parameters are given; these equations can be used in the preliminary design stage. Special effects of geometry on controlled force, dynamic range and MR damper performance are also discussed in this segment.

#### 4.4.1 Controlled force and dynamic range

The “controlled force” and the “dynamic range” are two most significant parameters in estimating the complete performance of the MR damper. As shown in Fig. 4.6, the over-all damping force can be divided into a controlled force \( F_c \) due to governable yield stress \( \tau_o \) and an uncontrolled force \( F_{uc} \). The uncontrolled force comprises a plastic viscous force \( F_\eta \) and a friction force \( F_f \). The “dynamic range” \( (D) \) is defined as the ratio of the damper controlled force \( F_c \) and the uncontrolled force \( F_{uc} \) as follows:

\[ D = \frac{F_c}{F_{uc}} = \frac{F_c}{\eta + F_f} \] (4.59)

With reference to the parallel-plate Bingham model, \( F_c \) and \( F_\eta \) are defined as:

\[ F_c = \frac{c\tau_0 A_p}{h} \text{sgn}(\nu_0) \] (4.60)

\[ F_\eta = \left( 1 + \frac{wh\nu_0}{2Q} \right)^{12\eta QL} \frac{v}{wh^3} \] (4.61)

and

\[ c \approx 2.07 + 1/(1 + 0.4 \Gamma) \] restricted in between the value of 2.07-3.07. (Spencer et. al. 1998).
The controlled force in Eq. (4.61) can furthermore be written by using Eq. (4.51) as:

\[
F_r = \left( 2.07 + \frac{12\eta}{12\eta + 0.4wh^2v_0} \right) \frac{\tau_0 A_p}{h} sgn(v_0)
\]

The above equation (4.62) specifies that the controlled force is inversely related to the gap dimension. The controlled force should be as large as possible to optimize the effectiveness of the MR damper; for that reason, a small gap size is compulsory.

However, a small gap size reduces the dynamic range. As presented in Eq. (4.60) and (4.61), the uncontrolled force (viscous force) rises two orders of magnitude faster than the controlled force with a small gap dimension if one assumes that the magnetic field is saturated; therefore, the dynamic range is reduced which tends to zero. As the gap dimension becomes large, the controlled force and the uncontrolled force are decreased. It is important to note that the friction force is a constant, so once more the dynamic range reduced which tends to zero. It is obvious that an optimum dynamic range must exist.

### 4.4.2 Geometry constraints

Eq. (4.60) and (4.61) are definitely suitable in the design of MR dampers; but, they frequently do not provide the finest understanding into the importance of various parameters. Hence, the “minimum active fluid volume” \( V \) is presented. Minimum active fluid volume \( V \) is the volume of MR fluids exposed to the magnetic field and thus
accountable for providing the desired MR effect. With the help of Eq. (4.60) and (4.61), we obtain the conclusion as:

\[ wh^2 = \frac{12k}{c} \left( \frac{\eta}{\tau_0} \right) \left( \frac{F_x}{F_\eta} \right) Q \]  (4.63)

Where \( k = 1 + (whv_0)/(2Q) \). Because eq. (4.60) can also be written as:

\[ \Delta p_\tau = \frac{F_x}{A_\tau} = \frac{c \tau_0 L}{h} \]  (4.64)

Eq. (4.63) can be further manipulated to give

\[ V = \frac{12k}{c^2} \left( \frac{\eta}{\tau_0^2} \right) \left( \frac{F_x}{F_\eta} \right) Q \Delta p_\tau \]  (4.65)

Where \( V = Lwh \) which is the “minimum active fluid volume”; and \( \Delta p_\tau \) is a pressure reduced due to the yield stress. Note that for most design cases, \( whv_0 \ll Q \) and as a result \( k \approx 1 \).

For preliminary geometric design of MR dampers, it can be assume that the friction force \( (F_f) \) has the same value as the uncontrolled force (plastic viscous force) \( (F_\eta) \). Therefore, \( \frac{F_x}{F_\eta} \approx 2D \), Where \( D \) is the necessary dynamic range. From the known value of the damper flow rate \( Q \), necessary dynamic range \( D \), yield stress \( \tau_0 \), housing dimension \( w \), plastic viscosity \( \eta \), and pressure drop \( \Delta p_\tau \), the gap dimension \( h \) and active pole length \( L \) can be achieved from Eqs. (4.63) and (4.65). However, this preliminary design needs to be confirmed by a more precise axisymmetric model. Typically, a complete design also includes iterations with the magnetic circuit design.