CHAPTER - 3
UNSTEADY FLUCTUATING PLANE FLOWS OF THERMO-VISCOUS FLUIDS

INTRODUCTION

The unsteady flows of a second order thermo-viscous incompressible fluid over a fluctuating flat plate are presented in this chapter. The pressure and temperature gradients are neglected. It is observed that the equations of momentum and energy are decoupled.

In this chapter, the following three problems are solved analytically.

A) Flow of an infinite fluid over a fluctuating bottom
B) Flow of a liquid of finite depth with bottom oscillating
C) Forced oscillations of a fluid bounded by a rigid bottom

Exact solutions are obtained in each case for the velocity and temperature distributions. It is noticed that the effect of cross-viscosity ($\mu_c$ due to Riener and Rivlin) is to produce normal stress while thermo-stress viscous nature of the fluid ($\alpha_s$) generates a force $F_c$ in the plane of motion, but perpendicular to the direction of motion. This is a significant feature of thermo-viscous fluids. It is noticed that these effects are independent of thermo-stress coefficient ($\alpha_s$). Both these two effects are not noticed in the classical Newtonian case.

The investigations of Stokes [24] on flows of this type for classical viscous fluids have been reported by Lamb [15]. Such flows of second order visco-elastic fluids were studied later by Pattabhi Ramacharyulu [22].
FORMULATION AND SOLUTION OF THE PROBLEM

Consider the unsteady motion of an incompressible thermo-viscous fluid over an infinite plate, fluctuating in its plane. The plate is maintained at a given temperature. Far away from the plate, the fluid is supposed to be at rest and maintained at zero temperature.

With reference to a rectangular Cartesian coordinate system \( O(x,y,z) \) with origin chosen on the plate, the \( x \)-axis in the direction of the plate and \( y \)-axis perpendicular to the plate. The excess of fluid temperature over that at infinity is \( \theta(y,t) - \theta_{\infty}(t) \) is denoted by \( \eta(y,t) \).

A) FLOW OF AN INFINITE FLUID OVER A FLUCTUATING BOTTOM

Considering a class of plane flows characterized by the velocity components \([u(y,t),0,0]\) and the temperature distribution \( \eta(y,t) \) and in the absence of pressure and temperature gradients and heat source within the flow region, the equations of momentum are in the \( x \)-direction:
\[ \rho \frac{\partial^2 u}{\partial t^2} = \mu \frac{\partial^2 u}{\partial y^2} \]  
(3.A.1)

in the \( y \)-direction:

\[ \mu \frac{\partial}{\partial y} (u \frac{\partial}{\partial y}) + \rho F_1 = 0 \]  
(3.A.2)

in the \( z \)-direction:

\[ -\alpha_s \frac{\partial}{\partial y} (u \eta) + \rho F_2 = 0 \]  
(3.A.3)

and the energy equation:

\[ \rho c_v \frac{\partial \eta}{\partial t} = \mu \left( \frac{\partial u}{\partial y} \right)^2 + k \frac{\partial^2 \eta}{\partial y^2} \]  
(3.A.4)

The boundary conditions are

\[ u(0,t) = u_0 \cos \sigma t = u_0 \text{Re.} \exp(i\sigma t) \]  
(3.A.5)

and

\[ \eta(0,t) = \eta_1^* \cos \sigma t = \eta_1^* \text{Re.} \exp(i\sigma t) \]  
(3.A.6)

and far away from the plate, the velocity and temperature are

\[ u(x,t) = 0, \quad \eta(x,t) = 0 \]  
(3.A.7)

In the above equations \( F_1 \) and \( F_2 \) are the body forces in \( y \) and \( z \) directions. \( u_0 \) and \( \eta_1^* \) are constants.

From the equations (3.A.1) and (3.A.4), we observe that these equations are decoupled in the absence of temperature gradient. The oscillations of the classical viscous fluid on the upper half of the plane \( y = 0 \), which vibrates harmonically have been discussed by Lamb and those of second order fluids by Pattabhi Rmacharyulu.

Let us assume the velocity field \( u(y,t) \) in the form

\[ u(y,t) = \text{Re.} f(y) \exp(i\sigma t) \]  
(3.A.8)

Substituting this velocity function (3.A.8) in the equation (3.A.1), we get the differential equation for \( f(y) \):

\[ (D^2 - m^2)f = 0 \]  
(3.A.9)
with the boundary conditions on \( f(y) \) are

\[
f(0) = u_0 \quad \text{and} \quad f(\infty) = 0 \tag{3.A.10}
\]

where

\[
m^2 = \frac{\sigma}{\nu}, \quad m = \delta(1 + i)
\]

and

\[
\delta = \sqrt{\frac{\sigma}{2\nu}} \tag{3.A.11}
\]

from the equations (3.A.9) and (3.A.10), we get

\[
f(y) = u_0 \exp(-my) \tag{3.A.12}
\]

Hence the velocity distribution is given by

\[
u(y, t) = \operatorname{Re} u_0 \exp(-my) \exp\left(i\sigma t\right)
\]

\[
= u_0 \exp(-\delta t) \cos\sigma \left(t - \frac{\delta}{\sigma} y\right) \tag{3.A.13}
\]

which is same as that in classical Newtonian case.

It can be noted that the fluid velocity is also fluctuating with the same period (\( = 2\pi / \sigma \)) as the bottom but with a phase-lag (\( = \delta / \sigma \)) and an exponentially decreasing amplitude with the characteristic distance

\[
\frac{1}{\delta} = \sqrt{\frac{\nu}{\rho \sigma}}
\]

and this can be taken as the thickness of boundary layer.

Let the temperature distribution be

\[
\eta(y, t) = \eta_1(y)e^{-\mu t} + \eta_2(y)e^{2\mu t} \tag{3.A.14}
\]

Substituting (3.A.14) in the energy equation (3.A.4), we get after separation of variables, the differential equations for \( \eta_1(y) \) and \( \eta_2(y) \):\n
\[
\eta_1'' - \frac{\rho c_i}{k} \sigma \eta_1 = 0 \tag{3.A.15}
\]

and

\[
\eta_2'' - \frac{2\rho c_i}{k} \sigma \eta_2 = -2i \frac{\mu}{k} u_0 \delta^2 e^{-2\mu t + i\phi} \tag{3.A.16}
\]

together with the boundary conditions:
The solutions of these equations (3.15) and (3.16) together with the boundary conditions (3.17), (3.18) are

\[ \eta_1(y) = -\eta_1^* \text{Re} \exp \left(-\sqrt{Pr} \delta(1+i)y\right) \]  

and

\[ \eta_2(y) = \frac{\mu}{k} \text{Re} \frac{u_0 \delta^2}{4\delta^2 + m^2 Pr} \left[ e^{-\sqrt{2Pr} \delta(1+i)y} - e^{-2(1+i)\delta} \right] \]  

where \( Pr = \frac{c\mu}{k} \) is the Prandtl number.

\[ \eta(y, t) = \text{Re} \, \eta_1^* \exp(-\sqrt{Pr} \delta(1+i)y) \exp(\sigma t) \]

\[ + \text{Re} \frac{\mu}{k} \frac{u_0 \delta^3}{4\delta^2 + m^2 Pr} \left[ \exp(-\sqrt{2Pr} \delta(1+i)y) - \exp(2\delta) \exp(2i\sigma t) \right] \]

The temperature distribution thus is given by

\[ \eta(y, t) = \eta_1^* \exp(-\sqrt{Pr} \delta y) \cos \left( t - \frac{\sqrt{Pr}}{\sigma} \delta y \right) \]

\[ + \frac{\mu u_0}{2k(2 - Pr)} \left[ \exp(-\sqrt{2Pr} \delta y) \cos \left( 2\sigma t - \frac{Pr \delta y}{2\sigma} \right) - \exp(-2\delta y) \cos \left( 2\sigma t - \frac{\delta y}{\sigma} \right) \right] \]

for \( Pr \neq 2 \)

\[ = \eta_1^* \exp(-\sqrt{2} \delta y) \cos \left( t - \frac{\sqrt{2}}{\sigma} \delta y \right) \]

\[ + \frac{\mu u_0}{2\sqrt{2}k} \left[ \exp(-2\delta y) \sin \left( 2\sigma t - \frac{\delta y}{\sigma} - \frac{\pi}{8\sigma} \right) \right] \]

for \( Pr = 2 \)

(3.A.22)

(3.A.23)

The temperature distribution is composed of two harmonics with periods \( \frac{2\pi}{\sigma} \) and \( \frac{\pi}{\sigma} \) with exponentially decreasing amplitude. From this, the thickness of the thermal boundary layer is min \( \left\{ \frac{1}{2\delta}, \frac{1}{\delta \sqrt{2Pr}} \right\} \).
The Nusselt number on the plate \((y = 0)\):
\[
\frac{\partial \eta}{\partial y} \bigg|_{y=0} = \eta^*_v \sqrt{2 Pr} \delta \left[ \sin (\sigma t - \frac{\pi}{4}) - A \sin (2 \sigma t - \frac{\pi}{4}) \right] \tag{3.A.24}
\]
where
\[
A = \frac{\mu u_0}{k(\sqrt{Pr} + \sqrt{2}) \eta^*_v \sqrt{Pr}}
\]
for all values of \(Pr \neq 0\)

The effect of cross-viscosity in the direction perpendicular to the plane is obtained from the equation of momentum (3.A.2) in \(y\)-direction as
\[
\rho \partial F_{\gamma} = -\mu \frac{\partial}{\partial y} (u_{\gamma}^*)
\]
\[
= 2 \mu u_0 \delta^3 \exp (-2 \delta \gamma) \left[ 1 - \sqrt{2} \cos 2 \sigma \left( t - \frac{\delta \gamma}{\sigma} - \frac{\pi}{8 \sigma} \right) \right] \tag{3.A.25}
\]
From the equation of motion (3.A.3) it is noticed that a force is generated perpendicular to the direction of the plate fluctuations and is given by
\[
\rho F_{\gamma} = \alpha_y \frac{\partial}{\partial y} (u, \eta_y)
\]
\[
= 2 \alpha_y \delta^3 u_0 \left[ -\eta^*_v \sqrt{2 Pr} \delta \gamma \exp \left[ (1 + \sqrt{2} Pr \delta \gamma) \right] \left[ \sin (\sigma t - \delta \gamma) \sin \left( \sigma t - \frac{\pi}{4} - \sqrt{Pr} \delta \gamma \right) \right. \right.
\]
\[
\left. \left. + \sqrt{2} \sin (\sigma t - \delta \gamma) \sin \left( \sigma t - \frac{\pi}{4} \right) \right] \right]
\]
\[
+ \frac{\mu u_0 \delta^3}{k(2 - Pr)} \left[ - \exp \left[ (1 + \sqrt{2} Pr \delta \gamma) \right] \sqrt{2} \sin (\sigma t - \delta \gamma) \sin \left( \sigma t - \frac{\pi}{4} \right) \right]
\]
\[
+ 2 \sqrt{2} \text{ Pr} e^{-1 + \frac{\text{Pr} \delta \gamma}{2} \sin \left( \sigma t - \frac{\text{Pr} \delta \gamma}{2} \right) \cos \left( \sigma t - \frac{\text{Pr} \delta \gamma}{2} \right) \cos \left( \sigma t - \frac{\pi}{4} - \delta \gamma \right) - \sqrt{2} \text{ Pr} e^{-1 + \frac{\text{Pr} \delta \gamma}{2} \sin \left( \sigma t - \delta \gamma \right) \sin \left( 2 \sigma t - \frac{\pi}{4} - 2 \delta \gamma \right) \right] \tag{3.A.26}
\]

On the plate \(y = 0\), the normal thrust generated due to the cross-viscosity \((\mu_y)\) is
\[ \rho F_{1, \omega} = 2 \mu \omega \sigma \left[ 1 - \sqrt{2} \sin \left( 2\sigma t + \frac{\pi}{4} \right) \right] \]  
(3.A.27)

and the transverse force perpendicular to the direction of the plate fluctuations is

\[ \rho F_{2, \omega} = 2 \alpha \omega \sigma \sin \left[ -\eta \sqrt{2\Pr(1 + \sqrt{\Pr}) \sin \left( \sigma t - \frac{\pi}{4} \right) \right] 
+ \frac{\mu \omega}{k(2 - Pr)} \left[ -\sqrt{Pr(1 + \sqrt{Pr}) \sin \left( 2\sigma t - \frac{\pi}{4} \right) + 2 \sin \left( \sigma t - \frac{\pi}{4} \right) } \right] \]  
(3.A.28)

We observe that the upward thrust vanishes for \( \gamma = 0 \) and the transverse force vanishes for \( \alpha = 0 \). i.e. for Newtonian-Fourier-heat conducting fluids. These two effects are not observed in classical case. The generation of the force is the significant feature of thermo-viscous fluids flows. We can observe that these effects are independent of thermo-stress coefficient (\( \eta \)).

**B) FLOW OF THE LIQUID OF FINITE DEPTH WITH BOTTOM OSCILLATING**

![Flow Configuration](image)

**Fig. 3.2  Flow Configuration**

The flow of a thermo-viscous fluid of finite depth with bottom oscillating is examined.
It is assumed that (i) the dissipation term in the energy equation is neglected (ii) the bottom is oscillating with a velocity \( u_0 \cos \sigma t \) (iii) the temperature of the bottom is also fluctuating with velocity \( \eta_0 \cos \sigma t \).

The equations of motion and energy reduce to

\[
\rho \frac{\partial^2 u}{\partial t^2} = \mu \frac{\partial^4 u}{\partial y^4}
\]

(3.B.1)

\[
\rho F_x = \mu \frac{\partial}{\partial y} (u \frac{\partial u}{\partial y})
\]

(3.B.2)

\[
\rho F_z = -\alpha_s \frac{\partial}{\partial y} (u_0 \eta)
\]

(3.B.3)

and

\[
\rho c \frac{\partial^2 \eta}{\partial t^2} = k \frac{\partial^2 \eta}{\partial y^2}
\]

(3.B.4)

The boundary conditions are

\[
\begin{align*}
\frac{\partial^2 u}{\partial y^2} (0, t) &= K_c, \\
\frac{\partial u}{\partial y} (h, t) &= R_c
\end{align*}
\]

(3.B.5)

and

\[
\begin{align*}
u(0, t) &= Re. \exp(i \sigma t) \\
\frac{\partial \eta}{\partial y} (0, t) &= Re. \eta_0 e^{i \sigma t} \\
\frac{\partial \eta}{\partial y} (h, t) &= 0
\end{align*}
\]

(3.B.6)

Let the velocity distribution be assumed as

\[
u(y, t) = Re. f(y) \exp(i \sigma t)
\]

(3.B.7)

substituting (3.B.7) in (3.B.1) we get

\[
(D^2 - m^2) f = 0
\]

(3.B.8)

where

\[
m^2 = \rho \sigma i / \mu
\]

(3.B.9)

the boundary conditions for \( f(y) \) are

\[
\begin{align*}
f(0) &= u_0 \end{align*} \text{ and } f'(h) = 0
\]

(3.B.10)

hence from (3.B.8) and (3.B.10), we have

\[
f(y) = u_0 \frac{\cosh m (h - y)}{\cosh m h}
\]

(3.B.11)
Hence the velocity distribution is

\[ u(y,t) = \text{Re} \, u_0 \frac{\cosh m(h - y)}{\cosh mh} \, e^{\iota \sigma t} \]

\[ = u_0 [G_i(y) \cos \sigma t - H_i(y) \sin \sigma t] \quad (3.\text{B}.12) \]

where

\[ G_i(y) = \frac{\cos \delta(2h - y) \cosh \delta \gamma + \cos \delta \gamma \cosh \delta(2h - y)}{\cos 2\delta \gamma + \cosh 2\delta h} \quad (3.\text{B}.13) \]

and

\[ H_i(y) = \frac{\sin \delta \gamma \sinh \delta(2h - y) - \sin \delta(2h - y) \sin \delta \gamma}{\cos 2\delta \gamma + \cosh 2\delta h} \quad (3.\text{B}.14) \]

The temperature distribution is assumed as

\[ \eta(y,t) = \eta_1 g(y) \, e^{\iota \sigma t} \quad (3.\text{B}.15) \]

from the equations (3.\text{B}.4) and (3.\text{B}.15) we get

\[ (D^2 - m_i^2) g = 0 \quad (3.\text{B}.16) \]

where

\[ m_i^2 = Pr \, m_t^2 \quad (3.\text{B}.17) \]

The boundary conditions for \( g(y) \) are

\[ g(0) = 1 \quad \text{and} \quad g(h) = 0 \quad (3.\text{B}.18) \]

From the equations (3.\text{B}.16) and (3.\text{B}.18), we get

\[ g(h) = \frac{\sinh m_i(h - y)}{\sinh m_i h} \quad (3.\text{B}.19) \]

Hence, the temperature distribution is

\[ \eta(y,t) = \eta_1 \text{Re} \frac{\sinh m_i(h - y)}{\sinh m_i h} \, e^{\iota \sigma t} \]

\[ = \eta_1 [G_2(y) \cos \sigma t - H_2(y) \sin \sigma t] \quad (3.\text{B}.20) \]

where

\[ G_2(y) = \frac{\cos \sqrt{Pr} \delta \gamma \cosh \sqrt{Pr} \delta(2h - y) - \cos \sqrt{Pr} \delta(2h - y) \cosh \sqrt{Pr} \delta \gamma}{\cosh 2\sqrt{Pr} \delta h - \cosh 2\sqrt{Pr} \delta h} \quad (3.\text{B}.21) \]

and

\[ H_2(y) = \frac{\sin \sqrt{Pr} \delta(2h - y) \sinh \sqrt{Pr} \delta \gamma - \sin \sqrt{Pr} \delta \gamma \sinh \sqrt{Pr} \delta(2h - y)}{\cosh 2\sqrt{Pr} \delta h - \cosh 2\sqrt{Pr} \delta h} \quad (3.\text{B}.22) \]
Introducing the non-dimensional quantities:

\[ u = u_0 \xi, \quad y = h \eta, \quad \beta = \delta h \]

\[ \eta(y, t) = \eta^* T(Y, t) \]  

(3.B.23)

The velocity and temperature distributions in the non-dimensional form can be expressed as

\[ u(Y, t) = G_1(Y) \cos \sigma t - H_1(Y) \sin \sigma t \]  

(3.B.24)

and

\[ T(Y, t) = G_1(Y) \cos \sigma t - H_1(Y) \sin \sigma t \]  

(3.B.25)

where

\[ G_1(Y) = \frac{\cos \beta(2 - Y) \cosh \beta' - \cos \beta' \cosh \beta(2 - Y)}{\cos 2 \beta + \cosh 2 \beta} \]  

(3.B.26)

\[ H_1(Y) = \frac{\sin \beta \cosh \beta(2 - Y) - \sin \beta' \sinh \beta(2 - Y) \cosh \beta'}{\cos 2 \beta + \cosh 2 \beta} \]  

(3.B.27)

and

\[ G_2(Y) = \frac{\cos \sqrt{Pr} \beta' \cosh \sqrt{Pr} \beta(2 - Y) - \cos \sqrt{Pr} \beta'(2 - Y) \cosh \sqrt{Pr} \beta'}{\cosh 2 \sqrt{Pr} \beta - \cos 2 \sqrt{Pr} \beta} \]  

(3.B.28)

\[ H_2(Y) = \frac{\sin \sqrt{Pr} \beta(2 - Y) \sinh \sqrt{Pr} \beta' \sinh \sqrt{Pr} \beta(2 - Y) \sinh \sqrt{Pr} \beta'}{\cosh 2 \sqrt{Pr} \beta - \cos 2 \sqrt{Pr} \beta} \]  

(3.B.29)

The effect of cross-viscosity in the direction perpendicular to the plane is obtained from the equation of momentum (3.B.2) in \( y \)-direction as

\[ \rho F_y = \frac{2 \mu \mu_y^2}{h^3} \left[ U_y U_{\eta y} \right] \]

\[ = \frac{\mu \mu_y}{h} \left[ G_1(Y)G_1''(Y)(1 + \cos 2\sigma \tau) - \left( G_1'(Y)H_1''(Y) + H_1'(Y)G_1''(Y) \right) \sin 2\sigma \tau \right] \]

\[ + H_1'(Y)H_1''(Y)(1 - \cos 2\sigma \tau) \]  

(3.B.30)

From the equation of motion (3.B.3), it is noticed that a force is generated perpendicular to the direction of the plate fluctuations and is given by
\[ \rho F_i = \frac{\alpha \mu \eta \beta^3}{h^3} \frac{\partial^2}{\partial y^2} (U, T_i) \]

\[ = \frac{\alpha \mu \eta \beta^3}{h^3} \left[ (G_i^{11} G_i^{1} + G_i^{1} G_i^{11}) \cos^2 \sigma t + (H_i^{11} H_i^{1} + H_i^{1} H_i^{11}) \sin^2 \sigma t \right. \]

\[ - \left. \left[ (G_i^{11} H_i^{11} + H_i^{11} G_i^{1}) + (G_i^{1} H_i^{11} + H_i^{1} G_i^{11}) \right] \cos \sigma t \sin \sigma t \right] \]

(3.31)

**On the plate** \( y = 0 \), the normal thrust generated due to the cross-viscosity \((\mu)\) is

\[ \rho F_i \bigg|_{y=0} = \frac{2 \mu \beta^3}{h} \frac{\cos 2\beta - \cosh 2\beta}{\cos 2\beta + \cosh 2\beta} \left[ (\sinh 2\beta - \sin 2\beta) (1 - \cos 2\sigma t) \right. \]

\[ \left. - (\sin 2\beta + \sin 2\beta) \sinh 2\sigma t \right] \]

(3.32)

and the force perpendicular to the direction of the plate fluctuations is

\[ \rho F_i \bigg|_{y=0} = \frac{2 \alpha \mu \eta \beta^3 \sqrt{Pr}}{h^3} \left[ \frac{\cos 2\beta - \cosh 2\beta}{\cosh 2\beta + \cosh 2\beta(\cosh 2\beta - \cos 2\beta)} \right. \]

\[ \left. + \sqrt{Pr} \frac{\sin 2\beta}{\cosh 2\beta + \sin 2\beta} \sin \sigma t (\sin \sigma t - \cos \sigma t) \right] \]

(3.33)

**C) FORCED OSCILLATIONS OF A FLUID BOUNDED BY A RIGID BOTTOM**

Let the thermo-viscous fluid of depth \( h \) bounded by the rigid bottom \( y = 0 \) be influenced by the external force \( F_y \exp (i\sigma t) \) in the \( x \)-direction. The equations of motion and energy now get modified as

\[ \rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + \rho F_y \exp \sigma t \]

(3.31)
\[0 = \mu \frac{\partial^2 u}{\partial y^2} + \rho F,\]  
\[0 = -\alpha \frac{\partial}{\partial y} (u_\eta) + \rho F,\]  
\[\rho c \frac{\partial \eta}{\partial t} = \mu \left( \frac{\partial u}{\partial y} \right)^2 + k \frac{\partial^2 \eta}{\partial y^2}\]  
with the no slip condition on the bottom
\[u(0,t) = 0\]  
and the free surface condition
\[S_{\eta, x} = 0\]  
the boundary conditions for temperature are
\[\eta(0,t) = \eta e^{\sigma t}\]  
and
\[\eta(h,t) = 0\]

Where \(\eta\) is the excess of the fluid temperature over that at the free surface.

Assuming the velocity distribution as
\[u(y,t) = g(y) e^{\sigma t}\]

from the equations (3.C.1), (3.C.9) and the boundary conditions (3.C.5), (3.C.6), the velocity distribution is obtained as
\[u(y,t) = \text{Re} \frac{F_\sigma}{\sigma} \left[ \frac{\cosh m(h-y)}{\cosh mh} - 1 \right] e^{\sigma t}\]
\[= \frac{F_\sigma}{\sigma} \left[ [1 - G_i(y)] \sin \sigma t - H_i(y) \cos \sigma t \right]\]  
where \(G_i(y)\) and \(H_i(y)\) are given in (3.B.13) and (3.B.14)

From the equations (3.C.4), (3.C.7) and (3.C.8) we get the temperature distribution as
\[\eta(y,t) = \eta_i \text{Re} \frac{\sinh \sqrt{Pr} \delta (1+i)(h-y)}{\sinh \sqrt{Pr} (1+i)h} e^{\sigma t}\]
\[= \eta_i [G_i(y) \cos \sigma t - H_i(y) \sin \sigma t]\]
where \( G_1(y), H_1(y) \) are given (3.B.21) and (3.B.22).

Further,

\[
\rho F_y = \Re \mu \frac{\partial}{\partial y} u_i^2 = \frac{2F_0}{\nu} \mu \left[ -G_1(y) \sin \sigma t - H_1(y) \cos \sigma t \right] \left[ -(\nu^{11}_1(y) \sin \sigma t - H_1^{11}(y) \cos \sigma t) \right] \tag{3.C.12}
\]

\[
\rho F_z = \Re \frac{\partial}{\partial z} (u, \eta, \) \\
\alpha \left[ -G_1^{11}(y) \cos \sigma t - H_1^{11}(y) \cos \sigma t \right] \left[ G_1^{11}(y) \cos \sigma t - H_1^{11}(y) \cos \sigma t \right] \\
+ \left[ G_1^{11}(y) \sin \sigma t - H_1^{11}(y) \cos \sigma t \right] \left[ G_1^{11}(y) \cos \sigma t - H_1^{11}(y) \sin \sigma t \right] \tag{3.C.13}
\]

On the plate \( y = 0 \)

\[
\rho F_{y, z = 0} = \frac{2\mu F_0}{\nu} \beta^2 \left[ \sin 2\beta + \sinh 2\beta \right] \sin 2\sigma t \\
+ \left( \sinh 2\beta - \sin 2\beta \right) \left( 1 + \cos 2\sigma t \right) \frac{(\cos 2\beta - \cosh 2\beta)}{(\cos 2\beta + \cosh 2\beta)} \tag{3.C.14}
\]

 Giving the cross-viscosity effect normal to the plate

\[
\rho F_{z, z = 0} = \alpha \left[ -(\cos 2\beta - \cosh 2\beta) \frac{(2\sqrt{\Pr} \beta + \sinh 2\sqrt{\Pr} \beta)}{\cosh 2\sqrt{\Pr} \beta - \cos 2\sqrt{\Pr} \beta} \right] \cos^2 \sigma t \\
+ \sqrt{\Pr} \left( \sin 2\beta + \sinh 2\beta \right) \sin^2 \sigma t \tag{3.C.15}
\]

Exhibiting the thermo-viscous effect perpendicular to the flow direction in the plane of the plate.
RESULTS AND DISCUSSIONS

The normal stress given by equations (3.A.27), (3.B.32) and (3.C.14) is the Reiner-Rivlin effect attributed to the cross-viscosity coefficient \( \mu \). Further the transverse force \( \rho F_i \) given by the equations (3.A.28), (3.B.33) and (3.C.15) can be attributed to the thermo-viscous nature of the fluid characterized by the coupling parameter thermo-stress-viscosity coefficient \( \alpha_s \) and this is independent of thermo-stress coefficient \( \alpha_c \). These effects are not felt in the case of classical viscous fluids i.e. when the fluid is Newtonian-viscous and Fourier-heat conducting.