CHAPTER - 8
UNSTEADY FLOW OF A THERMO-VISCOUS FLUID AROUND AND WITHIN AN OSCILLATING SPHERE

INTRODUCTION

In this chapter the unsteady flow of a second order thermo-viscous fluid filled in an oscillating sphere is examined. Under the hypothesis of slow motion the basic equations have been simplified by neglecting the terms which are non-linear in the derivatives of the velocity and temperature. However the non-linear terms are retained in the equations that determine the pressure. These terms are taken into consideration to highlight the effect of thermo-viscous parameters.

For small and large frequencies the velocity and temperature field are calculated. For small frequencies the stresses, pressure distribution the force required to sustain the flow, the couple and the drag force are influenced by the thermo-viscous parameters \( \alpha_v, \alpha_\kappa \) and the cross viscosity coefficient \( \mu_c \).

The oscillatory flows A) around and B) within an oscillating sphere are obtained. The rotary oscillations of a sphere surrounded by a Newtonian viscous liquid was investigated by Stokes [24] and Basset [3] and these have been reported by Lamb [15].

FORMULATION AND SOLUTION OF THE PROBLEM

Consider the unsteady flow of a second order thermo-viscous fluid filled in an oscillating sphere. Choosing a spherical coordinate system \((r, \theta, \phi)\), where \( r \) is the distance from the center of the sphere, \( \theta = 0 \) is the
axis of rotation and $\phi$ is the azimuthal angle. By axial symmetry all the physical quantities are independent of the azimuthal angle $\phi$. The boundary of the sphere is denoted by $r = a$.

The flow is characterized by the velocity distribution $[0,0,w(r,t)]$ and the temperature distribution $\eta(r,t)$. This choice of velocity satisfies the continuity equation. Under the hypothesis of slow motion as stated earlier, the non-linear terms in the equations of momentum in the azimuthal direction and energy are neglected to facilitate exact determination of the primary velocity and temperature fields.

The basic equations characterizing the flow are the following:

in the radial direction:

$$\frac{\partial p}{\partial r} = -\rho \frac{w^2}{r} + \mu \left( \frac{1}{r} \left( \frac{\partial w}{\partial r} \right)^2 + \frac{\partial}{\partial r} \left( \frac{\partial w}{\partial r} \right) \right)$$

$$+ \alpha_r \left( -\frac{2}{r} \frac{\partial \eta}{\partial r} \right) + \alpha_s \left( -\frac{\cot \theta \partial \eta}{r} \frac{\partial w}{\partial r} \right)$$

(8.1)

in the transverse direction:

$$pF_\theta = -\rho \frac{w^2 \cot \theta}{r} + \frac{2 \partial p}{\partial \theta}$$

(8.2)

in the azimuthal direction:

$$\rho \frac{\partial w}{\partial t} = \mu \left( \nabla^2 w - \frac{w}{r^2 \csc^2 \theta} \right)$$

(8.3)

and energy equation:

$$\rho c \frac{\partial \eta}{\partial t} = k \nabla^2 \eta$$

(8.4)

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{r} \frac{\partial}{\partial \theta}$$
The equation of motion in radial direction (8.1) is used to
determine the pressure distribution, whereas the thermo-viscous effect-
the force \( (\rho F_r) \) generated in the fluid due to its thermo-viscous nature can
be got from the equation of motion in the transverse direction.

A) UNSTEADY FLOW AROUND AN OSCILLATING SPHERE

The present investigation is on an unsteady flow of an
incompressible thermo-viscous fluid around a solid sphere oscillating
about one of its diameters with an angular velocity with small amplitude.
Further the temperature of the sphere is also pulsating with the period as
the angular velocity but with a different amplitude \( \eta_1 \).

In the absence of temperature gradient, the momentum and energy
equations (8.3) and (8.4) reduce to

\[
\rho \frac{\partial \mathbf{w}}{\partial t} = \mu \left[ \nabla \mathbf{w} - \frac{w}{r} \cos \theta \csc \theta \right] \tag{8.A.1}
\]

and

\[
\rho c \frac{\partial \eta}{\partial t} = k \nabla^2 \eta \tag{8.A.2}
\]
on the boundary of sphere \( r = a \), we have

\[
w(a, \theta, t) = \text{Re} \ a \dot{\Omega} e^{\alpha t} \sin \theta \tag{8.A.3}
\]
and

\[
\eta(a, \theta, t) = \text{Re} \ \eta_1 e^{\alpha t} \tag{8.A.4}
\]
and far away from the sphere

\[
w(\infty, \theta, t) = 0 \tag{8.A.5}
\]
and

\[
\eta(\infty, \theta, t) = 0 \tag{8.A.6}
\]
where \( \eta_1 \) is a constant amplitude of the oscillating temperature on the
sphere.

Introducing the following non-dimensional quantities:

\[
r = aR, \ w(r, \theta, t) = a\dot{\Omega}W(R, \theta, \tau) \text{ and } \eta(r, \theta, t) = T(R, \theta, \tau) \tag{8.A.7}
\]
The momentum and energy equations (8.A.1) and (8.A.2) using (8.A.7) reduce to

\[ \nabla^2 \omega - \frac{\omega}{R^2} \cos \theta \partial \partial \theta = \frac{\mu}{\Omega} \frac{\partial \theta}{\partial \tau} \]

(8.A.8)

and

\[ \nabla^2 T = Pr \frac{\partial \theta}{\partial \tau} \]

(8.A.9)

where \( Pr = \frac{\mu}{k} \) is the Prandtl number.

Using (8.A.7), the boundary conditions (8.A.3),(8.A.4),(8.A.5) and (8.A.6) reduced to

\[ W(1, \theta, \tau) = \Omega e^{\alpha} \sin \theta, \quad W(x, \theta, \tau) = 0 \]

(8.A.10)

and

\[ T(1, \theta, \tau) = \alpha e^{\alpha}, \quad T(x, \theta, \tau) = 0 \]

(8.A.11)

i.e. on the boundary of the sphere, the velocity and temperature are prescribed and far away these are assumed to be zero.

Let the velocity be assumed in the form

\[ W(R, \theta, \tau) = \Re F(R) e^{\alpha} \sin \theta \]

(8.A.12)

The equation satisfied by \( F(R) \) is

\[ F'' + \frac{2K^2}{R} F \left( \frac{2}{R^2} + m^2 \right) = 0 \]

(8.A.13)

with the boundary conditions

\[ F(1) = 1, \quad F(\infty) = 0 \]

(8.A.14)

Using these, the velocity distribution is

\[ W(R, \theta, \tau) = \Re \frac{J_{\frac{1}{2}}(\delta i^2 R)}{\sqrt{R}} \frac{J_{\frac{1}{2}}(\delta i^2 \tau)}{\sqrt{\tau}} e^{\alpha} \]

(8.A.15)

Let the temperature distribution be assumed in the form

\[ T(R, \theta, \tau) = \Re \left[ G(R) + H(R) \cos \theta \right] e^{\alpha} \]

(8.A.16)

128
Substituting this in the equation (8.A.9), we get the equations satisfied by $G(R)$ and $H(R)$

\[
G^{(1)} + 2 \frac{G^{(1)}}{R} - m^2 \text{Pr} G = 0 \tag{8.A.17}
\]

and

\[
H^{(1)} + 2 \frac{H^{(1)}}{R} - 2 \frac{H}{R^2} - m^2 \text{Pr} H = 0 \tag{8.A.18}
\]

with boundary conditions

\[
G(1) = 1, G(\infty) = 0 \tag{8.A.19}
\]

and

\[
H(1) = 1, H(\infty) = 0 \tag{8.A.20}
\]

we thus get

\[
\int_{R}^{1/R} \frac{J_{1/2}(\delta \sqrt{\text{Pr} R/r})}{J_{1/2}(\delta \sqrt{\text{Pr} R/r})} = 0 \tag{8.A.21}
\]

The solution of the energy equation (8.A.9) together with the boundary conditions (8.A.11) is

\[
T(R, \theta, \tau) = \text{Re} \frac{1}{\sqrt{R}} \int_{R}^{1/R} \frac{J_{1/2}(\delta \sqrt{\text{Pr} R/r})}{J_{1/2}(\delta \sqrt{\text{Pr} R/r})} e^{i\sigma \tau} \tag{8.A.22}
\]

Using the asymptotic expansions for Bessel functions, the velocity and temperature fields are obtained as

\[
W(R, \theta, \tau) = \frac{\sin \theta}{R^2 (X_{\theta1} + Y_{\theta1})} \left[ (X_{\theta1} X_{\theta1} + Y_{\theta1} Y_{\theta1}) \cos \sigma \tau \right] - \left[ Y_{\theta1} X_{\theta1} + X_{\theta1} Y_{\theta1} \right] \sin \sigma \tau \tag{8.A.23}
\]

\[
T(R, \theta, \tau) = \frac{1}{R} \frac{1}{(L_{\theta1}^2 + M_{\theta1}^2)} \left[ (L_{\theta1} L_{\theta1} + M_{\theta1} M_{\theta1}) \cos \sigma \tau \right] - \left[ (M_{\theta1} L_{\theta1} - L_{\theta1} M_{\theta1}) \right] \sin \sigma \tau \tag{8.A.24}
\]
where
\[ X_n = \cos \frac{\delta R}{2} \cosh \frac{\delta R}{2} - \sin \frac{\delta R}{2} \sinh \frac{\delta R}{2} - \sqrt{2} \delta R \sin \frac{\delta R}{2} \cosh \frac{\delta R}{2} \]  \hspace{1cm} (8.3.25)
\[ Y_n = \sin \frac{\delta R}{2} \sinh \frac{\delta R}{2} + \cos \frac{\delta R}{2} \cosh \frac{\delta R}{2} + \sqrt{2} \delta R \cos \frac{\delta R}{2} \sinh \frac{\delta R}{2} \]  \hspace{1cm} (8.3.26)

and \( X_n, Y_n \) are the values of \( X_n, Y_n \) at \( R = 1 \).

Also
\[ I_n = \cos \frac{\delta n Pr}{2} \cosh \frac{\delta n Pr}{2} \]  \hspace{1cm} (8.3.27)
\[ M_n = \sin \frac{\delta n Pr}{2} \sinh \frac{\delta n Pr}{2} \]  \hspace{1cm} (8.3.28)

with \( I_n, M_n \) are the values of \( I_n, M_n \) at \( R = 1 \).

**Approximations:**

Using the approximations of Bessel functions [26] for small values of \( \nu \)
\[ j_n(\nu) = \frac{2^n}{\nu^n} \]  \hspace{1cm} (8.3.29)

For small \( m \), the velocity distribution reduces to
\[ W(R, \theta, r) = \frac{\sin \theta \cos \sigma \tau}{R} \]  \hspace{1cm} (8.3.30)

and the temperature distribution to
\[ T(R, \theta, r) = \frac{\cos \sigma \tau}{R} \]  \hspace{1cm} (8.3.31)

From these, we observe that the velocity and temperature distributions do not depend on thermo-viscous parameters and are same as those of classical Newtonian-Fourier heat conducting fluids.

Using the equation of motion in \( r \)-direction and the calculated velocity and temperature fields, the pressure distribution is
\[ P = \left[ \frac{3}{R^4} \sin^2 \theta - \mu \frac{315 \sin^2 \theta}{R^4} + \alpha_s \frac{8 \eta_i \cos \theta}{\rho a^2 \Omega^2} \right] \cos^2 \sigma \tau \]  

(8.A.32)

The force required to sustain this flow is calculated from equation of motion in transverse direction and is given by

\[ \rho P = \left[ -\frac{1}{R^4} \sin 2\theta - \frac{6 \mu a \Omega^2}{R^4} \sin 2\theta - \frac{630 \mu}{R^4} \sin 2\theta \right. \]

\[ \left. - \frac{48 a \eta_i}{\rho a^2 \Omega R^3} \sin \theta + \frac{9 \mu \Omega^2}{a^2} \sin 2\theta - \frac{6 a \eta_i \Omega}{a^2 \sin \theta} \right] \cos^2 \sigma \tau \]  

(8.A.33)

The components of stresses:

\[ S_{xx} = -\frac{3}{R^4} \sin^2 \theta - \frac{315 \mu a \sin^2 \theta}{R^4} + \frac{8 \alpha_s \eta_i}{\rho a^2 \Omega^2} - \frac{\alpha_s 24 \eta_i \cos \theta}{\rho a^2 \Omega R^3} \cos^2 \sigma \tau \]  

(8.A.34.1)

\[ S_{rr} = \frac{2 \mu \Omega^2}{R^4} \sin \theta \cos \sigma \tau \]  

(8.A.34.2)

\[ S_{rr} = 0 \]  

(8.A.34.3)

\[ S_{xx} = -\frac{3}{R^4} \sin^2 \theta \cos^2 \sigma \tau + \mu \frac{\sin^2 \theta}{R^4} \left( \frac{9 \alpha_i \Omega^2}{R^4 \cos^2 \sigma \tau} + \frac{315}{R^4} \cos^2 \sigma \tau \right) \]

\[ + \frac{\alpha_s \eta_i \sin^2 \sigma \tau}{a^2} \left( 1 - \frac{8}{\rho a \Omega R} \right) - \frac{\alpha_s 24 \eta_i \cos \theta}{\rho a^2 \Omega R^3} \cos^2 \sigma \tau \]  

(8.A.34.4)

\[ S_{yy} = 0 \]  

(8.A.34.5)

\[ S_{xy} = -\frac{3}{R^4} \sin^2 \theta \cos \sigma \tau + \mu \frac{\sin^2 \theta}{R^4} \left( \frac{9 \alpha_i \Omega^2}{R^4 \cos^2 \sigma \tau} + \frac{315}{R^4} \cos^2 \sigma \tau \right) \]

\[ + \frac{\alpha_s \eta_i \sin^2 \sigma \tau}{a^2} \left( 1 - \frac{8}{\rho a \Omega R} \right) - \frac{\alpha_s 24 \eta_i \cos \theta}{\rho a^2 \Omega R^3} \cos^2 \sigma \tau \]  

(8.A.34.6)

From the above, we observe that the components of stresses, the force generated are influenced by thermo-viscous parameters \( \alpha_s, \alpha_i \) and cross-viscosity \( \mu \).
The couple on the sphere:

$$\int_{\phi=0}^{\phi=\pi} 2\pi r \left[ S_{\phi \phi} \sin^2 \theta \right] d\theta$$

$$= 8\pi a^4 \mu \Omega \cos \sigma t$$  \hspace{1cm} (8.A.35)

which is same as that for the Newtonian flow.

The drag force on the sphere:

$$\text{drag} \propto \frac{8\pi \mu}{\rho} \left[ r^2 \left[ S_{\phi \phi} \cos \theta \sin \theta - S_{\theta \theta} \sin^2 \theta \right] \right]$$

$$= \frac{32\pi \mu}{3} \left[ \frac{\mu}{\rho \Omega^2} + \frac{4}{3} \right] \cos^3 \sigma t$$  \hspace{1cm} (8.A.36)

The non-vanishing of the drag is a significant feature of thermoviscous fluids. The drag vanishes when the fluid is Newtonian and also when it is Reiner-Rivlin fluid. This is also true for the general Stokesian fluids.

The couple and the drag force are influenced by the thermo-stress parameter $\alpha_s$.

**B) FLOW INSIDE AN OSCILLATING SPHERE**

Consider the unsteady slow motion of a second order thermoviscous fluid filled in a rotating sphere about the axis of rotation with given angular velocity $\Omega$. The velocity and temperature at the center of the sphere are finite / regular.

The equations of motion in $\phi$-direction and the energy equations are respectively.

$$\rho \frac{\partial \psi}{\partial t} = \mu \left[ \nabla^2 w - \frac{w}{r^2} \cos \phi \right]$$  \hspace{1cm} (8.B.1)

and

$$\rho c \frac{\partial \eta}{\partial t} = k \nabla^2 \eta$$  \hspace{1cm} (8.B.2)
the boundary conditions in this case are

\[ w(0, \theta, t) = \text{finite}, \quad w(a, \theta, t) = a \Omega e^{-\gamma t} \sin \theta \]  

(8.B.3)

and

\[ \eta(0, \theta, t) = \text{finite}, \quad \eta(a, \theta, t) = \eta e^{-\gamma t} \]  

(8.B.4)

Employing the non dimensional quantities as

\[ r = a R, \quad w(r, \theta, t) = a \Omega W(R, \theta, \tau) \quad \text{and} \quad \eta(r, \theta, t) = \eta \mathcal{F}(R, \theta, \tau) \]  

(8.B.5)

Using the non dimensional quantities defined in (8.B.5) the momentum equation (8.B.1) can be written as

\[ \nabla^2 W - \frac{W}{R \cos \theta} \frac{1}{\mu \Omega} \frac{\partial^2 W}{\partial \tau^2} = \frac{P \sigma^2}{\mu \Omega} \nabla \cdot \mathcal{F} \]  

(8.B.6)

The boundary conditions (8.B.3) and (8.B.4) using (8.B.5) reduce to

\[ W(0, \theta, \tau) = \text{finite}, \quad W(1, \theta, \tau) = e^{-\gamma \tau} \sin \theta \]  

(8.B.7)

and

\[ \mathcal{F}(0, \theta, \tau) = \text{finite}, \quad \mathcal{F}(1, \theta, \tau) = e^{-\gamma \tau} \]  

(8.B.8)

Assuming the velocity distribution in the form,

\[ W(R, \theta, \tau) = \Re \chi \mathcal{F} e^{-\gamma \tau} \sin \theta \]  

(8.B.9)

we get

\[ F'' + \frac{2}{R} F' + \left( \frac{2}{R^2} + m^2 \right) F(R) = 0 \]  

(8.B.10)

where

\[ m^2 = \frac{\sigma^2}{\nu \Omega} \]  

the boundary conditions for \( F(R) \) are

\[ F(1) = 1 \quad \text{and} \quad F(0) = 0 \]  

(8.B.11)

from (8.B.10), we have

\[ F(R) = \frac{A}{\sqrt{R}} J_1(\delta i^2 R) + \frac{B}{\sqrt{R}} J_2(\delta i^2 R) \]  

(8.B.12)
using the conditions for $F(R)$ given in (8.B.11), we get

$$F(R) = \frac{1}{\sqrt{R}} \frac{J_1(\delta'^2 R)}{J_{1/2}(\delta'^2)}$$

(8.B.13)

where $\delta^2 = a \sigma \gamma^2 \Omega$

and $m = \delta(1 + i)$

Hence the velocity field is obtained as

$$W(R, \theta, \tau) = \frac{1}{\sqrt{R}} \frac{J_1(\delta'^2 R)}{J_{1/2}(\delta'^2)} e^{i m \sin \theta}$$

(8.B.14)

It is noted that the motion represented by equation (8.B.14) is not a rigid body rotation while that for Newtonian fluids, the motion is a rigid body rotation.

Using the non-dimensional quantities defined in (8.B.5), the energy equation (8.B.2) reduces to

$$\nabla^2 T = Pr \frac{\alpha^2}{\nu \Omega \dot{\gamma} \tau}$$

(8.B.15)

where $Pr$ is the Prandtl number.

Assuming the temperature distribution in the form

$$T(R, \theta, \tau) = Re \left[ G(R) + H(R) \cos \theta \right] e^{i m \tau}$$

(8.B.16)

Substituting this in the equation (8.B.15), we get after separating variables, the equations satisfied by $G(R)$ and $H(R)$ are:

$$G'' + 2 \frac{G'}{R} - m^2 Pr G = 0$$

(8.A.17)

and

$$H'' + 2 \frac{H'}{R} - \left( \frac{2}{R^2} + m^2 Pr \right) H = 0$$

(8.A.18)
with the boundary conditions

\[ G(0) = \text{regular}, G(1) = 1 \]  

and

\[ H(0) = \text{regular}, H(1) = 0 \]  

We thus get

\[ G(R) = \frac{1}{\sqrt{R}} \frac{J_\ell(\sqrt{\pi} R)}{J_\ell(\sqrt{\pi} \sqrt{R})} \]  

and

\[ H(R) = 0 \]

The temperature distribution is obtained as

\[ T(R, \phi) = \text{Re} \left( \frac{1}{\sqrt{R}} \frac{J_\ell(\sqrt{\pi} R)}{J_\ell(\sqrt{\pi} \sqrt{R})} \right) \]

Using the standard relation of Bessel functions [26] in terms of trigonometric functions,

\[ J_1(x) = \frac{2}{\pi} \sin x, \quad J_2(x) = \frac{2}{\pi} \cos x \]

and

\[ J_\ell(x) = \frac{2}{\pi} \left( \frac{\sin x - \cos x}{\sqrt{x}} \right), \quad J_\ell'(x) = -\frac{2}{\pi} \left( \sin x + \frac{\cos x}{x} \right) \]

Using these values for Bessel’s functions, the velocity distribution can be given as

\[ W(R, \theta, \tau) = \frac{\sin \theta}{R^2} \frac{1}{(C_{\ell 1} + S_{\ell 1})} \left\{ (C_{\omega} C_{\alpha 1} + S_{\omega} S_{\alpha 1}) \cos \sigma \tau - (C_{\alpha 1} S_{\omega} - S_{\alpha 1} C_{\omega}) \sin \sigma \tau \right\} \]  

(8.24)
where
\[ C_R = \sin \frac{\delta R}{2} \cosh \frac{\delta R}{2} + \cos \frac{\delta R}{2} \sinh \frac{\delta R}{2} - \sqrt{2} \delta R \cos \frac{\delta R}{2} \cosh \frac{\delta R}{2} \]
\[ S_R = \frac{1}{2} \sin \frac{\delta R}{2} \cosh \frac{\delta R}{2} - \cos \frac{\delta R}{2} \sinh \frac{\delta R}{2} - \sqrt{2} \delta R \sin \frac{\delta R}{2} \sinh \frac{\delta R}{2} \]
and \( C_R, S_R \) are the values of \( C_R, S_R \) at \( R = 1 \) respectively.

The temperature field is realized as
\[ T(R, \theta, \tau) = \frac{1}{R(P_R Q_R + Q_R P_R)} \left[ (P_R Q_R + Q_R P_R) \cos \sigma \tau - (P_R Q_R + Q_R P_R) \sin \sigma \tau \right] \]
(8.B.25)

where
\[ P_R = \sin \frac{\delta R}{2} \cosh \frac{\delta R}{2} - \cos \frac{\delta R}{2} \sinh \frac{\delta R}{2} \]
\[ Q_R = \cos \frac{\delta R}{2} \cosh \frac{\delta R}{2} + \sin \frac{\delta R}{2} \sinh \frac{\delta R}{2} \]
and \( P_R, Q_R \) are the values of \( P_R, Q_R \) at \( R = 1 \).

Approximations:

For small values of \( \delta \), we have the following approximations from [26]
\[ J_\ell(x) = \frac{1}{2^n n!} x^n \]
\[ J_{-\ell}(x) = \frac{(-1)^n}{2^n n!} x^n \]

Hence for small frequency, i.e., for small values of the parameter \( \delta \), the velocity and temperature fields are approximately:
\[ W(R, \theta, \tau) = R \sin \theta \cos \sigma \tau \]
(8.B.26)
and
\[ T(R, \theta, \tau) = \cos \sigma \tau \]
(8.B.27)
we observe that these are same as that of the classical Newtonian-Fourier-heat conducting fluids.
The components of stresses:

\[ S_{zz} = S_{rr} = S_{\theta\theta} = -P \]  \hspace{1cm} (8.8.28)
\[ S_{r\theta} = S_{r\phi} = S_{\phi\phi} = 0 \]  \hspace{1cm} (8.8.29)

The pressure distribution:

\[ P = \frac{-R^2}{2} \sin^2 \theta \cos^2 \sigma \theta \]  \hspace{1cm} (8.8.30)

The drag force on the sphere:

\[ \text{drag}\; \eta = \frac{8 \pi \mu}{\rho} \int_0^\pi r^3 \left( R_{zz} \cos \phi \sin \theta - R_{r\theta} \sin^2 \theta \right) d\theta = 0 \]  \hspace{1cm} (8.8.31)

The couple on the sphere:

\[ \int_{\theta=0}^{\theta=\pi} 2\pi r^2 \sin^2 \phi \sin \theta d\theta = 0 \]  \hspace{1cm} (8.8.32)

The external force generated to sustain this flow is calculated as

\[ \rho \mathbf{F}_e = \frac{R}{2} \sin 2\theta \cos \sigma \theta \]  \hspace{1cm} (8.8.33)

From the above it can be seen that for the flow of thermo-viscous fluids in the absence of temperature gradient, the thermo-viscous parameter does not have any effect on the flow but it is observed that a force is generated in transverse direction to sustain the flow.