3.1 INTRODUCTION

In chapter-2, the classical and Bayes estimates of the reliability characteristics and parameters of a probabilistic form of a composite hazard-rate model with random proportion have been obtained using the complete sample information. However, there are several situations where it is neither possible nor desirable to obtain the complete sample information on the life testing experiment due to the cost and time constraints. Note that the life testing experiments are usually destructive in that the items get destroyed at the end of the experiment and cannot be used again. This limits the number of items which are put to test. Thus, n items can be put on the test and the experiment may be terminated when a pre-assigned number of items, say r (<n) have failed. The sample obtained from such experiment are called 'failure censored' sample. Failure censored sample is almost mandatory in dealing with high cost sophisticated items.

Another factor that affects the life-testing experiment is the amount of time required to obtain the complete sample. To limits this factor, we may put n items to test and terminate the experiment at a pre-assigned time \( t_0 \). The sample obtained from such experiment is called 'time censored' sample. Time censored sampling is almost essential in dealing with life testing experiments in which the cost of experiments increases heavily with time.

Following the concept, the present study proposes the method of estimation the parameters and reliability characteristics of the probabilistic bath-tub hazard-rate model with type-I and type-II censored information in classical and Bayesian setups. Theoretical developments have further been highlighted
with simulated data arising from the three phases of the bathtub shaped hazard-rate model (BSHM).

### 3.2 NOTATIONS

- \( T \) : Random variable (r.v.) denoting lifetime of the device
- \( h(t) \) : Hazard-rate function (HRF)
- \( H(t) = \int_0^t h(u) \, du \) : Cumulative hazard function
- \( f(t) \) : p.d.f. of \( T \)
- \( F(t) \) : c.d.f. of \( T \)
- \( R(t) = P[T > t] \) : Reliability/survival function for a mission time \( t \)
- \( E(T) = \int_0^\infty R(t) \, dt \) : Mean time to system failure (MTSF)
- \( n \) : Total number of items on test
- \( t_0 \) : Pre-assigned time to terminate the test under type-I censoring
- \( r \) : Pre-assigned number of failure \( r \) to terminate the test under type-II censoring
- \( t_n = \{t_{(1)}, t_{(2)}, \ldots, t_{(m)}\} \) : Failure times of \( m \) items under type-I censoring
- \( t_r = \{t_{(1)}, t_{(2)}, \ldots, t_{(r)}\} \) : Failure times of \( r \) items under type-II censoring
- \( L(t_n | \lambda, p_i) \) : Sample likelihood function under type-I censoring
- \( L(t_r | \lambda, p_i) \) : Sample likelihood function under type-II censoring
- M.L.E.: Maximum likelihood estimate
- SELF: Squared error loss function
- \( \hat{\lambda} \) : ML [Bayes] estimate of \( \lambda \)
- \( \hat{p}_i \) : ML [Bayes] estimate of \( p_i \)
\[ h(t) = \sum_{i=1}^{3} \lambda_i (t^{\delta_i-1}) ; \quad t > 0 \]  

... (3.3.1)

Here, the parameters \( \lambda_i \) and \( \delta_i \) (\( i = 1, 2, 3 \)) are such that \( \lambda_i > 0 \) and \( \delta_i < 1 \), \( \delta_2 = 1 \) and \( \delta_3 > 1 \) for the states \( S_1 \), \( S_2 \) and \( S_3 \) respectively. The arbitrarily fixed durations corresponding to the three states \( S_1 \), \( S_2 \) and \( S_3 \) are \( (0 - 0.8) \), \( (0.8 - 2.0) \) and \( (2.0 - 3.2) \). Also let \( p_i \left( 0 < p_i < 1; \sum_{i=1}^{3} p_i - 1 \Rightarrow p_3 = (1 - p_1 - p_2) \right) \) be the probability associated to the state \( S_i \).

(b) The cumulative hazard function is

\[ H(t) = \sum_{i=1}^{3} \lambda_i (t^{\delta_i}) ; \quad t > 0 \]  

... (3.3.2)

(c) For the Bayesian analysis of the reliability characteristics, we treat \( \lambda_i \) and \( (p_1, p_2) \) as random variables with their respective prior p.d.f.s as

\[ g(\lambda) = \frac{e^{-\lambda}}{\Gamma(\beta) \lambda^{\beta-1}} \exp(-\alpha \lambda) ; \quad (\lambda, \alpha, \beta) > 0 \]  

... (3.3.3)

and

\[ h(p_1, p_2) = \frac{\Gamma(a_1 + a_2 + a_3) \Gamma(1) \Gamma^2}{\Gamma(a_1 \Gamma a_2 \Gamma a_3)} p_1^{a_1-1} p_2^{a_2-1} (1 - p_1 - p_2)^{a_3-1} ; \quad (p_1, p_2) > 0, p_1 + p_2 < 1, (a_1, a_2, a_3) > 0 \]  

... (3.3.4)
(d) $\delta_i$ is taken as known constant.

(e) SELF is considered in its standard form, i.e.

$$L(0, \hat{\theta}) = (0 - \hat{\theta})^2$$ (3.3.5)

Here, $\hat{\theta}$ is an estimated value of the parameter $\theta$.

### 3.4 CHARACTERISATION OF THE LIFETIME DISTRIBUTION AND ITS RELIABILITY CHARACTERISTICS

On using (3.3.1) and (3.3.2) in a well known relationship

$$f(t) = h(t) \exp \left\{ H(t) \right\}$$

one gets,

$$f(t) = \left( \sum_{i=1}^{n} \lambda \frac{p_i}{\delta_i} \right) \exp \left\{ - \left( \lambda \sum_{i=1}^{n} \frac{p_i}{\delta_i} \right) t \right\} ; \quad t > 0$$ (3.4.1)

The reliability function is

$$R(t) = \exp \left\{ - \left( \lambda \sum_{i=1}^{n} \frac{p_i}{\delta_i} \right) t \right\}$$ (3.4.2)

and

$$M(t) = \int_{0}^{t} \exp \left\{ - \left( \lambda \sum_{i=1}^{n} \frac{p_i}{\delta_i} \right) \right\} dt$$ (3.4.3)

### 3.5 M.L.E.s FOR PARAMETERS AND OTHER RELIABILITY CHARACTERISTICS

#### 3.5.1 CASE-I: TYPE-I CENSORING:

Let $t_{(1)}, t_{(2)}, \ldots, t_{(m)}$ be the failure times of $m$ items that failed up to time $t_0$ and $(n-m)$ items survived beyond $t_0$. The likelihood function of the sample is:

$$L(t_m | \lambda, p_i) = \frac{n! \lambda^m \left( \prod_{j=1}^{m} p_j \right) \exp \left\{ - \left( \lambda \sum_{j=1}^{m} \frac{p_j \delta_j}{\delta_j} \right) \right\} \exp \left\{ - \left( \lambda (n-m) \sum_{i=1}^{m} \frac{p_i \delta_i}{\delta_i} \right) \right\}}{(n-m)!}$$ (3.5.1.1)
\[
\log L \left( t_{m|\lambda, p_1} \right) = \log n! - \log(n-m)! - m \log \lambda + \frac{m}{\lambda} \log \left( \sum_{i=1}^{m} p_i t_i^{\delta_1} \right) - n \log \left( \sum_{i=1}^{m} \frac{p_i t_i^{\delta_1}}{\delta_1} \right) - \lambda \left( (n-m) \sum_{i=1}^{m} \frac{p_i t_i^{\delta_1}}{\delta_1} \right)
\]

\[
= \log n! + \log (n-m)! - m \log \lambda + \sum_{j=1}^{m} \log \left[ t_j^{\delta_1} + p_1 \left( t_j^{\delta_1} - t_j^{\delta_2} \right) + p_2 \left( t_j^{\delta_3} - t_j^{\delta_4} \right) \right]
\]

\[
= \lambda \sum_{j=1}^{m} \frac{t_j^{\delta_1}}{\delta_1} + p_1 \left( \frac{t_j^{\delta_1}}{\delta_1} - \frac{t_j^{\delta_2}}{\delta_2} \right) + p_2 \left( \frac{t_j^{\delta_3}}{\delta_3} - \frac{t_j^{\delta_4}}{\delta_4} \right) + \lambda (n-m) \left[ \frac{t_j^{\delta_2}}{\delta_2} - \frac{t_j^{\delta_3}}{\delta_3} \right] + p_1 \left( \frac{t_j^{\delta_1}}{\delta_1} - \frac{t_j^{\delta_3}}{\delta_3} \right) + p_2 \left( \frac{t_j^{\delta_2}}{\delta_2} - \frac{t_j^{\delta_4}}{\delta_4} \right)
\]

... (3.5.1.2)

To find the simultaneous M.L.E.s of \( \lambda, p_1 \) and \( p_2 \), we consider

\[
\frac{\partial \log L \left( t_{m|\lambda, p_1} \right)}{\partial \lambda} = 0,
\]

which gives

\[
\frac{m}{\lambda} - \sum_{j=1}^{m} \left[ \frac{t_j^{\delta_1}}{\delta_1} + p_1 \left( \frac{t_j^{\delta_1}}{\delta_1} - \frac{t_j^{\delta_2}}{\delta_2} \right) + p_2 \left( \frac{t_j^{\delta_3}}{\delta_3} - \frac{t_j^{\delta_4}}{\delta_4} \right) \right] - (n-m) \left[ \frac{t_j^{\delta_3}}{\delta_3} + p_1 \left( \frac{t_j^{\delta_1}}{\delta_1} - \frac{t_j^{\delta_3}}{\delta_3} \right) + p_2 \left( \frac{t_j^{\delta_2}}{\delta_2} - \frac{t_j^{\delta_4}}{\delta_4} \right) \right] = 0
\]

\[
\Rightarrow \lambda = \frac{m}{\sum_{j=1}^{m} \left[ \frac{t_j^{\delta_1}}{\delta_1} + p_1 \left( \frac{t_j^{\delta_1}}{\delta_1} - \frac{t_j^{\delta_2}}{\delta_2} \right) + p_2 \left( \frac{t_j^{\delta_3}}{\delta_3} - \frac{t_j^{\delta_4}}{\delta_4} \right) \right] - (n-m) \left[ \frac{t_j^{\delta_3}}{\delta_3} + p_1 \left( \frac{t_j^{\delta_1}}{\delta_1} - \frac{t_j^{\delta_3}}{\delta_3} \right) + p_2 \left( \frac{t_j^{\delta_2}}{\delta_2} - \frac{t_j^{\delta_4}}{\delta_4} \right) \right]}
\]

... (3.5.1.3)

Similarly,

\[
\frac{\partial \log L \left( t_{m|\lambda, p_1} \right)}{\partial p_1} = 0
\]

gives

\[
\sum_{j=1}^{m} \left[ \frac{t_j^{\delta_1}}{\delta_1} + p_1 \left( \frac{t_j^{\delta_1}}{\delta_1} - \frac{t_j^{\delta_2}}{\delta_2} \right) + p_2 \left( \frac{t_j^{\delta_3}}{\delta_3} - \frac{t_j^{\delta_4}}{\delta_4} \right) \right] - \lambda \left( (n-m) \frac{t_j^{\delta_3}}{\delta_3} - \frac{t_j^{\delta_4}}{\delta_4} \right) = 0
\]

... (3.5.1.4)

And \( \frac{\partial \log L \left( t_{m|\lambda, p_1} \right)}{\partial p_2} = 0 \) gives
\[
\sum_{i=1}^{m} \frac{t_i^{\delta_i-1} - 1}{t_i^{\delta_i-1} \cdot \ln(t_i^{\delta_i-1})} + p_2 \left( \frac{t_i^{\delta_i-1} - 1}{t_i^{\delta_i-1}} \right) + p_3 \left( \frac{t_i^{\delta_i-1} - 1}{t_i^{\delta_i-1}} \right) \] 
\[= \lambda \left( \sum_{i=1}^{m} t_i^{\delta_2} \sum_{i=1}^{m} t_i^{\delta_3} \right)^{-\lambda} \left[ \frac{\delta_2}{\delta_2 - 1} \right] \left[ \frac{\delta_3}{\delta_3 - 1} \right] - 0 \] 
(3.5.1.5)

Equations in (3.5.1.3), (3.5.1.4) and (3.5.1.5) can be solved simultaneously by using suitable iterative procedure to get M.L.E.s for \( \lambda \), \( p_1 \) and \( p_2 \). Finally, on using the invariance property of M.L.E.s one gets-

\[ \hat{p}_1 = 1 - \hat{p}_1 - \hat{p}_2 \] 
(3.5.1.6)

\[ \hat{h}(t) = \frac{\lambda}{3} \sum_{i=1}^{m} t_i^{\delta_i-1} \] 
(3.5.1.7)

\[ \hat{R}(t) = \exp \left( - \frac{3 \hat{p}_1 t_i^{\delta_i}}{\delta_i} \right) \] 
(3.5.1.8)

and

\[ MTF = \hat{M} \exp \left( - \frac{3 \hat{p}_1 t_i^{\delta_i}}{\delta_i} \right) \] 
(3.5.1.9)

3.5.2 CASE-II: TYPE-II CENSORING

Let \( t_{(1)}, t_{(2)}, \ldots, t_{(r)} \) be the failure times of \( r \) (<n) items that failed and \( n-r \) items survived until \( t_{(r)} \). The likelihood function of the sample is:

\[ L \left( \lambda, p_i \right) = \frac{n! \lambda^r \left( \prod_{i=1}^{r} p_i t_i^{\delta_i} \right) \exp \left( - \left( \lambda \sum_{i=1}^{r} t_i^{\delta_i} \right) \right) \exp \left( - \left( \lambda n - r \sum_{i=1}^{r} t_i^{\delta_i} \right) \right)}{(n-r)!} \]

Here, the estimates of the parameters \( \lambda, p_1, p_2, p_3 \) and \( h(t), R(t) \) and MTSF can be obtained by simple replacing \( m \) by \( r \) and \( t_i \) by \( t_{(r)} \) in the corresponding results of the section 3.5.1.
Further, by using general theory of M.L.E. Rao, [1973], the asymptotic distribution of

\[
\begin{pmatrix}
\hat{\lambda} - \lambda \\
\hat{p}_1 - p_1 \\
\hat{p}_2 - p_2 \\
\hat{p}_3 - p_3
\end{pmatrix}
\] is \( N_4(0, \Sigma^{-1}) \)

Where \( \Sigma \) is the Fisher's information matrix having elements:

\[
\Sigma_{11} = -E \left( \frac{\partial^2 \log L}{\partial^2 \lambda} \right)
\]

\[
\Sigma_{22} = -E \left( \frac{\partial^2 \log L}{\partial^2 p_1} \right)
\]

\[
\Sigma_{33} = -E \left( \frac{\partial^2 \log L}{\partial^2 p_2} \right)
\]

\[
\Sigma_{44} = -E \left( \frac{\partial^2 \log L}{\partial^2 p_3} \right)
\]

\[
\Sigma_{ij} = E \left( \frac{\partial^2 \log L}{\partial \lambda \partial p_i} \right) - E \left( \frac{\partial \log L}{\partial \lambda} \right) E \left( \frac{\partial \log L}{\partial p_i} \right)
\]

Here, the exact values of various elements of \( \Sigma \) are very difficult to obtain. However, following recommendations in [Cohen, 1965], the approximated values of these elements can be easily obtained as:

\[
\Sigma_{11} = \left( \frac{\partial^2 \log L}{\partial^2 \lambda} \right)_{\hat{\lambda}, \hat{p}_1, \hat{p}_2}
\]

\[
\Sigma_{ij} = E \left( \frac{\partial^2 \log L}{\partial \lambda \partial p_i} \right)_{\hat{\lambda}, \hat{p}_1, \hat{p}_2}
\]

\[
\Sigma_{ij} = E \left( \frac{\partial^2 \log L}{\partial p_j \partial p_i} \right)_{\hat{\lambda}, \hat{p}_1, \hat{p}_2}
\]

\[\text{etc.}\]

**Lemma 1:** Let \( \hat{R}(t) = g(\hat{\lambda}, \hat{p}_1, \hat{p}_2, \hat{p}_3) \), then the asymptotic distribution of \( \hat{R}(t) \), for fixed \( t \), is \( N_1 \left( R(t), G' \Sigma^{-1} G \right) \) where \( G = (\partial R(t)/\partial \lambda, \partial R(t)/\partial p_1, \partial R(t)/\partial p_2, \partial R(t)/\partial p_3) \) and \( \Sigma^{-1} \) as defined above.

**Lemma 2:** Let \( \hat{h}(t) = h(\hat{\lambda}, \hat{p}_1, \hat{p}_2, \hat{p}_3) \), then the asymptotic distribution of \( \hat{h}(t) \), for fixed \( t \), is \( N_1 \left( h(t), H' \Sigma^{-1} H \right) \) where \( H = (\partial h(t)/\partial \lambda, \partial h(t)/\partial p_1, \partial h(t)/\partial p_2, \partial h(t)/\partial p_3) \) and \( \Sigma^{-1} \) is the same as above.

**Note:** The proofs for lemmas 1 and 2 can be seen in Rao [1973].
Similarly, the Bayes estimate of $p_i$ ($i = 1, 2$) under SELF will be

$$p_i^* = \mathbb{E}(p_i | t)$$

$$= \frac{1}{\Gamma(m + \beta)\phi(t_m, p_i)} \int_0^t \int_0^t p_i \pi(\lambda, p_1, p_2 | t_m) d\lambda dp_2 dp_i$$

$$= \frac{1}{\Gamma(m + \beta)\phi(t_m, p_i)} \int_0^t \int_0^t p_i \lambda^{m+\beta-1} \exp \left[ -\left( \alpha + \sum_{j=1}^3 \sum_{i=1}^3 \frac{P_{ij}}{\delta_i} + (n-m)\sum_{i=1}^3 \frac{P_{i0}}{\delta_i} \right) \right]$$

$$\times \left( \prod_{j=1}^3 \sum_{i=1}^3 \frac{P_{ij}}{\delta_i} \right) \left( \prod_{j=1}^3 \sum_{i=1}^3 \frac{P_{i0}}{\delta_i} \right) \lambda^{m+\beta} dp_2 dp_i$$

The Bayes estimate of $p_3$ becomes

$$p_3^* = 1 - p_1^* - p_2^*$$

and the Bayes estimates of $h(t)$, $R(t)$ and MTSF comes out to be

$$h^*(t) = \int_0^t \int t \pi(\lambda, p_1, p_2 | t_m) d\lambda dp_2 dp_i$$
\[ -\frac{1}{\Gamma(m+\beta)\phi(\lambda_m, \lambda_1)} \int_0^\infty \int_0^\infty 2^{m-\beta} \left( \sum_{i=1}^{m} \frac{\lambda_i^{\delta_i}}{\delta_i} \right) \exp \left[ -\left( \lambda + \sum_{j=1}^{m} \frac{\lambda_j^{\delta_j}}{\delta_j} + (n-m) \sum_{i=m+1}^{n} \frac{\lambda_i^{\delta_i}}{\delta_i} \right) \right] \left( \prod_{j=1}^{m} \lambda_j^{\delta_j} \right)^{-1} p_1^{a_1-1} p_2^{a_2-1} (1-p_1-p_2)^{a_3-1} d\lambda dp_1 dp_2 dp_3 \]

\[ R(t) = \int_0^\infty \int_0^\infty R(1) \pi(\lambda, p_1, p_2 | \lambda_m) \beta \lambda dp_2 dp_1 \]

\[ -\frac{1}{\Gamma(m+\beta)\phi(\lambda_m, \lambda_1)} \int_0^\infty \int_0^\infty 2^{m-\beta} \left( \sum_{i=1}^{m} \frac{\lambda_i^{\delta_i}}{\delta_i} \right) \exp \left[ -\left( \lambda + \sum_{j=1}^{m} \frac{\lambda_j^{\delta_j}}{\delta_j} + (n-m) \sum_{i=m+1}^{n} \frac{\lambda_i^{\delta_i}}{\delta_i} \right) \right] \left( \prod_{j=1}^{m} \lambda_j^{\delta_j} \right)^{-1} \exp \left[ -\left( \sum_{i=1}^{m} \frac{\lambda_i^{\delta_i}}{\delta_i} \right) \right] p_1^{a_1-1} p_2^{a_2-1} (1-p_1-p_2)^{a_3-1} d\lambda dp_1 dp_2 \]

\[ -\frac{1}{\phi(\lambda_m, \lambda_1)} \int_0^\infty \left( \alpha + \sum_{j=1}^{m} \frac{\lambda_j^{\delta_j}}{\delta_j} + \sum_{i=m+1}^{n} \frac{\lambda_i^{\delta_i}}{\delta_i} + (n-m) \sum_{i=m+1}^{n} \frac{\lambda_i^{\delta_i}}{\delta_i} \right) \left( \prod_{j=1}^{m} \lambda_j^{\delta_j} \right)^{-1} p_1^{a_1-1} p_2^{a_2-1} (1-p_1-p_2)^{a_3-1} dp_1 dp_2 \]

\[ \text{and} \]

\[ MTSF^* = \int_0^\infty \int_0^\infty R(t) \pi(\lambda, p_1, p_2 | \lambda_m) \beta \lambda dp_2 dp_1 \]
\[ \frac{1}{\Gamma(m+\beta)} \int_0^1 \int_0^1 \frac{\prod_{i=1}^{m+\beta} \lambda^{m+\beta}}{\prod_{i=1}^{m+\beta} \delta_i^{n_i}} \exp \left\{ -\left( \sum_{j=1}^{3} \sum_{i=1}^{n_i} \delta_i \right) \left( \alpha + \sum_{j=1}^{3} \frac{\lambda t_j^{\delta_i}}{\delta_i} \right) + \left( n - m \right) \sum_{j=1}^{3} \frac{\lambda \delta_j}{\delta_i} \right\} \left( \prod_{j=1}^{m+\beta} \delta_j \right) \] 
\[ \times \exp \left\{ -\left( \sum_{i=1}^{n_i} \frac{\lambda t_i^{\delta_i}}{\delta_i} \right) \right\} \left( p_1^{\alpha_1} p_2^{\alpha_2} (1-p_1-p_2)^{\alpha_3-1} \right) \left( t dt dp_2 dp_1 \right) \]

\[ \frac{1}{\phi(t_0 p_1)} \int_0^1 \left( \alpha + \sum_{j=1}^{3} \frac{\lambda t_j^{\delta_j}}{\delta_j} \right) \left( \prod_{j=1}^{m+\beta} \delta_j \right) \left( \prod_{i=1}^{n_i} \frac{\lambda t_i^{\delta_i}}{\delta_i} \right) \left( t dt dp_2 dp_1 \right) \]

\[ \frac{(n+\beta)}{\phi(t_0 p_1)} \int_0^1 \left( \alpha + \sum_{j=1}^{3} \frac{\lambda t_j^{\delta_j}}{\delta_j} \right) \left( \sum_{i=1}^{n_i} \frac{\lambda t_i^{\delta_i}}{\delta_i} \right) \left( t dt dp_2 dp_1 \right) \]

\[ \left( p_1^{\alpha_1} p_2^{\alpha_2} (1-p_1-p_2)^{\alpha_3-1} \right) \]

... (3.6.7)

The Bayes estimates of \( \lambda, p_1, p_2, p_3, h(t), R(t) \) and MTSF for the type-II censored case can be obtained again by putting \( m=r \) and \( t_0 = t(t) \) in the corresponding results.

### 3.7 AN EXAMPLE

For analyzing variations in the classical and Bayesian estimates of reliability characteristics under type-I and type-II censoring, we consider the following data set:

1. Under type-I censoring, three sets of random failures, each of size 20 (out of three sets of random samples, each of size 30, generated from the lifetime distribution in (3.4.1) on assuming parametric values as listed in Table-3.1 for the states 1, 2 and 3 respectively) have been recorded up to a pre-assigned time \( t_0 \). Using the recorded sample information and the corresponding expressions in section 3.5.1, M.L.E.s of the involved parameters and reliability characteristics (\( \lambda, p_1, p_2, p_3 \) and MTSF) in states 1, 2 and 3 have been obtained and listed in Table-3.1. The uncertainties in estimating the parameters are given in the form of variance-covariance matrices I, II and III corresponding to the states 1, 2 and 3 respectively.
Again, for getting the estimates in the Bayesian setup, three sets of random failures, each of size 5, out of 10, were recorded from (3.4.1) up to pre-assigned time $t_n$. This information has been used to obtain Bayes estimates of the involved parameters and the same too are listed in Table-3.1.

2. Under type-II censoring, three sets of first 20 ordered failures (out of three sets of random samples, each of size 30, generated from the lifetime distribution in (3.4.1) on assuming parametric values as listed in Table-3.2 for the states 1, 2 and 3 respectively) have been taken. Using the recorded sample information and the corresponding expressions in section 3.5.2, M.L.E.s of the involved parameters and reliability characteristics ($\lambda, p_1, p_2, p_3$ and MTSF) in states 1, 2 and 3 have been obtained and listed in Table-3.2. The uncertainties in estimating the parameters are given in the form of variance-covariance matrices $I_4, V$ and $VI$ corresponding to the states 1, 2 and 3 respectively.

Now, for getting the estimates in the Bayesian setup, three sets of first ordered failures, each of size 5, out of 10, were taken from (3.4.1). This information has been used to obtain Bayes estimates of the involved parameters and the same too are listed in Table-3.2

3. Using the expressions in (3.3.1), (3.5.1.7) and (3.6.5) and the data set discussed above, the curves for $h(t), \hat{h}(t)$ and $h^*(t)$ for varying $t$ have been plotted in Fig. 3.1 and Fig.3.2. A comparison of these curves reveals a consistent uniform behavior of hazard rates in all the three states. Similarly, on using (3.4.2), (3.5.1.8) and (3.6.6), the curves for $R(t), \hat{R}(t), R^*(t)$ for varying $t$ have been plotted in Fig. 3.3 and Fig.3.4.

4. The variances $V(\hat{h}(t))$ and $V(\hat{R}(t))$ are listed in Tables 3.3 and 3.4 for analyzing the large sample properties of the corresponding classical estimates.
5. Posterior variances \( V\left( h^*(t) \right) \) and \( V\left( R^*(t) \right) \) are given in Table 3.5 and 3.6 for analyzing the behavior of the corresponding estimates in the Bayesian set-up.

6. The trends in Fig. 3.1 and 3.2 clearly reveal that in the present data set, Bayes estimates are observed to be less consistent as compared with M.L.E.s in states 1 and 2.

Coding in C++ language have been developed enclosed as in appendix-3.

3.8 CONCLUSION

Over-all notable feature of the results may be summarized as –

1. With observed sample information, the M.L.E.s of the parameters \( \lambda, p_1, p_2 \) and \( p_3 \) varies in terms of their large sample properties and varying duration of the three states.

2. Even experimental data can be used to update the respective Bayesian estimates of \( \lambda, p_1, p_2 \) and \( p_3 \).

3. \( h(t) \) and \( R(t) \) and their classical and Bayesian estimates also vary with arbitrarily fixed durations of the three states. Thus, for obvious reasons, the estimates obtained may also be analyzed with the changing pattern of these durations.
Table-3.1: True and estimated values of the parameters and MTSF (using type-I censored information)

<table>
<thead>
<tr>
<th>States</th>
<th>True Values</th>
<th>M.L.E.s</th>
<th>Bayes Estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>$P_3$</td>
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<td></td>
<td>MTSF</td>
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<td>13.10322</td>
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<td>$P_1$</td>
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<td>$P_2$</td>
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<td>$P_3$</td>
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<tr>
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<td>MTSF</td>
<td>7.989357</td>
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</table>

Table-3.2: True and estimated values of the parameters and MTSF (Using type-II censored information)

<table>
<thead>
<tr>
<th>States</th>
<th>True Values</th>
<th>M.L.E.s</th>
<th>Bayes Estimates</th>
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<tbody>
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<tr>
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<td>0.6999</td>
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<td>$P_2$</td>
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<td>0.13438</td>
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<td>$P_3$</td>
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<td>0.16572</td>
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<tr>
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<td>MTSF</td>
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<td>$P_1$</td>
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<td>$P_2$</td>
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<td>$P_3$</td>
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<td>MTSF</td>
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<td>8.574745</td>
</tr>
</tbody>
</table>
Variance – Co-Variance matrices: (In case of type-I censoring)

Matrix-I
\[
\begin{pmatrix}
0 & .0181 & -0.0138 & .0070 \\
.0181 & 1.7498 & -1.9592 & .15076 \\
.0182 & -1.9592 & 2.3592 & -.07497 \\
.0070 & .1508 & -0.0749 & .0654
\end{pmatrix}
\]

Matrix-II
\[
\begin{pmatrix}
.0015 & .0028 & -0.0192 & .0210 \\
.0028 & .8972 & -1.4654 & -.1963 \\
-.0192 & -1.0465 & 2.8239 & .6939 \\
-.0210 & .1963 & .6939 & .4362
\end{pmatrix}
\]

Matrix-III
\[
\begin{pmatrix}
.0188 & -.6977 & 1.0260 & -.2041 \\
-.6977 & 29.7069 & -44.7365 & 6.9075 \\
1.0260 & -44.7365 & 68.2556 & -9.6623 \\
-.2041 & 6.9075 & -9.6623 & 2.5228
\end{pmatrix}
\]

Variance – Co-Variance matrices: (In case of type-II censoring)

Matrix-IV
\[
\begin{pmatrix}
.00039 & .00070 & .00048 & -6.08E-06 \\
.00070 & .02129 & -.00096 & .00780 \\
.00048 & -.00096 & .00018 & -.01910 \\
-6.08E-06 & .00782 & -.01010 & 5.34E-05
\end{pmatrix}
\]

Matrix-V
\[
\begin{pmatrix}
.00198 & .00472 & -.02817 & -.02733 \\
.00472 & .90357 & -1.48622 & -.21671 \\
-.02817 & -1.48622 & 2.94772 & .80919 \\
-.02733 & -.21672 & .80919 & .52579
\end{pmatrix}
\]

Matrix-VI
\[
\begin{pmatrix}
.00185 & .00466 & -.02725 & -.034925 \\
.00466 & 1.11113 & -2.47683 & -.54826 \\
-.02725 & -2.47683 & 6.22248 & 1.79899 \\
-.03492 & -.54826 & 1.79899 & 1.02774
\end{pmatrix}
\]
### Table-3.3: Asymptotic Variances of $\hat{h}(t)$ and $\hat{R}(t)$ for varying $t$.
(Using type-I censored information)

<table>
<thead>
<tr>
<th>t</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V(\hat{h}(t))$</td>
<td>0.0190</td>
<td>0.0070</td>
<td>0.0044</td>
<td>0.0035</td>
<td>0.0064</td>
<td>0.0075</td>
<td>0.0086</td>
<td>0.0096</td>
</tr>
<tr>
<td>$V(\hat{R}(t))$</td>
<td>0.0045</td>
<td>0.0061</td>
<td>0.0089</td>
<td>0.0103</td>
<td>0.0029</td>
<td>0.0040</td>
<td>0.2116</td>
<td>0.0077</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>t</th>
<th>1.8</th>
<th>2.0</th>
<th>2.2</th>
<th>2.4</th>
<th>2.6</th>
<th>2.8</th>
<th>3.0</th>
<th>3.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V(\hat{h}(t))$</td>
<td>0.0105</td>
<td>0.0114</td>
<td>0.1308</td>
<td>0.1387</td>
<td>0.1459</td>
<td>0.1528</td>
<td>0.1591</td>
<td>0.1649</td>
</tr>
<tr>
<td>$V(\hat{R}(t))$</td>
<td>0.0105</td>
<td>0.0053</td>
<td>0.0639</td>
<td>0.0999</td>
<td>0.1370</td>
<td>0.2226</td>
<td>0.2276</td>
<td>0.2810</td>
</tr>
</tbody>
</table>

### Table-3.4: Asymptotic Variances of $\hat{h}(t)$ and $\hat{R}(t)$ for varying $t$.
(Using type-II censored information)

<table>
<thead>
<tr>
<th>t</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V(\hat{h}(t))$</td>
<td>0.0022</td>
<td>0.0012</td>
<td>0.0009</td>
<td>0.0007</td>
<td>0.0067</td>
<td>0.0080</td>
<td>0.0091</td>
<td>0.0102</td>
</tr>
<tr>
<td>$V(\hat{R}(t))$</td>
<td>0.003</td>
<td>0.0006</td>
<td>0.0009</td>
<td>0.0012</td>
<td>0.0027</td>
<td>0.0036</td>
<td>0.2428</td>
<td>0.0072</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>t</th>
<th>1.8</th>
<th>2.0</th>
<th>2.2</th>
<th>2.4</th>
<th>2.6</th>
<th>2.8</th>
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<th>3.2</th>
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<tbody>
<tr>
<td>$V(\hat{h}(t))$</td>
<td>0.0111</td>
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<td>0.0373</td>
<td>0.0401</td>
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<td>0.0419</td>
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<tr>
<td>$V(\hat{R}(t))$</td>
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<td>0.0668</td>
<td>0.0813</td>
<td>0.0968</td>
<td>0.1167</td>
<td>0.0416</td>
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</table>
Table-3.3: Asymptotic Variances of $\hat{h}(t)$ and $\hat{R}(t)$ for varying $t$.  
(Using type-I censored information)

<table>
<thead>
<tr>
<th>$t$</th>
<th>.2</th>
<th>.4</th>
<th>.6</th>
<th>.8</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{V}(\hat{h}(t))$</td>
<td>0.0190</td>
<td>0.0070</td>
<td>0.0044</td>
<td>0.0035</td>
<td>0.0064</td>
<td>0.0075</td>
<td>0.0086</td>
<td>0.0096</td>
</tr>
<tr>
<td>$\text{V}(\hat{R}(t))$</td>
<td>0.0045</td>
<td>0.0061</td>
<td>0.0089</td>
<td>0.0103</td>
<td>0.0029</td>
<td>0.0040</td>
<td>0.2116</td>
<td>0.0077</td>
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</table>

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<th>2.2</th>
<th>2.4</th>
<th>2.6</th>
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<th>3.2</th>
</tr>
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<tbody>
<tr>
<td>$\text{V}(\hat{h}(t))$</td>
<td>0.0105</td>
<td>0.0114</td>
<td>0.1308</td>
<td>0.1387</td>
<td>0.1459</td>
<td>0.1528</td>
<td>0.1591</td>
<td>0.1649</td>
</tr>
<tr>
<td>$\text{V}(\hat{R}(t))$</td>
<td>0.0105</td>
<td>0.0053</td>
<td>0.0689</td>
<td>0.0999</td>
<td>0.1370</td>
<td>0.2226</td>
<td>0.2276</td>
<td>0.2810</td>
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Table-3.4: Asymptotic Variances of $\hat{h}(t)$ and $\hat{R}(t)$ for varying $t$.  
(Using type-II censored information)

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<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
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</thead>
<tbody>
<tr>
<td>$\text{V}(\hat{h}(t))$</td>
<td>0.0022</td>
<td>0.0012</td>
<td>0.0009</td>
<td>0.0007</td>
<td>0.0007</td>
<td>0.0080</td>
<td>0.0091</td>
<td>0.0102</td>
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<tr>
<td>$\text{V}(\hat{R}(t))$</td>
<td>0.0003</td>
<td>0.0009</td>
<td>0.0009</td>
<td>0.0012</td>
<td>0.0027</td>
<td>0.0036</td>
<td>0.2428</td>
<td>0.0072</td>
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<th>2.6</th>
<th>2.8</th>
<th>3.0</th>
<th>3.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{V}(\hat{h}(t))$</td>
<td>0.0111</td>
<td>0.0121</td>
<td>0.0339</td>
<td>0.0356</td>
<td>0.0373</td>
<td>0.0401</td>
<td>0.0404</td>
<td>0.0419</td>
</tr>
<tr>
<td>$\text{V}(\hat{R}(t))$</td>
<td>0.0098</td>
<td>0.0052</td>
<td>0.0814</td>
<td>0.0668</td>
<td>0.0813</td>
<td>0.0968</td>
<td>0.1167</td>
<td>0.0416</td>
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</table>
Fig.-3.1: Actual and Estimated Plots of Hazard Function in Type-I Censoring

Fig.-3.2: Actual and Estimated Plots of Hazard function in Type-II Censoring
Fig. 3.3: Actual and Estimated Reliability Function
in Type-I Censoring

Fig. 3.4: Actual and Estimated Plots of Reliability Function
in Type-II Censoring