CHAPTER 2

DESIGN OF UNIFORM DFT FILTER BANKS
2. DESIGN OF UNIFORM DFT FILTER BANKS

Slow convergence and high computational complexity are the drawbacks of time domain adaptive filters. To improve the convergence rate while reducing the computational complexity associated with time domain adaptive filters subband adaptive filters have been proposed. Subband processing introduces transmission delay caused by the filters in the filter bank and signal degradations due to aliasing effects. To allow a higher sample rate than critically needed in the subbands and thus reduce subband signal degradation, is one efficient way to reduce the aliasing effect. This result in introducing optimally exploited additional degree of freedom in the design of filter banks. This chapter suggests a design method for modulated uniform filter banks with any oversampling, where the analysis filter bank and the total filter bank delay may be specified with minimized aliasing and magnitude/phase distortions.

Uniformly modulated filter banks are designed and performance evaluations are conducted. Filter banks from the suggested method are compared with conventional filter banks.

2.1 Introduction:

Filter banks have been of great interest in a number of signal processing applications. A large group of these applications comprise those utilizing subband adaptive filtering. Reduction in computational complexity and the increase in convergence speed for the adaptive algorithm, is achieved by dividing the algorithm into subbands, [11]. Because of this reason Subband Adaptive filtering has been proposed as an alternative for
conventional time domain adaptive filter [108]. The computational savings comes from the fact that time domain convolution becomes decoupled in the subbands, at a lower sample rate, [120]. Examples of applications where subband adaptive filtering is successfully applied are acoustic echo cancellation [107, 108, 109, 110, 111], speech enhancement [80], signal separation [112], and beam-forming.

The sampling rate can be reduced in the subbands, to decrease the complexity of filter bank structures. These filter banks, referred to as decimated filter banks, are afflicted with three major types of distortions: amplitude, phase and aliasing distortion. These distortions degrade the application performance of the filter bank used.

Several design methods have been proposed and evaluated on subband adaptive filtering applications. Prototype filters for modulated filter banks are designed by interpolation of a two-channel Quadrature Mirror Filter (QMF), and evaluated in a real-time acoustic echo cancellation application in [106]. Using an iterative least-squares algorithm a design method is proposed where a reconstruction error and the stopband energy are simultaneously minimized in [113]. Other methods have been proposed for the design of perfect reconstruction or para-unitary filter banks, which is a class of perfect-reconstruction filter banks. A method using non-constrained optimization is described in [114], for the design of perfect reconstruction polyphase filter banks with arbitrary delay, aimed at applications in audio coding. Kliewer proposed a method for linear-phase prototype FIR filters with power complementary constraints for cosine modulated filter banks, based on an improved frequency-sampling design [115]. In [116], the design of perfect reconstruction cosine-modulated para-unitary filter banks is discussed. Heller presented a design method for prototype filters for perfect reconstruction cosine-modulated filter banks with arbitrary prototype filter length and delay in [117].
Subband analysis and synthesis is often performed using multirate filter banks [67]. Non-ideal filters in the filter bank cause aliasing of the subband signals. This aliasing can be cancelled in the synthesis bank when certain conditions are met by the synthesis filters and in the subband processing. However, even if aliasing distortion in the filter bank output is cancelled in this way, the inband aliasing is still present in the subband adaptive filter input signals. Consequently, the adaptive filters are perturbed and the overall performance of the system is reduced. [109].

Several solutions to the subband filtering problem have been suggested in literature. Non-critical decimation has been suggested in [108], where filter bank delay aspects, and amplitude distortions, have not especially been taken into consideration. The use of cross filters, [109], has been suggested to explicitly filter out the aliasing components. A delayless structure has been proposed in [41], where the actual filtering with higher computational complexity is performed in the time domain. The computational complexity also increases significantly with cross band filters.

Because of their efficient implementation and their simplicity [118, 67] due to the polyphase implementation, uniformly modulated filter banks have been of special interest. In this chapter, using unconstrained quadratic optimization, a design method for analysis and synthesis uniformly modulated filter banks is presented. The goal of the design method is to minimize magnitude, phase, and aliasing distortion in the reconstructed output signal, caused by the filter bank, as well as to minimize aliasing in the subband signals. Aliasing affects the performance of subband adaptive filters. The goal is to design filter banks, taking into consideration that adaptive filtering in the subbands should cause minimal degradation.
The proposed filter bank design method uses pre-specified design parameters which control the filter bank properties, such as:

- Number of subbands
- Decimation factor
- Analysis and synthesis filter lengths
- Analysis filter bank delay
- Total delay of the analysis-synthesis structure

2.2 Design of $M$-Channel Filter Banks & Uniformly Modulated Filter Banks:

2.2.1 Decimated Filterbanks with Parallel Structure:

An $M$-channel analysis filter bank is a structure transforming an input signal to a set of $M$ subband signals. A corresponding $M$-channel synthesis filter bank transforms $M$ subband channels to a full band signal, in other words the inverse operation. Different kinds of structures of filter banks exist, for example the tree structure and the parallel structure [6]. The parallel structure is more general and favourable because of its simplicity. In Fig. 2.1, the analysis and synthesis filter banks with parallel structure are illustrated.

![Fig. 2.1: Parallel structure of general M-channel analysis and synthesis filter banks](image-url)
The analysis filter bank consists of $M$ analysis filters, $H_m(z)$, with non-overlapping bandwidth. The first analysis filter, $H_0(z)$, is usually of lowpass type and the other filters are of bandpass type. All filters may generally have arbitrary bandwidth. A special case is the uniform filter bank, where all filters have the same bandwidth.

The input signal, $x(n)$, is filtered by the analysis filters to form the signals $v_m(n)$, $m = 0, \ldots, M - 1$,

$$v_m(n) = h_m(n) * x(n) \leftrightarrow V_m(z) = H_m(z)X(z)$$

(2.1)

The reduced band width of $v_m(n)$ allows for a reduction of the sample rate. Reducing the sample rate by decimation gives rise to efficient implementation of the filter bank. The decimation factor, $D_m$, generally depends on the bandwidth of the corresponding signal $v_m(n)$,

$$x_m(l) = v(ID_m) \leftrightarrow X_m(z) = \frac{1}{D_m} \sum_{d=0}^{D_m-1} V_m(z^{D_m} e^{-j2\pi dl/D_m})$$

(2.2)

where a new discrete time variable $l$ is introduced for the signals with lower sample rate. Note that $x_m(l)$ consists of $D_m$ terms, which will be referred to as aliasing terms. In subband adaptive filtering, filters are applied on the subband signals $x_m(l)$,

$$y_m(l) = \xi_m(l) * x_m(l) \leftrightarrow Y_m(z) = \xi_m(z)X_m(z)$$

(2.3)

The output signals, $y_m(l)$ of the subband filters $\xi_m(l)$, are then transformed to the fullband output signal $y(n)$ by a synthesis filter bank. The synthesis filter bank consists of interpolators and synthesis filters $G_m(z)$. The input subband signals, $y_m(l)$, are interpolated,
\begin{equation}
    u_m(n) = \begin{cases} 
    y_m(n/D_m), & n = 0, \pm D_m, \pm 2D_m, \ldots \\
    0, & \text{otherwise}
    \end{cases} \leftrightarrow U_m(z) = Y_m(z^{1/n}),
\end{equation}

and then filtered by the synthesis filters. Finally, the signals are summed together to form the output signal, \( y(n) \).

\begin{equation}
    y(n) = \sum_{n=0}^{M-1} g_m(n) \ast u_m(n) \leftrightarrow y(z) = \sum_{n=0}^{M-1} G_m(z)U_m(z)
\end{equation}

### 2.2.2 Design Issues of Decimated Filter Banks:

Different types of distortions inflict the Analysis-Synthesis filter bank structures. These distortions among other important filter bank properties are illustrated in Fig. 2.2. Without any filtering in the subbands, i.e. \( \xi_m(z) = 1 \), the desired response of the total analysis-synthesis structure has unit amplitude and linear phase. Henceforth this response will be referred to as the total response. Deviations from these requirements are referred to as amplitude distortion and phase distortion.

![Fig. 2.2: Important properties in filter bank design](image-url)
The decimation and interpolation operations cause aliasing and imaging distortion. The stopband of the analysis filters attenuates the spectral portions of the input signal, which appear in the subband signals, $x_m(l)$, as undesired aliasing components. The stopband of the synthesis filters attenuates the multiple images of the subband spectrum, which appear in the signals, $u_m(n)$, after interpolation. Depending on how the analysis and synthesis filters are designed, and on the subband filtering, residual aliasing might be present in the output signal, $y(n)$.

Filter design procedures can be regarded as optimization procedures with error functions, which are minimized in a certain way. One of the ways in which error functions may be minimized, is by using the least squares criterion.

In the design of $M$-channel filter banks, the least squares criterion can be stated as

$$\min_{h_m, g_m} \int_{\omega} |E_{h_m, g_m}(\omega)|^2 \, d\omega$$

(2.6)

where $E_{h_m, g_m}(\omega)$ is the error function which depends on the impulse responses of the analysis and synthesis filters $h_m$ and $g_m$, and contains the desired properties of the filter bank. The least squares error in Eq. (2.6) needs generally to be solved using non-linear optimization procedures. However, when the analysis and synthesis filters are defined from single modulated prototype filters, in uniformly modulated filter banks, and when the analysis filters are designed prior to the synthesis filters, the design problem can be divided into two sequential quadratic optimization problems. This design procedure will be described in detail in Sections 2.3 and 2.4.
2.2.3 Uniformly Modulated Filter Banks:

An M-channel modulated filter bank consists of a set of M branches. Each branch consists of an analysis filter $H_m(z)$, a decimator with decimation factor $D$, an interpolator with interpolation factor $D$, and a synthesis filter $G_m(z)$. An illustration of such a filter bank is given in Fig. 2.3, where subband filtering is applied in the subband signals.

All subbands have equal bandwidth and the same decimation and interpolation factors.

The analysis and synthesis prototype filters are FIR filters of length $L_h$ and $L_g$, respectively.

In order to construct a uniform filter bank, which means that all subbands have the same bandwidth, low pass analysis and synthesis prototype filters, $H(z)$ and $G(z)$, are defined. All analysis filters in the filter bank are modulated versions of the prototype analysis filter according to

$$h_m(n) = h(n)W^{-mn}_M = h(n)e^{j2\pi mn/M} \leftrightarrow H_m(z) = H(zW^m_M) \quad (2.7)$$

Similarly, all synthesis filters are modulated versions of the prototype synthesis filter according to

$$g_m(n) = g(n)W^{-mn}_M = g(n)e^{j2\pi mn/M} \leftrightarrow G_m(z) = G(zW^m_M) \quad (2.8)$$
where $W_M = e^{-j2\pi M}$. Observe that the analysis and synthesis filters in the first branch ($m=0$) are the same as the prototype filters, $H_0(z) = H(z)$ and $G_0(z) = G(z)$. An example of a modulated prototype filter is given in Fig. 2.4.

![Analysis Filter](image)

**Fig. 2.4: Analysis filters $H_0(z)$, ..., $H_3(z)$ for a modulated filter bank with $M = 4$ subbands.**

### 2.2.4 Filter Bank Response:

The input-output relation is derived in order to analyze filter bank properties. Each branch signal, $V_m(z)$, will simply be a filtered version of the input signal as

$$V_m(z) = H_m(z)X(z) = H(zW_M^m)X(z) \quad (2.9)$$

The decimators will expand the spectra of the branch signals according to

$$X_m(z) = \frac{1}{D^m} \sum_{\omega=0}^{D-1} V_m(z\frac{1}{D}W_M^\omega) = \frac{1}{D^m} \sum_{\omega=0}^{D-1} H(z\frac{1}{D}W_M^\omega W_M^\omega)X(z\frac{1}{D}W_M^\omega) \quad (2.10)$$
where \( W_D = e^{j2\pi D} \). The summation in Eq. (2.10) shows that the subband signals consist of \( D \) aliasing terms. Depending on the subband index and the decimation factor, the desired spectral content is present in one or more aliasing terms. This is shown in Fig. 2.5 for a critically sampled case and in Fig. 2.6 for an oversampled case.

Fig. 2.5: The influence of \( H_m(z) \) on the inband-aliasing terms for a case with \( M = D = 4 \).
The plots represent \( |H(e^{j2\pi mW_D})|^2 \) in dB for different \( m \) and \( d \). The sum over \( d \) of the terms is plotted in the right column.

Fig. 2.6: The influence of \( H_m(z) \) on the inband-aliasing terms for a case with \( M=4 \) and \( D=2 \). The plots represent \( |H(e^{j2\pi mW_D})|^2 \) in dB for different \( m \) and \( d \). The sum over \( d \) of the terms is plotted in the right column.
The subband signals $X_m(z)$ are processed in many filter bank applications. The filters in the subbands are denoted by $\xi_m(z)$. The processed subband signals, $Y_m(z)$, are filtered versions of the input subband signals, $X_m(z)$, according to

$$Y_m(z) = \xi_m(z)X_m(z) \quad (2.11)$$

The interpolators compress the spectra of the processed subband signals according to

$$U_m(z) = Y_m(z^{m'}) = \frac{1}{D} \xi_m(z^{m'}) \sum_{m=0}^{M-1} H(zW_{m}^*W_{m}^{'})X(zW_{m}) \quad (2.12)$$

After interpolation, multiple images of the signal spectrum will be present in $U_m(z)$, due to the repetitive character of the discrete signal spectrum and the compressing effect of the interpolator. The undesired images are attenuated by the reconstruction filters $G_m(z)$, whereafter the signals are added to form the output signal, $Y(z)$.

$$Y(z) = \sum_{m=0}^{M-1} U_m(z)X_m(z) \quad (2.13)$$

Inserting Eq. (2.12) into Eq. (2.13), the relation between the input signal, $X(z)$, and the output signal, $Y(z)$, becomes

$$Y(z) = \frac{1}{D} \sum_{m=0}^{M-1} X(zW_{m}^{*}) \sum_{m=0}^{M-1} \xi_m(z^{m'})H(zW_{m}^{*}W_{m}^{'})G(zW_{m}) \quad (2.14)$$

which is expressed in terms of the input signal, $X(z)$, the prototype filters, $H(z)$ and $G(z)$, and the subband filters, $\xi_m(z)$. For simplicity, the product of filters in Eq. (2.14) is defined as $A_m(z)$, according to
The influence of $A_{m,d}(z)$, with $\zeta_m(z) = 1$, on the aliasing terms is shown in Fig. 2.7 for a critically sampled filter bank and in Fig. 2.8 for an oversampled filter bank. The figures show how number modulated versions of the input signal are filtered and added together to form the output. Clearly the non-modulated input signal contributes with the desired spectral content while the terms for $d > 0$ are undesired aliasing terms which arise from the modulated versions of the input signal.

\begin{equation}
A_{m,d}(z) = \frac{1}{D} \zeta_m(z^{-1}) H(z W_d^m W_d^d) G(z W_d^m)
\end{equation}

The output signal in Eq. (2.14) can be rewritten in the more convenient form

\begin{equation}
Y(z) = \sum_{d=0}^{D-1} A_d(z) X(z W_d^d)
\end{equation}
where

$$A_d(z) = \sum_{m=0}^{M-1} A_{m,d}(z), \quad d = 0, \ldots, D-1$$

The transfer functions, \(A_d(z)\) for \(d = 1, \ldots, D-1\), can be viewed as the transfer functions which give rise to the residual aliasing terms in the output signal. The function, \(A_0(z)\), is the transfer function which gives rise to the desired output signal spectrum.

Fig. 2.8: The influence of \(A_{m,d}(z)\) on the aliasing terms in \(Y(z)\) for a filter bank with \(M = 4\) and \(D = M/2 = 2\). The plots represent \(|A_{m,d}(e^{j\omega})|^2\) in dB for different \(m\) and \(d\). The sum over \(d\) of the terms is plotted in the right column and the sum of all terms, \(T(z)\), is given below.

The total response function, \(T(z)\), for the filter-bank is given by the sum of the desired and undesired aliasing transfer functions, \(A_d(z)\)

$$T(z) = \sum_{d=0}^{D-1} A_d(z)$$
2.3 Design of Analysis Filter Bank:

2.3.1 Design Strategy:

The analysis filter bank design problem reduces to the design of a single prototype analysis filter, \( H(z) \), when the analysis filters are modulated versions of the prototype analysis filter according to Eq. (2.7). The purpose of the analysis filters is to split the original signal into a set of subband signals. Ideal analysis filters are bandpass filters with normalized center frequencies \( \omega_m = 2\pi \frac{m}{M}, m = 0, \ldots, M-1 \), and with bandwidth \( 2\pi/M \). The ideal filters have unit magnitude and zero phase in the passband while the stopband magnitude is zero. While zero phase filters require non-causality, the requirements need to be relaxed by using linear phase filters. FIR filters may have exact linear phase but they cannot possess the ideal magnitude requirements. Therefore approximations need to be made.

A straightforward way to design the prototype analysis filter is to design a lowpass filter, with a passband region centered around \( \omega = 0 \), and a minimum magnitude stopband region using filter design methods such as window techniques or the Parks-McClellan optimal equi-ripple FIR filter design method [86]. The bandwidth is controlled by the passband region and the attenuation of signal components in the stopband region leads to low-energy inband-aliasing terms. These methods cannot control the delay properties of the resulting filter. However, filter design techniques with complex approximation for filters with arbitrary phase exist [119].

The analysis filter bank may be designed so that the transfer functions of the analysis filters have power-complementary transfer functions, i.e. the sum of the squared filter magnitudes is unity [67].
If a prototype analysis filter is designed with this constraint in a modulated structure, the obtained bandwidth will be the maximum bandwidth in order to keep constant magnitude in the analysis filter bank. If the application does not demand the analysis filter bank to possess the power-complementary property, i.e. when a smaller bandwidth of the prototype analysis filters is allowed, a more general approach would be to define an arbitrary boundary frequency of the passband region.

An appropriate design criterion may be to minimize the following objective function

$$
\varepsilon_k = \frac{1}{2\omega_p} \int_{\omega_p}^{\omega_p} |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega
$$

(2.20)

where $H_d(z)$ is a desired frequency response of the prototype analysis filter in the passband region $\Omega_p = [-\omega_p, \omega_p]$. The desired frequency response is defined as

$$
H_d(e^{j\omega}) = e^{-j\tau_H\omega}, \quad \omega \in \Omega_p
$$

(2.21)

where $\tau_H$ is the desired group delay of the prototype analysis filter and, consequently, the desired delay of the whole analysis filter bank.

By only specifying the power complementary constraint, or more general, a complex specification for the passband region, a suitable prototype filter cannot be obtained. The stopband regions of the prototype analysis filters must also be defined otherwise significant inband-aliasing distortion may occur in the subbands.
One approach to combat the undesired inband-aliasing is to minimize the energy in the stopband which in turn leads to minimal energy in the aliasing terms. A more appealing approach would be to address the inband-aliasing directly in the objective function. This can be done by complementing the design criterion with the minimization of inband-aliasing distortion. In Section 2.2.3, the $M \times D$ aliasing terms in all subband signals were identified. The design criterion is complemented by adding the energies in all $M \times (D-1)$ aliasing terms into the objective function in Eq. (2.20). For a critically sampled filter bank, the objective function becomes

$$\varepsilon_h = \alpha_h + \beta_h$$  \hspace{1cm} (2.22)

with the Passband Response Error

$$\alpha_h = \frac{1}{2\omega} \int_{-\omega_s}^{\omega_s} |H(e^{i\omega}) - H_d(e^{i\omega})|^2 \, d\omega$$  \hspace{1cm} (2.23)

and the Inband-Aliasing Distortion

$$\beta_h = \frac{1}{2\pi D^2} \sum_{m=0}^{M-1} \sum_{d=D-1}^{D-1} \int_{-\omega_s}^{\omega_s} |H(e^{i\omega})W_mW_d|^2 \, d\omega$$  \hspace{1cm} (2.24)

where all inband-aliasing terms are included. Taking a closer look at the summation of aliasing terms, for the critically sampled case, $M$ equal terms can be identified, which are included $M$ times in the summation. So, due to the modulated structure, it is sufficient to include only the terms in the first subband ($m = 0$), i.e. the terms for $d = 1, \ldots, D-1$. Therefore, $\beta_h$ in Eq. (2.22) can be reduced to
\[
\beta_s = \frac{1}{2\pi D^2} \sum_{d=1}^{\lfloor D/2\rfloor} |H(e^{i\omega_d})| \quad (2.25)
\]

Aliasing terms for a critically sampled filter bank case with \( M = D = 4 \) were shown in Fig. 2.5. When oversampling is applied, it is not trivial to distinguish the desired or undesired parts in the aliasing terms since the desired spectral content may appear in more than only one aliasing term. This is shown for the two-times oversampled case \( M = 4, D = M/2 = 2 \) in Fig. 2.6. It can be seen from this example that the desired spectral content is spread over a maximum of two aliasing terms. Fig. 2.6 shows that inclusion of the aliasing terms only for the first subband is a reasonable approach, and in the two-times oversampled case also, since the undesired spectral portions are suppressed. Generally, the approach of minimizing inband aliasing with Eq. (2.25) holds for oversampling by any decimation factor. Henceforth, only the critically sampled and two times oversampled \( (D = M/2) \) filter banks are considered. In the next sections, the objective function of the analysis filter-bank design is derived in terms of the prototype analysis filter, \( H(z) \).

### 2.3.2 Passband Response Error:

In this section, the quadratic form of the passband response error in Eq. (2.23) is derived, expressed in terms of the impulse response of the analysis prototype filter. The passband response error can be rewritten as

\[
\alpha_s = \frac{1}{2\omega_p} \int_{-\omega_p}^{\omega_p} |H(e^{i\omega}) - H_s(e^{i\omega})|^2 \, d\omega \\
= \frac{1}{2\omega_p} \int_{-\omega_p}^{\omega_p} \left( H(e^{i\omega}) - H_s(e^{i\omega}) \right) \left( H(e^{i\omega}) - H_s(e^{i\omega}) \right)^* \, d\omega \\
= \frac{1}{2\omega_p} \int_{-\omega_p}^{\omega_p} \left| H(e^{i\omega}) \right|^2 - 2 \text{Re} \left\{ H_s(e^{i\omega}) H(e^{i\omega}) \right\} + \left| H_s(e^{i\omega}) \right|^2 \, d\omega \\
= \frac{1}{2\omega_p} \int_{-\omega_p}^{\omega_p} \left( H(e^{i\omega}) \right)^2 \, d\omega (2.26)
\]
where $H_d(z)$ is the desired frequency response, defined in Eq. (2.21). The prototype analysis filter response $H(z)$ is expressed in terms of its impulse response, $h(n)$, according to

$$H(z) = \sum_{n=0}^{L_h-1} h(n)z^{-n} = h^T \phi_k(z)$$ (2.27)

where $h = [h(0), \ldots, h(L_h-1)]^T$ and $\phi_k(z) = [1, z^{-1}, \ldots, z^{-L_h+1}]^T$. Substituting Eqs. (2.27) and (2.21) into Eq. (2.26) yields

$$\alpha_h = \frac{1}{2\omega_p} \int_{-\omega_p}^{\omega_p} \left[ h^T \phi_k(e^{j\omega}) \phi_k^H(e^{j\omega}) h - 2 \Re \left[ e^{j\omega \theta} h^T \phi_k(e^{j\omega}) \right] + 1 \right] d\omega$$ (2.28)

The passband response error, Eq. (2.28), can be rewritten in the quadratic form

$$\alpha_h = h^T A h - 2h^T b + 1$$ (2.29)

where the $L_h \times L_h$ matrix $A$ is

$$A = \frac{1}{2\omega_p} \int_{-\omega_p}^{\omega_p} \phi_k(e^{j\omega}) \phi_k^H(e^{j\omega}) d\omega$$ (2.30)

and the $L_h \times 1$ vector $b$ is

$$b = \frac{1}{2\omega_p} \int_{-\omega_p}^{\omega_p} \Re \left[ e^{j\omega \theta} \phi_k(e^{j\omega}) \right] d\omega$$ (2.31)

Calculating the integrals for all entries in matrix $A$ and vector $b$, the following expression for the matrix entries $A_{mn}$ is obtained
\[ A_{m,n} = \frac{\sin(\omega_p(n-m))}{\omega_p(n-m)} \]  

(2.32)

and the vector entries \( b_m \)

\[ b_m = \frac{\sin(\omega_p(r_m - m))}{\omega_p(r_m - m)}. \]  

(2.33)

where \( m = 0, \ldots, M-1 \) and \( n = 0, \ldots, L_h - 1 \).

### 2.3.3 Inband-Aliasing Distortion:

The inband-aliasing distortion term of the objective function, Eq. (2.25), is given by

\[
\beta_h = \frac{1}{2\pi D^2} \int_{\omega} \left| H(e^{j\omega t}W^d_{n}) \right|^2 d\omega
\]

\[
= \frac{1}{2\pi D^2} \int_{\omega} \sum_{d=1}^{L_h} \sum_{n=1}^{L_h} H(e^{j\omega t}W^d_{n})H^*(e^{j\omega t}W^d_{n}) d\omega
\]

(2.34)

Using Eq. (2.27), it can be rewritten as

\[
\beta_h = \frac{1}{2\pi D^2} \sum_{d=1}^{L_h} \sum_{n=1}^{L_h} \mathbf{h}^T \left[ \int_{\omega} \phi_n(e^{j\omega t}W^d_{n})\phi_n^*(e^{j\omega t}W^d_{n}) d\omega \right] \mathbf{h}
\]

(2.35)

which can be written in quadratic form

\[
\beta_h = \mathbf{h}^T \mathbf{C} \mathbf{h}
\]

(2.36)

where the \( L_h \times L_h \) hermitian matrix \( \mathbf{C} \) is defined as
Calculating the integral for the entries in matrix \( C \), the following expression for the matrix entry \( C_{mn} \) is obtained

\[
C_{mn} = \frac{\phi(n-m) \sin(\pi(n-m)/D)}{\pi D(n-m)}
\]

where

\[
\phi(n) = D \sum_{k=-\infty}^{\infty} \delta(n-kD) - 1.
\]

### 2.3.4 The Optimal Prototype Analysis Filter:

The objective function for the prototype analysis filter design, Eq. (2.22), expressed in terms of \( h \) is

\[
\varepsilon_h = \alpha_h + \beta_h
\]

\[
= h^T A h - 2h^T b + 1 + h^T C h
\]

\[
= h^T (A + C) h - 2h^T b + 1.
\]

The solution to the design problem

\[
h_{\text{opt}} = \arg \min_h h^T (A + C) h - 2h^T b + 1.
\]
2.4 Design of Synthesis Filter Bank:

2.4.1 Design Strategy:

Similar to the design of analysis filter banks, presented in Section 2.3, the design of synthesis filter banks reduces to the design of a single synthesis prototype filter. When designing the synthesis filter bank, the focus is on the performance of the analysis-synthesis filter bank as a whole. This implies that the synthesis filter bank is designed given an analysis filter bank, i.e., given the prototype analysis filter, designed using the method presented in the previous section. Different applications of filter banks require different strategies. The focus in the proposed method is on applications, which uses filtering operations in the subbands.

In the proposed design of the synthesis filter bank, the goal is to minimize amplitude and phase distortion of the analysis-synthesis filter bank and to minimize aliasing distortion in the output signal \( Y(z) \). The objective function is the least square error

\[
\mathcal{E}_s(h) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |E(e^{j\omega})|^2 d\omega
\]  

where

\[
E(z) = T(z) - D(z)
\]
denotes the total response error. Here, \( T(z) \) is the complex-valued system response defined in Eq. (2.18). The desired complex-valued analysis-synthesis filter bank response \( D(z) \) is defined for the special case with \( \zeta_m(z) = 1 \).

\[
D(z) = z^{-\tau_T}
\]  

(2.45)

where \( \tau_T \) is the desired total analysis-synthesis filter bank delay.

The complex error function in Eq. (2.44) can be written as

\[
E(z) = E_0(z) + \sum_{m=0}^{M-1} \sum_{d=1}^{D-1} E_{m,d}(z).
\]

The first term, \( E_0(z) \), contains the desired spectral components and the summation in the second term contains all the undesired aliasing terms. The error function for the desired spectral content, \( E_0(z) \), is defined as

\[
E_0(z) = \sum_{m=0}^{M-1} E_{m,0}(z) = A_0(z) - D_0(z)
\]

(2.46)

where the subband filters are unity, i.e. \( \zeta_m(z) = 1 \). The total response excluding the undesired aliasing terms, \( A_0(z) \), was defined in Eq. (2.17). The desired transfer function for \( d = 0 \) is

\[
D_0(z) = \sum_{m=0}^{M-1} D_{m,0}(z) = z^{-\tau_T}
\]

(2.47)

Note that \( D_0(z) = D(z) \) according to (2.45). This implies that the desired transfer functions, \( D_{m,d}(z) \), for \( d > 0 \) are zero, i.e. \( D_{m,d} = 0 \). The error functions for the undesired aliasing terms are defined as
\[ E_{m,d}(z) = A_{m,d}(z) - D_{m,d}(z) = A_{m,d}(z). \] (2.49)

Aliasing content may cancel out at the final reconstruction summation, so that the residual aliasing in the output signal \( Y(z) \) is zero, even though the individual terms in the error function may have large energy, refer Fig. 2.9 (a). In applications without subband filtering, a direct minimization of Eq. (2.43) may be a reasonable design strategy. In that case the filter bank will be a nearly-perfect reconstruction filter bank.

In applications where subband adaptive filtering is used, i.e. the subband filters, \( \xi_m(z) \), are time-variant and unknown, the cancellation of aliasing terms at the reconstruction summation cannot be controlled by a fixed prototype synthesis filter. To achieve this, the synthesis filter bank need to be time-variant and matched to the filters in the subbands.

However a fixed synthesis prototype filter may be designed so that the energy in the individual aliasing terms is minimized. The motive for this approach is illustrated in Fig. 2.9. The summation of errors in Eq. (2.46) may sum to zero in a perfect reconstruction case when the subband filters are known, as shown in Fig. 2.9 (a).

Fig. 2.9: Summation of complex errors, (a) shows error cancellation with large errors, (b) shows a large error case, and (c) shows a case where individually minimized vectors, compared to (b) yield a small total error.
When the magnitude and phase of error terms are altered by subband filters, which are depicted in Fig. 2.9 (b), the error may be large since the individual vectors are large. A simplified example is shown in Fig. 2.10, for two terms. The perfect reconstruction case with $\zeta_n(z) = 1$ is shown in Fig. 2.10 (a). In Fig. 2.9 (b), the maximum error is shown when the error vectors may be altered by the subband filters, $\zeta_n(z)$. The grey zone denotes the range in which the vectors may be altered. If the synthesis filter bank is designed so that the aliasing errors individually are small, the sum will also be small.

![Fig. 2.10](image)

Fig. 2.10: Simple case with two errors, (a) shows the zero error case, and (b) shows the worst case where the grey zones denote the range in which the amplitude and phase of the vectors may be altered by subband filtering.

The objective function is defined as

$$
\mathcal{E}_s(h) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| E_0(e^{j\omega}) \right|^2 d\omega + \sum_{n=0}^{N-1} \sum_{d=1}^{n} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| E_{n,d}(e^{j\omega}) \right|^2 d\omega
$$

(2.50)

where the energies of all undesired aliasing terms will be individually minimized independently of $\zeta_n(z)$. In summary, the proposed objective function of the design method is

$$
\mathcal{E}_s(h) = \mathcal{E}_y(h) + \mathcal{E}_\delta(h)
$$

(2.51)
where
\[
\chi (h) = \frac{1}{2\pi} \int_{\pi}^{\pi} \sum_{m=0}^{M-1} \left| A_{n,0}(e^{im\omega}) - D_0(e^{im\omega}) \right|^2 \, d\omega
\]  
(2.52)

is the Total Response Error and
\[
\delta (h) = \frac{1}{2\pi} \int_{\pi}^{\pi} \sum_{x=1}^{L-1} \sum_{n=0}^{M-1} \left| A_{n,x}(e^{im\omega}) \right|^2 \, d\omega
\]  
(2.53)

is the Residual Aliasing Distortion.

In the next two sections, the two terms \( \chi (h) \) and \( \delta (h) \) will be derived in terms of the impulse response of the synthesis prototype filter.
2.4.2 Total Response Error:

The total response error is

\[
\gamma(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| A_0(e^{j\omega}) - D_0(e^{j\omega}) \right|^2 d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( A_0(e^{j\omega}) - D_0(e^{j\omega}) \right) \left( A_0(e^{j\omega}) - D_0(e^{j\omega}) \right) d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ A_0(e^{j\omega})^2 - 2 \text{Re}\{D_0(e^{j\omega})A_0(e^{j\omega})\} + |D_0(e^{j\omega})|^2 \right] d\omega.
\]

(2.54)

where \(A_0(z)\) is defined in Eq. (2.17), and \(\zeta_m(z) = 1\) for \(m = 0, \ldots, M - 1\). As with the prototype analysis filter in Eq. (2.27), the prototype synthesis filter can be written in terms of its impulse response as follows

\[
G(z) = \sum_{n=0}^{L-1} g(n) z^{-n} = g' \phi_0(z)
\]

(2.55)

where \(g = [g(0), \ldots, g(L-1)]^T\) and \(\phi_0(z) = [1, z^{-1}, \ldots, z^{-L+1}]^T\).

The summation over \(m\) in Eq. (2.54) can be written in terms of the impulse responses \(h\) and \(g\). Inserting Eq. (2.27) and Eq. (2.55) into \(A_0(z)\), with \(\zeta_m(z) = 1\), yields

\[
A_0(z) = \frac{1}{D} \sum_{n=0}^{L-1} H(zW_M^n)G(zW_M^n)
\]

\[
= \frac{1}{D} \sum_{n=0}^{L-1} h' \phi_0(zW_M^n) \psi_0(zW_M^n) g
\]

\[
= h' \Psi(z) g.
\]

(2.56)
\[ \Psi(z) = \frac{1}{D} \sum_{m=0}^{D-1} \Phi_{m,d}(z) \]  

(2.57)

and

\[ \Phi_{m,d}(z) = \phi_m(zW_M^mW_P^d)\phi^\dagger_m(zW_M^m) \]  

(2.58)

Substituting Eq. (2.48), and Eq. (2.56) into Eq. (2.54) yields

\[ \gamma_{\text{g}}(h) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ h^\dagger \Psi(e^{im})g \right]^2 - 2 \text{Re} \left\{ e^{im} h^\dagger \Psi(e^{im})g \right\} + 1 \right] d\omega, \]

(2.59)

which can be rewritten as

\[ \gamma_{\text{g}}(h) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ g^\dagger \Psi^\dagger(e^{im})h^* h^\dagger \Psi(e^{im})g - 2 \text{Re} \left\{ e^{im} h^\dagger \Psi(e^{im})g \right\} + 1 \right] d\omega \]

(2.60)

Finally, the following quadratic form is obtained

\[ \gamma_{\text{g}}(h) = g^\dagger E g - 2 g^\dagger \mathbf{f} + 1, \]

(2.61)

where the \( L \times L \) hermitian matrix \( E \) is defined as

\[ E = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi^\dagger(e^{im})h^* h^\dagger \Psi(e^{im}) d\omega, \]

(2.62)

and the \( L \times 1 \) vector \( \mathbf{f} \) is defined as

\[ \mathbf{f} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Re} \left\{ e^{im} \Psi^\dagger(e^{im})h \right\} d\omega. \]

(2.63)
Calculating the integral for all matrix entries, the following expression for the matrix entries $E_{p,q}$ is obtained

$$E_{p,q} = \frac{M^2}{D^2} \sum_{k=\infty}^{k=\infty} h^*(kM-p)h(kM-q)$$  \hspace{1cm} (2.64)

Similarly, for the vector entries $f_p$

$$f_p = \frac{\lambda}{\pi D} h(\tau_r - p)$$  \hspace{1cm} (2.65)

with

$$\lambda = \sum_{m=0}^{M-1} \cos(2\pi \tau_r m / M)$$  \hspace{1cm} (2.66)

Here the total delay, $\tau_r$, is assumed to be an integer.

### 2.4.3 Residual Aliasing Distortion:

The residual aliasing distortion term in Eq. (2.51) is given by

$$\delta_q(h) = \frac{1}{2\pi} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \int_{\omega} |A_{m,n}(e^{j\omega})|^2 \, d\omega$$  \hspace{1cm} (2.67)

Eq. (2.67) can be rewritten using the impulse response vectors $h$ and $g$, defined in Eq. (2.27) and Eq. (2.55), respectively, and the matrix $\Phi_{m,d}(\omega)$, defined in Eq. (2.58) according to
This can be rewritten as

\[ \delta_\lambda(h) = \frac{1}{2\pi D^2} \sum_{d \neq \lambda} \sum_{m=0}^{(n_1-1)} \int_\pi g' \Phi_{m,\lambda}(\epsilon(\omega)) h * h' \Phi_{m,\lambda}(\epsilon(\omega)) g \, d\omega. \]  

(2.69)

which finally leads to the quadratic form

\[ \delta_\lambda(h) = g' P g \]  

(2.70)

where the \( L_n \times L_n \) hermitian matrix \( P \) is defined as

\[ P = \frac{1}{2\pi D^2} \sum_{d \neq \lambda} \sum_{m=0}^{(n_1-1)} \int_\pi \Phi_{m,\lambda}(\epsilon(\omega)) h * h' \Phi_{m,\lambda}(\epsilon(\omega)) \, d\omega. \]  

(2.71)

Calculating the integral for all matrix entries, the following expression for \( P_{pq} \) is obtained

\[ P_{pq} = \frac{M}{D^2} \sum_{l=0}^{\omega} h^* (l+q) h(l+p) \varphi(p-q), \]  

(2.72)

where

\[ \varphi(n) = D \sum_{k=-\infty}^{\infty} \delta(n-kD) - 1. \]  

(2.73)
2.4.4 The Optimal Synthesis Prototype Filter:

The optimal prototype analysis filter, in terms of minimal total response error and minimal energy in the aliasing components, is found by minimizing the objective function from Eq. (2.51)

$$\varepsilon_{\tau}(h) = \gamma_{\tau}(h) + \nu \delta_{\tau}(h). \quad (2.74)$$

Here, a weighting factor \( \nu \) is introduced in order to enable emphasis on either the total response error (0 < \( \nu \) < 1) or the residual aliasing distortion (\( \nu \geq 1 \)). Inserting Eq. (2.61) and Eq. (2.70) into (2.74) yields

$$\varepsilon_{\tau}(h) = g^T(E + \nu P)g - 2g^Tf + 1, \quad (2.75)$$

The solution

$$g_{\text{opt}} = \arg \min_g \varepsilon_{\tau}(h). \quad (2.76)$$

can be found by solving the set of linear equations

$$(E + \nu P)g = f. \quad (2.77)$$

2.5 Design Parameters:

2.5.1 Pre-Specified Parameters:

A number of parameters need to be set, in order to use the design method. The most important parameter is the number of subbands, \( M \). The filter lengths of the analysis and synthesis filters, \( L_a \) and \( L_s \), and the decimation factor, \( D \), are then chosen. The
filter lengths are commonly set to multiples of the decimation factor because it gives rise to the efficient polyphase implementation. The parameters mentioned here influence both performance and complexity, as will be seen in the following sections. Other design parameters are the passband boundary frequency, optional weighting in the synthesis filter bank optimization, the delay of the analysis filter bank, and the total delay.

2.5.2 Passband Boundary Frequency:

The bandwidth of the prototype analysis filter is set by choosing the passband boundary frequency \( \omega_p \). The parameter determines the bandwidth of all analysis filters and it is therefore important, since it might influence the performance of the application. Prototype analysis filters, obtained from Eq. (2.41), with different passband boundary frequencies are shown in Fig. 2.11. The figure illustrates that a lower stop band level, and thus lower inband-aliasing distortion, can be obtained by choosing a smaller passband boundary frequency.

![Prototype Analysis Filters](image)

Fig. 2.11: Prototype analysis filter responses with \( L_h = 8, M = 4 \) and \( D = 2 \). The frequency response of the filters is shown for passband boundaries \( \omega_p = \frac{\pi}{M}, \omega_p = \frac{\pi}{2M} \) and \( \omega_p = \frac{\pi}{4M} \).

The corresponding inband-aliasing distortion, \( \beta_m \), is also shown.
Fig. 2.12: Filter bank performance as a function of the passband boundary frequency. The solid line is the inband-aliasing distortion, the striped line is the residual aliasing distortion and the dotted line is the response error. The number of subbands is $M=4$ and the decimation factor is $D=2$. In the left figure, the filter lengths are set to $L_h=L_g=4$ and in the right figure $L_h=L_g=8$.

Fig. 2.13: Filter bank as a function of the passband boundary frequency. The solid line is the inband-aliasing distortion, the striped line is the residual aliasing distortion and the striped line is the total response error. The number of subbands is $M=8$ and the decimation factor is $D=4$. In the left figure, the filter lengths are set to $L_h=L_g=8$ and in the right figure $L_h=L_g=16$.

The passband boundary frequency also affects the total response error and the residual aliasing distortion since the synthesis filter bank design has a dependency on the analysis filter bank design. Figs. 2.12 and 2.13 show how the inband-aliasing distortion, $\beta_h$ in Eq. (2.25), total response error, $\gamma_h(h)$ in Eq. (2.52), and the residual aliasing distortion,
\( \delta_k(h) \) in Eq. (2.53), are affected by the passband boundary frequency parameters in the analysis filter bank design. All figures show that a passband boundary frequency lower than \( \omega_p = \frac{\pi}{M} \) generally yields lower aliasing distortions. The amplitude error has local minima while the inband and residual aliasing distortion decrease monotonically when the passband boundary frequency is decreased.

### 2.5.3 Weighting in the Synthesis Filter Bank Design:

The error weighting parameter, \( \nu \), can be used to control the emphasis in the optimization on either the filter bank response error or the residual aliasing distortion. Figs. 2.14 and 2.15 show the total response error and the residual aliasing distortion as a function of the weighting factor. When the weighting factor is decreased (\( \nu < 1 \)), the total response error decreases while the residual aliasing distortion converges to a certain level. By increasing the weighting factor the residual aliasing distortion can be decreased at the expense of the total response error, which converges to the maximal level of 0 dB.

![Design Measures](image1.png)

![Design Measures](image2.png)

Fig. 2.14: Synthesis filter bank performance as a function of the weighting factor. The striped line is the residual aliasing distortion and the dotted line is the total response error. The number of subbands is \( M=4 \) and the decimation factor is \( D=2 \). The passband boundary frequency is set to \( \omega_p = \frac{\pi}{8M} \). In the left figure, the filter lengths are set to \( L_h = L_g = 4 \) and in the right figure \( L_h = L_g = 8 \).
Fig. 2.15: Filter bank performance as a function of the weighting factor. The striped line is the residual aliasing distortion and the dotted line is the total response error. The number of subbands is $M = 8$ and the decimation factor is $D = 4$. The passband boundary frequency is set to $\omega_p = \pi/8M$. In the left figure, the filter lengths are set to $L_h = L_g = 8$ and in the right figure $L_h = L_g = 16$

2.5.4 Critical Sampling and Oversampling:

Oversampling ($D < M$) will increase the degrees of freedom in the synthesis filter design. The total response error and the residual aliasing distortion can be decreased by choosing a smaller decimation factor than the critical decimation factor ($D = M$). The minimum values of the total response errors and the residual aliasing distortions for different decimation factors in some specific filter-bank design scenarios are shown in Fig. 2.16.
Fig. 2.16: Minimum total response error, $\gamma_y(h)$, and minimum residual aliasing distortion, $\delta_y(h)$, as a function of the decimation factor D. The filter lengths are set to $L_h = L_q = 2M$. Each curve corresponds to a specific number of subbands, $M = 4, \ldots, 64$. In all cases, the passband boundary frequency is set to $\omega_p = \pi/8M$ and the prototype filters have linear phase. The black dots denote decimation factors chosen as powers of two.

2.5.5 Analysis Filter Bank Delay and Total Filter Bank Delay:

Fig. 2.17 illustrates the relation between reduced total delay and the filter bank performance. When setting the system delay at a fixed value, it can be of great importance to study how the delay of the analysis filter bank can be set in order to obtain low-distortion. This is shown in Figs. 2.18 and 2.19. The inband-aliasing distortion has a unique minimum at $\tau_{ff} = \frac{(L_k - 1)}{2}$ in all evaluation cases. Here the analysis prototype filters are linear phase filters with even-symmetric impulse responses, so that all degrees of freedom are used to obtain low inband-aliasing distortion.
Synthesis Filter Bank Design Measures

Fig. 2.17: Total response error and residual aliasing distortion as a function of \( \tau_H \) and \( \tau_T \). The number of subbands is \( M = 4 \). The decimation factor is set to \( D = 2 \). The filter lengths are set to \( L_h = L_g = 8 \) in the left figure and \( L_h = L_g = 16 \) in the right figure. The delay of the analysis filter-bank is set to \( \tau_H = \frac{1}{2} \).

Analysis Filter Bank Design Measure

Fig. 2.18: Filter bank performance measures as a function of the analysis prototype filter delay, \( \tau_0 \). The solid line is the inband-aliasing distortion, the striped line is the residual aliasing distortion, and the dotted line is the total response error. The number of subbands is \( M = 4 \), the decimation factor is \( D = 2 \) and the prototype filter lengths are set to \( L_h = L_g = 2M = 8 \). The desired system delay is set to \( \tau_T = 4 \) for the left figure and \( \tau_T = 8 \) for the right figure. The group-delay of the prototype analysis filter \( \tau_H \) is varied from 0 to \( \tau_T \), which means that \( \tau_0 \) varies from \( \tau_T \) to 0 since \( \tau_T = \tau_H + \tau_0 \).
Fig. 2.17: Total response error and residual aliasing distortion as a function of \( \tau_{\text{IH}} \) and \( \tau_T \). The number of subbands is \( M = 4 \). The decimation factor is set to \( D = 2 \). The filter lengths are set to \( L_h = L_g = 8 \) in the left figure and \( L_h = L_g = 16 \) in the right figure. The delay of the analysis filter-bank is set to \( \tau_{\text{IH}} = \frac{1}{2} L_h \).

Fig. 2.18: Filter bank performance measures as a function of the analysis prototype filter delay, \( \tau_a \). The solid line is the inband-aliasing distortion, the striped line is the residual aliasing distortion, and the dotted line is the total response error. The number of subbands is \( M = 4 \), the decimation factor is \( D = 2 \) and the prototype filter lengths are set to \( L_h = L_g = 2M = 8 \). The desired system delay is set to \( \tau_T = 4 \) for the left figure and \( \tau_T = 8 \) for the right figure. The group-delay of the prototype analysis filter \( \tau_{\text{IH}} \) is varied from 0 to \( \tau_T \), which means that \( \tau_G \) varies from \( \tau_T \) to 0 since \( \tau_T = \tau_{\text{IH}} + \tau_G \).
Fig. 2.19: Design errors as a function of the analysis prototype filter group-delay. The solid line is the inband aliasing distortion, the striped line is the residual aliasing distortion, and the dotted line is the total response error. The number of subbands is $M = 4$, the decimation factor is $D = 2$ and the prototype filter lengths are set to $L_h = L_g = 4M = 16$. The desired system delay is set to $\tau_f = 8$ for the left figure and $\tau_f = 16$ for the right figure. The delay of the prototype analysis filter $\tau_1$ is varied from 0 to $\tau_f$, which means that $\tau_1$ varies from $\tau_1$ to 0 since $\tau_f = \tau_1 + \tau_1$.

2.6 Efficient Implementation and Computational Complexity of Modulated Filter Banks:

In this section an efficient structure for two-times oversampled ($D = M/2$) analysis and synthesis polyphase modulated filter banks is derived.

2.6.1 Type I Polyphase Decomposition:

First the Type I Polyphase Decomposition is introduced. The prototype analysis filter $H(z)$ can be decomposed into polyphase components according to

$$H(z) = \sum_{k=-\infty}^{\infty} h(nK)z^{-nk} + z^{-1} \sum_{k=-\infty}^{\infty} h(nK + 1)z^{-nk}$$
\[
+: \quad z^{-K+1} \sum_{n=-\infty}^{\infty} h(nL + K - 1)z^{-nk} = \sum_{k=0}^{K-1} z^{-k} \sum_{n=-\infty}^{\infty} h(nK + k)z^{-nk} = \sum_{k=0}^{K-1} z^{-k} E_k(z^k).
\]

(2.78)

where \(K\) is the number of elements in the decomposition. The filters \(e_k(n) = h(nK + k)\) are called the type I polyphase components of \(H(z)\).

### 2.6.2 Analysis Filter Bank:

Inserting the definition of Eq. 2.7 into Eq. 2.78, the polyphase decomposition of the \(m\)-th analysis filter in the analysis filter bank becomes

\[
H_m(z) = H(zW_m^m) = \sum_{k=0}^{K-1} (zW_m^m)^k E_k([zW_m^m]^k) = \sum_{k=0}^{K-1} z^{-k} W_m^{-mk} E_k([zW_m^m]^k).
\]

(2.79)

When the number of decompositions, \(K\), is set equal to the decimation factor, i.e. \(K = D = M/2\), the polyphase decomposition becomes

\[
H_m(z) = \sum_{k=0}^{\frac{M}{2}-1} z^{-k} W_m^{-mk} E_k([zW_m^m]^k).
\]

(2.80)

Since
\[ W_{m}^{m_m} = e^{-j\pi m} = \begin{cases} -1, & \text{when } m \text{ is odd} \\ 1, & \text{when } m \text{ is even} \end{cases} \]  

(2.81)

it implies that odd and even subbands have to be treated separately. Sets of even and odd indices are defined by \( M_{\text{even}} = \{0, 2, 4, \ldots, M - 2\} \) and \( M_{\text{odd}} = \{1, 3, 5, \ldots, M - 1\} \). The decomposition of the analysis filters in the even subbands and the analysis filters in the odd subbands are

\[ H_m(z) = \sum_{k=0}^{M_{\text{even}}} z^{-k} W_{m}^{m_k} E_k(z^D), \quad m \in M_{\text{even}}, \]  

(2.82)

and

\[ H_m(z) = \sum_{k=0}^{M_{\text{odd}}} z^{-k} W_{m}^{m_k} E_k([-z]^D), \quad m \in M_{\text{odd}}. \]  

(2.83)

The polyphase components for odd and even subbands are defined as \( E_a(z) \) and \( E_i(z) \), respectively,

\[ E_a(z) \leftrightarrow e_a(n) = h(nD + k), \]

\[ E_i(z) = E_a(-z) \leftrightarrow e_i(n) = h(nD + k)(-1)^n. \]  

(2.84)

The analysis filter bank structure with filters given in Eq. (2.82) and Eq. (2.83) consists of a delay line, the polyphase components, \( E_a(z) \) and \( E_i(z) \), and two summations over \( k \) with coefficients \( W_{m}^{m_k} \).

The implementation becomes efficient when the filter outputs are decimated by \( D = M/2 \) and the noble identity is applied, meaning that the decimation
operators have been placed before the polyphase components, and thereby giving low rate filtering, refer Fig. 2.20.

\[ E(z^0) \xrightarrow{\text{D}} E(z) \]  

![Diagram](image.png)

**Fig. 2.20: Noble Identity 1**

This involves that \( E_i(z^j) \) are replaced by \( E_i(z) \) and, similarly, for the odd subbands \( E'_i(z^j) \) are replaced by \( E'_i(z) \). The summation over \( k \) for each subband \( m \) with coefficients \( W_{M}^{-mk} \) can be implemented efficiently using the IFFT algorithm. Since

\[ W_{M}^{-mk} = e^{j2\pi mk/M} = e^{j2\pi m/k} \]  \hspace{1cm} (2.85)

for \( k = 0, \ldots, D - 1 \) and \( m \in \mathbb{M}_{\text{even}} \), an \( D \)-length FFT can be used for the even subbands. Similarly for the odd subbands,

\[ W_{M}^{-mk} = e^{j2\pi mk/M} = e^{j2\pi (m + 1) k / m} e^{j2\pi k / M} \]  \hspace{1cm} (2.86)

for \( k = 0, \ldots, D - 1 \) and \( m \in \mathbb{M}_{\text{odd}} \), which enables the use of an \( D \)-length FFT when each channel \( k \) is multiplied with the correction factor \( e^{j2\pi k/M} = W_{M}^{k} \). Fig. 2.21 shows how the analysis filter bank is implemented.
2.6.3 Type II Polyphase Decomposition:

In the synthesis filter bank, the Type II Polyphase Decomposition is used. The decomposition is basically the same as the Type I Decomposition with a slight difference

\[
G(z) = \sum_{k=0}^{K-1} z^{-k-i} F_{k+1}(z^k) \\
= \sum_{k=0}^{K-1} z^{-i} F_{k-i}(z^k),
\]

(2.87)

where \( F_i(z) \), \( k = 0, \ldots, K - 1 \) denote the type II polyphase components of \( G(z) \). The first part of the equation is the original Type I Decomposition and the second part is the Type II equation. The difference is the order of summation.
2.6.4 Synthesis Filter Bank:

The Type II Polyphase Decomposition of the synthesis filter in subband \( m \) is

\[
G_m(z) = \sum_{k=0}^{l-1} z^{-\left(1+D-k-1\right)} W_{ml}^{mD} F_k^p(z^{D}) \quad m \in \mathcal{M}_{even},
\]

\[
G_m(z) = \sum_{k=0}^{l-1} z^{-\left(1+D-k-1\right)} W_{ml}^{mD} F_k^p([-z]^{D}) \quad m \in \mathcal{M}_{odd},
\]

where

\[
F_k^p(z) \leftrightarrow f_k(n) = g(nD + D - k - 1),
\]

\[
F_k^p(z) = F_k(-z) \leftrightarrow f_k'(n) = g(nD + D - k - 1)(-1)^n.
\]

Note that the number of elements in the decomposition is set to \( K = D \), similar to the analysis filter bank. The output \( Y(z) \) of the synthesis filter bank may be written in terms of the subband signals \( Y_m(z) \) and the filters \( G_m(z) \) according to

\[
Y(z) = \sum_{m \in M_{even}} \sum_{k=0}^{l-1} W_{ml}^{mD} Y_m(z) F_k^p(z^{D}) z^{-\left(1+D-k-1\right)}
\]

\[
+ \sum_{m \in M_{odd}} \sum_{k=0}^{l-1} W_{ml}^{mD} Y_m(z) F_k'(z^{D}) z^{-\left(1+D-k-1\right)},
\]

which may be rearranged as

\[
Y(z) = \sum_{k=0}^{l-1} \sum_{m \in M_{even}} W_{ml}^{mD} Y_m(z) F_k^p(z^{D}) + \sum_{m \in M_{odd}} W_{ml}^{mD} Y_m(z) F_k'(z^{D}) z^{-\left(1+D-k-1\right)}
\]

When the signals \( Y_m(z) \) are interpolated and when Noble Identity 2, as shown in Fig. 2.22, is applied, the polyphase components and the interpolation operations trade places.
As with the analysis filter bank, the summation over $m$ for each $k$ with coefficients $w_m^a$ can be implemented efficiently using the FFT algorithm. For the odd subbands a correction by $e^{j2\pi kM} = w_m^a$ is needed after the transform operation. Fig. 2.23 shows the polyphase implemented synthesis filter bank.

Fig. 2.23: Polyphase implementation of a modulated synthesis filter bank with two-times oversampling

2.6.5 Computational Complexity:

This section discusses the computational complexity of critically sampled modulated filter banks and two-times oversampled modulated Filterbanks. In the previous
sections, the filter banks are implemented using the polyphase implementation was described.

The complexity of signal processing structures is usually measured in terms of the number of additions and multiplications. These additions and multiplications can either be real or complex valued. Complex additions can be decomposed into two real additions and complex multiplications can be decomposed into four real multiplications and two real additions.

The number of real and complex operations in critically sampled filter banks are shown in Table 2.1. The number of operations in the oversampled filter bank are given in Table 2.2.

<table>
<thead>
<tr>
<th>Implementation Part</th>
<th>Real Additions</th>
<th>Real Multiplications</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polyphase Components</td>
<td>L-M</td>
<td>L</td>
</tr>
<tr>
<td>IFFT</td>
<td>2M log₂ M</td>
<td>2M log₂ M</td>
</tr>
<tr>
<td>FFT</td>
<td>2M log₂ M</td>
<td>2M log₂ M</td>
</tr>
<tr>
<td>Polyphase Components</td>
<td>L-M</td>
<td>L</td>
</tr>
<tr>
<td>Delay + Sum Line</td>
<td>M-1</td>
<td>0</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>2L+M(4 log₂ M-1)-1</td>
<td>2L+4M log₂ M</td>
</tr>
</tbody>
</table>

Table 2.1: Computational complexity for the critically sampled polyphase filter bank structure

<table>
<thead>
<tr>
<th>Implementation Part</th>
<th>Real Additions</th>
<th>Real Multiplications</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polyphase Components</td>
<td>2(L-M)</td>
<td>2L</td>
</tr>
<tr>
<td>Complex Correction</td>
<td>M</td>
<td>2M</td>
</tr>
<tr>
<td>IFFTs</td>
<td>4M log₂ M/2</td>
<td>4M log₂ M/2</td>
</tr>
<tr>
<td>FFTs</td>
<td>4M log₂ M/2</td>
<td>4M log₂ M/2</td>
</tr>
<tr>
<td>Complex Correction</td>
<td>M</td>
<td>2M</td>
</tr>
<tr>
<td>Polyphase Components</td>
<td>2(L-M)</td>
<td>2L</td>
</tr>
<tr>
<td>Delay + Sum Line</td>
<td>M⁻¹/₂</td>
<td>0</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>4L+M(8 log₂ M⁻¹/₂-1½)-1</td>
<td>4L+4M(1+2 log₂ M⁻¹/₂)</td>
</tr>
</tbody>
</table>

Table 2.2: Computational complexity for the oversampled polyphase filter bank structure
The computation complexity of the polyphase implementation of an oversampled filter bank in relation to the complexity of a critically sampled filter bank is shown in Fig. 2.24. The ratio of the number of operations is plotted for $M = 4, \ldots, 512$, where $M$ is a power of 2. The filter length is chosen as $L_b = L_s = 2M$. The plots show that the oversampled structure is not twice as complex as the critically sampled structure. This is because by the FFT operations being less than half as complex in the oversampled polyphase implementation.

The computational complexity of the subband filtering structure is evaluated in Fig. 2.25. The subband filtering structure consists of analysis and synthesis polyphase implemented filter banks and filters in the subbands. The number of subbands varies between $M = 4, \ldots, 512$ while the length of the subband filters is correspondingly reduced from $L_c = 128, \ldots, 1$. The ratio between the complexity of the oversampled filter bank and the critically sampled filter bank is shown as a function of the number of subbands, $M$.

Fig. 2.24: Filter bank complexity. Comparison between critically sampled and oversampled filter bank implementations. The ratio describes how much more demanding the oversampled implementation is compared to the critical sampled implementation.
Fig. 2.25: Filter bank complexity. The figures show the complexity ratio including the subband filtering operations. In the critical sampled case, the least number of subbands is 4 with length 128 subband filters and the maximum number of subbands is 512, with 1-coefficient subband filters. In the oversampled case, the subband filters are twice as long.

2.7 Filter Bank Examples:

2.7.1 Filter Bank Cases:

In this section four different filter banks are designed using the proposed method and they were compared against four conventional filter banks with known structure as reference. The number of subbands is set to \( M = 8 \) in all filter banks. The four reference filter banks are:

Case 1. A perfect reconstruction FFT filter bank with \( L_h = L_R = M \) and \( D = M \) (critically sampled). The filter bank is implemented using delay lines and FFT and IFFT operations. The prototype filters are rectangular windows.

Case 2. A perfect reconstruction filter bank with \( M = L_h = L_R \) and \( D = M/2 \) (oversampling). The prototype analysis filter is an \( M \)-point Hanning window, and the prototype synthesis filter is an \( M \)-point rectangular window.
Case 3. A critically sampled, modulated filter bank with $L_h = L_g = 2M$ and $D = M$. The analysis and synthesis filters are modulated from the same prototype lowpass linear phase filter designed using the window method with a Hamming window.

Case 4. An oversampled modulated filter bank with $L_h = L_g = 2M$ and $D = M/2$. The prototype analysis and synthesis filters are obtained as for the reference filter bank in case 3.

Four filter banks with the same number of subbands were designed, according to the proposed method presented in this chapter. Note that these filter banks have the same structure as the filter banks in case 3 and 4 and that only the prototype filters differ.

Case 5. A critically sampled modulated filter bank with $L_h = L_g = 2M$ and $D = M$. The prototype analysis and synthesis filters are designed using the proposed method. The desired total delay is set to $\tau_T = 2M$.

Case 6. An oversampled modulated filter bank with $L_h = L_g = 2M$ and $D = M/2$. The prototype analysis and synthesis filters are designed using the proposed method. The desired total delay is set to $\tau_T = 2M$.

Case 7. A critically sampled modulated filter bank with $L_h = L_g = 2M$ and $D = M$. The prototype analysis and synthesis filters are designed using the proposed method. The desired total delay is reduced to $\tau_T = M$. 
Case 8. An oversampled modulated filter bank with $l_{\alpha} = l_{\beta} = 2M$ and $D = M/2$. The prototype analysis and synthesis filters are designed using the proposed method. The desired total delay is reduced to $\tau_f = M$.

Note that the filter banks in case 5 and 6 have the same delay as in case 3 and 4. The delay reduced to half for the filter banks in cases 7 and 8.

2.7.2 Performance Evaluation:

In Fig. 2.26 to Fig. 2.33, illustrations are presented for all eight filter bank cases. The frequency response and the impulse response of the prototype analysis and synthesis filters are plotted. All filter banks have $M=8$ subbands. The following spectral properties of the filter banks are plotted.

- **Amplitude Error** - The amplitude error function, refer Eq. (2.44)

\[ \left| |T(e^{j\omega})| - |D(e^{j\omega})| \right|^2 \]

- **Group Delay** - The group delay of the total filter bank structure

\[ \tau(\omega) = -\frac{d}{d\omega} \angle T(e^{j\omega}) \]

- **Phase Error** - The phase error function

\[ \left| \angle T(e^{j\omega}) - \angle D(e^{j\omega}) \right|^2 \]
* Inband-Aliasing - The inband-aliasing spectrum, refer Eq. (2.10)

$$\frac{1}{D} \sum_{d=0}^{B-1} H(e^{j\omega_d} W_d W_d^*)$$

* Residual Aliasing - The residual aliasing spectrum, refer Eq. (2.15)

$$\left| \sum_{d=0}^{B-1} \sum_{m=0}^{M-1} A_{m,d}(e^{j\omega_d}) \right|^2$$

* Residual Aliasing Distortion - The residual aliasing distortion, refer Eq. (2.53)

$$\sum_{d=0}^{B-1} \sum_{n=0}^{N-1} |A_{m,d}(e^{jn\omega_d})|^2$$
Fig. 2.26(a): Impulse response and frequency response of the prototype analysis filter and prototype synthesis filter

Fig. 2.26(b): Spectral properties of the total filter bank.

Fig. 2.26: FFT Filter Bank
Fig. 2.27(a): Impulse response and frequency response of the prototype analysis filter and prototype synthesis filter.

Fig. 2.27(b): Spectral properties of the total filter bank.

Fig. 2.27: Hanning Filter Bank.
Fig. 2.28(a): Impulse response and frequency response of the prototype analysis filter and prototype synthesis filter.

Fig. 2.28(b): Spectral properties of the total filter bank.

Fig. 2.28: Critically Sampled Filter Bank - Window Design Method.
Fig. 2.29(a): Impulse response and frequency response of the prototype analysis filter and prototype synthesis filter.

Fig. 2.29(b): Spectral properties of the total filter bank.

Fig. 2.29: Oversampled Filter Bank - Window Design Method.
Impulse Response of $H(z)$

Impulse Response of $G(z)$

Frequency Response of $H(z)$

Frequency Response of $G(z)$

Fig. 2.30(a): Impulse response and frequency response of the prototype analysis filter and prototype synthesis filter

Fig. 2.30(b): Spectral properties of the total filter bank.

Fig. 2.30: Critically Sampled Filter Bank - Proposed Method.
Fig. 2.31(a): Impulse response and frequency response of the prototype analysis filter and prototype synthesis filter.

Fig. 2.31(b): Spectral properties of the total filter bank.

Fig. 2.31: Oversampled Filter Bank - Proposed Method.
Fig. 2.32(a): Impulse response and frequency response of the prototype analysis filter and prototype synthesis filter.

Fig. 2.32(b): Spectral properties of the total filter bank.

Fig. 2.32: Critically Sampled Filter Bank with less delay - Proposed Method.
Fig. 2.33(a): Impulse response and frequency response of the prototype analysis filter and prototype synthesis filter.

Fig. 2.33(b): Spectral properties of the total filter bank.

Fig. 2.33: Oversampled Filter Bank with less delay - Proposed Method.
2.7.2.1 Inband-Aliasing:

Fig. 2.34 shows the inband-aliasing distortion, which is one of the objectives of minimization in the analysis filter bank design. The measure is defined in Eq. (2.25). Clearly, the oversampled filter banks (even case numbers) have less inband-aliasing distortion compared to the critically sampled filter banks (odd case numbers). Comparing the oversampled filter bank cases 4, 6, and 8, which have the same structure, it can be seen that the optimal filter bank in case 6 has the least distortion. In case 8, the reduction of delay requires more degrees of freedom, which results in higher inband-aliasing distortion.

Fig. 2.34: Inband-Aliasing Distortion for the filter banks in the evaluation
2.7.2.2 Residual Aliasing:

\[
\text{Residual Aliasing: } \xi(c) = \sum_{m=0}^{M-1} A_m(c^m) \quad \text{(2.92)}
\]

Fig. 2.35: (a) Residual aliasing and (b) Residual Aliasing Distortion measure for the filter banks in the evaluation.

The residual aliasing energy, i.e. energy of the aliasing present in the output of the synthesis filter bank, is calculated according to

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{m=0}^{M-1} A_m(c^m) \right|^2 d\omega.
\]

Here, no subband filtering is applied, i.e. \( \xi(c) = 1 \). Fig. 2.35 (a) shows the aliasing energies for the eight filter bank cases. Clearly, the perfect reconstruction filter banks in case 1 and 2 have zero reconstruction aliasing. In the other cases, the oversampled filter banks have lower aliasing than the critically sampled filter banks. The residual aliasing distortion measure, defined in Eq. (2.53), which is one of the objectives of minimization in the synthesis filter bank design, is shown in Fig. 2.35 (b). In this case, the perfect reconstruction filter banks have the highest bound, while the optimal filter banks have the lowest bound, especially the oversampled filter banks.
2.7.2.3 Amplitude and Phase Distortion:

The amplitude and phase distortion of the total filter bank are evaluated separately. Both measures are included in the total response error in the synthesis filter bank design. The amplitude distortion is defined by

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| T(e^{j\omega}) - |D(e^{j\omega})| \right| d\omega,$$

(2.93)

with $T(z)$ defined as in Eq. (2.18), and $D(z)$ defined as in Eq. (2.45). The measures are presented in Fig. 2.36. Clearly, the perfect reconstruction filter banks in cases 1 and 2 have zero amplitude distortion. The optimal filter banks have lowest amplitude distortion, especially the oversampled filter banks. The phase distortion, which is defined as

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \angle T(e^{j\omega}) - \angle D(e^{j\omega}) \right) d\omega,$$

(2.94)

is illustrated in Fig. 2.36. The optimal filter banks with oversampling have the lowest phase distortion, except for the perfect reconstruction filter banks in case 1 and 2, which have zero phase error.
2.8 Conclusions:

A two-step method for the design of modulated filter banks in subband filtering applications has been proposed. The method determines two low-pass prototype filters for the analysis filter bank and the synthesis filter bank, using unconstrained quadratic optimization. The delay of the analysis filter bank and the delay of the total filter bank may be controlled while the effects of inband-aliasing in the subband signals, and the residual aliasing in the output signal are minimized.

Filter banks with critical sampling and oversampling are designed using the suggested method, and they are compared with conventional filter bank types. Also filter banks with reduced delay have been included in the evaluation.

The filter banks have been evaluated in subband adaptive filtering applications, such as Acoustic Echo Cancellation and Speech Enhancement. It has been shown that the optimal filter banks, design with the proposed method can improve performance in these applications. Also the negative side effects, which are caused by the filter bank, such as aliasing and transmission delay, can be reduced without significant reduction of the performance.

Hence, an efficient design method has been proposed for a uniform DFT filter bank with the possibility of a pre-specified filter bank group delay. The method minimizes the inband and output aliasing error as well as the overall filter-bank transfer function's phase and amplitude deviation. Subband oversampling allows for a decrease in aliasing and amplitude errors, which in turn increases the performance significantly.