SECTION-I

INTRODUCTION
CHAPTER - 1
later studied some steady state problems dealing with certain flows of thermo-viscous fluids.

The present thesis deals with a sub-class of fluids called thermo-viscous fluids characterized by Koh and Eringen [12]. We briefly summarize here under the basic equations of continuum mechanics and the constitutive equations of thermo-visco-elastic fluids. Equations of thermo-viscous fluids are obtained as a special case from the general theory.

**Conservation Principles**

Any material regardless of its mechanical and thermal properties must satisfy certain conservation principles, namely conservation of mass, momentum, angular momentum and energy. The mathematical expressions of these principles for a continuum are listed hereunder for ready reference.

a) **Conservation of mass:**

\[
\dot{\rho} + (\rho \nu_k)_{,k} = 0
\]  

(1.1)

where \(\rho\) is the mass density and \(\nu_k\) is the velocity in \(k\)-th direction, the comma(,) preceding a subscript indicates covariant differentiation and a dot (.) on the top denotes the differentiation with respect to time.

b) **Conservation of linear momentum:**

\[S_{mk,m} + \rho (f_k - a_k) = 0\]

(1.2)

where \(S_{mk}\), \(f_k\) and \(a_k\) are respectively the stress tensor, the body force and acceleration in the \(k^{th}\)-direction.

c) **Conservation of angular momentum:**

In the absence of body couples, this law reduces to

\[S_{mk} = S_{km}\]

(1.3)
i.e. the stress tensor is symmetric. This law is identically satisfied by all non-polar materials for which incidentally form a special case, the thermo-viscous fluids that are being dealt in the present work.

(d) Conservation of energy

\[ \rho \dot{\varepsilon} = S_{ni} d_{km} - q_{k,i} + \rho \gamma \]  

(1.4)

where \( \dot{\varepsilon} \) is the time rate of change of the specific internal energy \( \varepsilon \), \( q_k \) are the components of the heat flux vector \( \vec{q} \), \( \gamma \) is the supply of energy, and \( d_{km} \) are the components of the deformation rate tensor \( \ddot{\varepsilon} \) defined as

\[ 2d_{km} = v_{k,m} + v_{m,k} \]  

(1.5)

The specific internal energy change rate \( \dot{\varepsilon} \) in the equation (1.4) reduces for a fluid to a function of the specific entropy \( \eta \) only, which reduces for an incompressible fluid to

\[ \dot{\varepsilon} = c \dot{\theta} \]  

(1.6)

where \( c \) is the specific heat of the material. Substituting this in (1.4), we get for an incompressible fluid

\[ \rho c \dot{\theta} = S_{ni} d_{km} - q_{k,i} + \rho \gamma \]  

(1.7)

Constitutive Equations

Different materials having the same geometry and subjected to identical external forces would, in general, respond differently. This difference in response may be attributed to the constitution of the material. The mathematical equations which characterize such response-variations in a body may be called as the constitutive equations for the material.

A properly formulated constitutive equation must satisfy certain invariant principles [26]. Of these, the following are particularly
important in the formulation of the constitutive equations of thermo-viscous fluids.

a) **Principle of determinism:**

This principle is based on physical considerations and it may be expressed as follows.

The stress tensor $\tilde{S}(\tilde{x}, \tau)$ and the heat flux vector $\tilde{q}(\tilde{x}, \tau)$ at the spatial point $\tilde{x}$ at time $\tau$ are determined by the past history of motion of an arbitrarily small neighborhood of the material point $\tilde{x}$ (which occupies the position $\tilde{x}$ at time $\tau$) and the past thermodynamic history of this neighbourhood. Both $\tilde{S}$ and $\tilde{q}$ must be functionals of some kinematic and thermodynamic variables such as the rate of deformation tensor and thermal bigradient vector which characterize the materials with memory.

The thermo-viscous fluids belong to a class of materials that are conscious of the present state $\tilde{x}$ and at that state possesses a perfect memory of some past state $\tilde{x}$, but are oblivious to the intervening configurations of the body.

b) **Principle of material objectivity:**

This principle in effect states that the response of the material in a given event must be independent of an observer i.e., the constitutive equations must be invariant with respect to any rigid rotation of spatial coordinates.

c) **Principle of equipresence:**

The constitutive theory of thermo-viscous fluids stems out of strict adoption of this principle. This principle states that an independent variable present in either the stress equation or in the heat flux equation that would make up a set of thermo-mechanical constitutive relation must
also be present in the other. In other words, we express the stress tensor \( \vec{S} \) and heat flux vector \( \vec{q} \) as functions of the same kinematic and thermodynamic variables i.e. deformation rate tensor \( \vec{d} \) and the temperature bigradient vector \( \vec{b} \) as presented here under.

**Constitutive equations for thermo-viscous fluids**

A thermo-viscous fluid (i.e. a viscous fluid in a thermal state) is characterized by two sets of constitutive equations, one for the stress and the other for the heat.

The stress tensor \( \vec{S} \) and heat flux bivector \( \vec{h} \) are polynomial functions of the kinematic tensor: the deformation rate tensor \( \vec{d} \), the thermal bigradient vector \( \vec{b} \), the density \( \rho \) and the temperature \( \theta \).

\[
b = \| b_y \| = |\varepsilon_{iik} \theta_i |
\]

\[
h = \| h_y \| = |\varepsilon_{iik} q_i |
\]  
(1.8)  
(1.9)

with \( \varepsilon_{ik} \) representing the permutation symbol.

In other words, for thermo-viscous fluids,

\[
\vec{S} = \vec{S}(d, h; \rho, \theta)
\]

\[
\vec{h} = \vec{h}(d, h; \rho, \theta)
\]

(1.10)  
(1.11)

These equations can be seen to be in harmony with the principles of determinism, material objectivity and equipresence. Further, these functions are hemitropic polynomial functions of tensors \( d_{km} \) and \( h_{km} \) [11].

It can thus be noticed that

\[
S = \alpha_1 I + \alpha_2 d + \alpha_3 d^2 + \alpha_4 b^2 + \alpha_5 (db - bd)
\]

\[
+ \alpha_{16} (db^2 - b^2 d) + \alpha_{17} (bd^2 - d^2 b) + \alpha_{18} (d^2 b^2 - b^2 d^2)
\]

\[
+ \alpha_{20} (dbd^2 - d^2 bd) + \alpha_{21} (bd^2 - d^2 bd) + \alpha_{24} (bd^2 b^2 - b^2 d^2 b)
\]

\[
h = \beta I b + \beta_3 (bd + db) + \beta_4 (db^2 - b^2 d)
\]

(1.12)

5
the constitutive coefficients $\alpha_i$ and $\beta_i$ above are polynomials in the invariants of $d$ and $h$, namely

$$tr \, d, tr \, d^2, tr \, d^3, tr \, h^2, tr \, db^2, tr \, d^2h^2, tr \, hdb^2d^2$$

with coefficients depending on $\rho$ and $\theta$.

Further

$$\alpha_i tr \, d + \alpha_i tr \, d^2 + \alpha_i tr \, d^3 \geq 0$$

and

$$\beta_i \leq 0$$

These thermo-viscous fluids form an extended class of Stokesian fluids. The equations (1.10) and (1.11) reduce to constitutive equations of Stokesian fluid under isothermal conditions ($h = 0$). The Newtonian viscous fluid theory can be subsequently derived from these equations by linearization.

As is done by Rivlin and Erickson [23] for visco-elastic fluids and by Noll [20, 21] for simple materials, the thermo-viscous materials may be classified according to the combined degree of the independent variables $d$ and $h$ appearing in each term of the constitutive equations. If the degrees of $d$ and $h$ appearing in a particular term of the above constitutive equations be $N$ and $P$, then the combined degree of that term is $N + P$.

**a) Zero-order theory:**

For the case $N + P = 0$, (i.e., $N = 0$ and $P = 0$) all the constitutive coefficients with exception of $\alpha_i$ are equal to zero. The constitutive equations now reduce to

$$S = \alpha_i l$$

$$h = 0$$
where \( \alpha_i \) is independent of \( d \) and \( b \) is simply a function of \( \rho \) and \( \theta \). The materials characterized by these equations are nothing but the so called ideal (non-viscous and non-heat conducting) fluids with \( \alpha_i \) \((<0)\) representing the hydrostatic pressure.

**b) First-order theory:**

In this case the \( \text{Max} |N+P|=1 \). Therefore, the constitutive equations for this case become

\[
S = \alpha_1 I + \alpha_3 d
\]

\[
h = \beta_0 b
\]

where the constitutive coefficients \( \alpha_1, \alpha_3 \), and \( \beta_0 \) are scalar polynomials in \( \text{tr}d \) with coefficients which are functions of \( \rho \) and \( \theta \). For this order of approximation, we obtain

\[
\alpha_1 = \alpha_{1000} + \alpha_{1010} \text{tr}d
\]

\[
\alpha_3 = \alpha_{3010}
\]

\[
\beta_0 = \beta_{001}
\]

where the secondary coefficients \( \alpha_{n,n} \) and \( \beta_{n,n} \) are functions of \( \rho \) and \( \theta \).

The fluids characterized by these equations \((1.19)\) and \((1.20)\) may be termed as linear thermo-viscous fluids. The equation \((1.19)\) may be recognized as the constitutive equations of the first order simple fluid of Noll's type or that of the classical Newtonian-viscous fluids. The equation \((1.20)\) is same as the linear law of heat conduction i.e. Fourier law. For this reason, the fluids characterized by these equations \((1.19)\) and \((1.20)\) have been referred to as classical Newtonian-viscous and Fourier-heat conducting fluids in the foregoing chapters.

It may be noticed that from the zeroth and first order theories, the constitutive relations are decoupled in deformation rate tensor \( d \) and thermal bigradient-vector \( b \) and therefore both these theories do not
truly describe the interaction between the viscous and thermal characteristics of the medium. It is therefore necessary to consider higher order theories which exhibit such interactions.

c) **Second-order thermo-viscous fluids:**

For this case, we have \( \text{Max } |N + P| = 2 \). Thus we have the constitutive equations as

\[
S = \alpha_1 + \alpha_2 d + \alpha_3 d^2 + \alpha_4 h^2 + 2\alpha_5 (dh - hd) \\
(1.22)
\]

\[
h = \beta_1 h + 2\beta_2 (dh + hd) \\
(1.23)
\]

where the constitutive coefficients \( \alpha_i \) and \( \beta_i \) are scalar polynomials in \( trd, trd^2 \) and \( tr^2 \). Explicit expressions for the constitutive coefficients \( \alpha_i \) and \( \beta_i \) for the second order theory may be obtained as follows \[11\]

\[
\begin{align*}
\alpha_1 &= \alpha_{1000} + \alpha_{1010} tr d + \alpha_{1020} tr d^2 + \alpha_{1020} (tr d)^2 + \alpha_{1020} tr h^2 \\
\alpha_3 &= \alpha_{3010} + \alpha_{3020} tr d \\
\alpha_5 &= \alpha_{5020} \quad \alpha_6 = \alpha_{6002} \\
\alpha_8 &= \alpha_{8011} \\
\beta_1 &= \beta_{1011} + \beta_{1011} tr d \\
\beta_3 &= \beta_{3011}
\end{align*}
\]

(1.24)

The secondary coefficients \( \alpha_{mn} \) and \( \beta_{mn} \) are functions of \( \rho \) and \( \theta \).

The fluids defined by the constitutive equations (1.22) and (1.23) with constant values for the constitutive coefficients \( \alpha_i \) and \( \beta_i \) may be called Koh-Eringen second order thermo-viscous fluids (shortly Koh-Eringen fluids). This is the simplest model for a thermo-viscous fluid which could contain the interaction between mechanical and thermal phenomena in both the constitutive equations.

Higher order theories of thermo-viscous fluids can be realized in a similar fashion. It needs no special elaboration to recognize that the interaction between the mechanical and thermal responses would only be evident from the second order theory onwards.
Block diagrams of the constitutive equations (1.22) and (1.23) of second order thermo-viscous fluids:

A) Stress constitutive equation (1.22):

\[
\text{Stress} = S = \text{Isotropic stress} = -pl \\
+ \text{Newtonian stress due to rate of strain} = 2\mu d \\
+ \text{Reiner -Rivlin cross-stress} = 4\mu d^2 \\
+ \text{Stress generated due to temperature variations (gradients)} = \alpha_\nu \beta \beta^T \\
+ \text{Stress generated due to interaction between (mechanical) strain-rates and thermal variation} = 2\alpha_s (\beta \delta - \delta \beta) \\
\]

This is not taken care of when fluid is Newtonian-viscous

\[
S = -pl + 2\mu d + 4\mu_c d^2 + \alpha_\nu \beta \beta^T + 2\alpha_s (\beta \delta - \delta \beta)
\]
B) Thermal constitutive equation (1.23):

\[
\text{Heat flux bivector } = h, \quad h = -k\beta + 2\alpha_1(hd + db)
\]

Heat flux due to the interaction between the thermal variations and (mechanical) rate of strain = \(2\alpha_1(hd + db)\).

This is not taken care of when the fluid is Newtonian viscous and Fourier heat-conducting.

In the above constitutive equations.

\[
\begin{align*}
\alpha_i &= -p : \quad p &= \text{fluid pressure} \\
\alpha_3 &= 2\mu : \quad \mu &= \text{coefficient of classical (Newtonian) viscosity} \\
\alpha_s &= 4\mu_c : \quad \mu_c &= \text{coefficient of (Reiner-Rivlin) cross-viscosity} \\
\alpha_e &\quad &\text{thermo-stress coefficient} \\
\alpha_s &\quad &\text{thermo-stress viscosity coefficient.} \\
-\beta_1 &= k : \quad k &= (\text{Fourier}) \text{ thermal conductivity coefficient.} \\
\beta_3 &\quad &\text{strain thermal conductivity coefficient.}
\end{align*}
\]
The aim of present thesis is to study some steady and unsteady flows of a second order thermo-viscous incompressible fluids characterized by the equations [1.22] and [1.23] with the constitutive coefficients \( \alpha, \beta \) as constants without bearing any dependence on the invariants of \( d \) and \( b \).

The basic equations which describe the physical situations handled in the foregoing chapters are the following.

**Equation of continuity:**

\[
\nu_{ij} = 0
\]  \hspace{1cm} (1.25)

**Equation of momentum:**

\[
\rho \nu_i \nu_j = \rho F_i + S_{n,j}
\]  \hspace{1cm} (1.26)

**Equation of energy:**

\[
\rho c \nu_i \theta_{,j} = S_{m} d_{km} - q_{d,k} + \rho y
\]  \hspace{1cm} (1.27)

together with the constitutive equations (1.22) and (1.23)

Solving a specific boundary value problem would mean, finding the solution of these equations with the appropriate boundary conditions, the no slip condition (i.e. the velocity of fluid relative to the boundary is zero) and the prescription of the wall temperature. The later condition may also be replaced by the prescription of heat flux on the boundary.
The author's contribution to the subject incorporated in this thesis is broadly divided into seven chapters (2 to 8), the contents of which are presented here under.

**Chapter-2:**

The steady flow of a second order thermo-viscous fluid between two parallel plates in relative motion under the influence of a constant pressure gradient is investigated. The differential equations characterizing the temperature and velocity fields have been solved with the appropriate boundary conditions using a finite difference scheme [2]. Numerical work has been carried out for various values of thermo-viscous parameters \( \alpha_v \) and \( \tau \).

It is observed that the point of maximum velocity is shifted towards the hotter plate, whereas in the classical case the velocity attains its maximum in the middle of the channel. The convection effects are considerably more for thermo-viscous fluids when compared to the classical Newtonian-Fourier heat conducting flows. A reversal of the velocity profile is noticed for smaller values of thermo-viscous parameter.

**Chapter-3:**

The unsteady fluctuating plane flows of a second order thermo-viscous fluid over a flat plate is examined. The equations of motion and energy are decoupled in the absence of a constant temperature gradient.

The following three problems are solved analytically. A) Flow of an infinite fluid over a fluctuating bottom B) Flow of a fluid of finite depth with its bottom oscillating C) Forced oscillations of a fluid of finite depth over a fixed bottom.
From the equation of momentum in \( y \)-direction, the effect of cross-viscosity in the direction perpendicular to the plane is obtained. It is also noticed that a force is generated perpendicular to the direction of the plate fluctuations. It can be noted that \( \rho F_\gamma \) is due to the cross viscosity \( \mu_c \) (Riener-Rivlin coefficient) while \( \rho F_z \) is due to the thermo-viscous nature of the fluid. Both these two force components vanish when the fluid is Newtonian-viscous. For Riener-Rivlin fluids \( F_\gamma = 0 \) and for thermo-viscous fluids \( F_\gamma \neq 0, F_z \neq 0 \). This is a significant feature of thermo-viscous fluids.

**Chapter 4:**

The steady flow of a thermo-viscous fluid through a long moving circular pipe is examined. The velocity and temperature distributions are obtained as closed form solutions, involving modified Bessel functions, of the equations of motion and energy and their profiles are illustrated graphically. It is observed that for smaller values of thermo-viscous parameter, the velocity of the fluid is more than that of velocity of the pipe at the center and there after gradually decreases.

It is also noticed, there exists a value for the thermo-viscous parameter \( (m^2 = 0.5) \) for which the velocity remains the same as that of the velocity of the pipe. For \( m^2 < 0.5 \) the temperature increases at a faster rate. This effect may be attributed to the greater conversion of the fluid kinetic energy to the thermal energy. It is noticed by Erickson that a purely rectilinear flow down a circular pipe can be maintained under the influence of a constant axial pressure gradient. This would not be the case for a thermo-viscous fluid flow in a circular pipe.
Chapter-5:

The steady flow of a thermo-viscous fluid around a moving circular pipe, moving with a given velocity is examined. The velocity and temperature profiles are illustrated graphically. For small values of the thermo-viscous parameter, the velocity of the fluid increases from that of the velocity of the pipe, with the increase of the distance from the pipe. For the values of the radius >> 10, the velocity approaches a constant value. For large values of thermo-viscous parameter, the velocity decreases from that of the velocity of the pipe and then remains constant at large distances. It is also observed that the velocity remains same as that of the velocity of the pipe, for the value of the thermo-viscous parameter \( m^2 = 0.5 \).

Chapter-6:

The steady rectilinear flow of a thermo-viscous fluid in an annulus is examined. The expressions for the velocity and temperature distributions are obtained in terms of modified Bessel functions [26] and these are illustrated graphically. The expressions for the pressure, flow rate, components of stresses and the Nussult number, drag force on the inner and outer boundaries are obtained. It is noticed that these are influenced by the parameters namely Reiner-Rivlin cross viscosity \( \mu_r \), thermo stress coefficient \( \alpha_s \), thermo stress viscosity \( \alpha_s \) and thermal strain conductivity \( \beta_t \).

The force generated to sustain the flow is calculated from the equation of motion in the transverse direction. Also, for small and large values of thermo-viscous parameters, the velocity and temperature distributions are derived.
The following special cases are discussed:

(i) The steady flow under the influence of a constant pressure gradient and in the absence of temperature gradient is examined. (ii) The flow under the influence of a constant temperature gradient and in the absence of pressure gradient is studied.

**Chapter-7:**

The steady flow of a thermo-viscous fluid around and within a rotating sphere is examined. Analytical expressions for the velocity and the temperature distributions as well as the components of stresses, the pressure, the frictional couple and drag force on the sphere and the force generated to sustain purely rotary motion are obtained. The body force generated depends on the thermo-viscous parameters $\alpha_v$ and $\alpha_k$ and also on the Reiner-Rivlin cross viscosity parameter $\mu_t$. When the fluid is Newtonian $F_{\mu} = 0$, which is the significant feature of thermo-viscous fluids.

**Chapter-8:**

An unsteady flow of a thermo-viscous fluid around and within an oscillating sphere is studied. The expressions for the velocity and temperature distributions are obtained and these are same as those of Newtonian-Fourier heat conducting fluids. The components of stresses, pressure and the drag force are obtained and these are influenced by the thermo-viscous parameters. The non-vanishing of the drag is a significant feature of thermo-viscous fluids. The drag vanishes when the fluid is Newtonian and also for Reiner-Rivlin fluids. The viscous couple on the sphere is obtained, which is same that for the Newtonian flow.