CHAPTER-2

INFERENCES ON A PROBABILISTIC
HAZARD-RATE MODEL

2.1 INTRODUCTION

In view of some practical consideration in reliability analysis, the studies like Mann, Schaffer and singpurwall [1974], Bain [1978], Hjorth [1980], and Sharma, Krishna & Singh [1997] advocated the use of hazard-rate functions for charactering a lifetime distribution. The study like Kao [1959], Barlow, Marshall and Proschan [1963], Falls[1970], Bain [1974], Hjorth [1980], and Berger & Sun [1993] used deterministic forms of composite hazard-rate models in the analysis of system reliability. In such analysis we come across a typical hazard-rate model, which generally has a bathtub shape with three segments respectively representing the initial, chance and wear out phases, e.g., Mann, Schaffer and singpurwall [1974]. The reliability characteristics have been analyzed in classical and Bayesian setup by considering any or some of these phases of the bathtub shaped hazard-rate model as analyzed in Barlow, Marshall and Proschan [1963], and Berger & Sun [1993]. Here, it should be recognized that, by using past experiences or information, the duration (or lifetime) of these three phases may be arbitrarily fixed in advance and the system, in these durations, can fail from any of the three failure patterns in a bathtub shaped hazard model respectively called as initial or chance or wear-out phase. For example, in a standard human mortality table, it is assumed that up to the age of ten years a child can die of hereditary defects. Further, it is assumed that the death between ages ten to forty years are due to chance or accidents and after the age of forty years the deaths are attributed to old age. Thus, for characterizing a probabilistic hazard rate model, a high chance of failure is attributed to the system failure when it occurs in its corresponding phase/duration and relatively low chance of
failure is attached to system failure when it occurs in a phase/duration other than its own phase/duration.

Following the concept, the present study proposes a probabilistic form of a composite hazard-rate model with random proportions and uses the same for estimating the random proportions and analyzing the system reliability characteristics in the fixed durations of the three phases in the classical and Bayesian set-up. Theoretical developments have further been highlighted with simulated data arising form the three phases of the bathtub shaped hazard-rate model (BSHM). Results analyzed are found to have strong practical applications.

2.2 NOTATIONS

\[ T \] Random variable (r.v.) denoting life-time of the device
\[ h(t) \] Hazard-rate function (HRF)
\[ H(t) = \int_0^t h(u) du \] Cumulative hazard function
\[ f(t) \] p.d.f. of \( T \)
\[ F(t) \] c.d.f. of \( T \)
\[ R(t) = P[T > t] \] Reliability/survival function for a mission time \( t \)
\[ E(T) = \int_0^\infty R(t) dt \] Mean time to system failure (MTSF)

\[ n \] Total number of item on test
\[ t = (t_1, t_2, t_3, ..., t_n) \] Failure times of \( n \) items under test
\[ L(\lambda, p_i) \] Sample likelihood function
\[ M.L.E. \] Maximum likelihood estimate
\[ SELF \] Squared error loss function
\[ \hat{\lambda} \] ML (Bayes) estimate of \( \lambda \)
\[ \hat{p}_i \] ML (Bayes) estimate of \( p_i \)
\( \hat{h}(\mathbf{t}) \)\( \hat{h}^*(\mathbf{t}) \) \( \hat{R}(\mathbf{t}) \)\( \hat{R}^*(\mathbf{t}) \) \( \text{MTSF}^2 \) \( \text{MTSF}^2 \)

\( \text{ML [Bayes] estimate of } h(t) \)

\( \text{ML [Bayes] estimate of } R(t) \)

\( \text{ML [Bayes] estimate of MTSF^2} \)

### 2.3 ASSUMPTIONS

It is assumed that:

(a) The initial, chance and wear-out phases in a bathtub shaped hazard rate model be respectively denoted as \( S_1, S_2 \) and \( S_3 \). Further, let the probabilistic form of the composite hazard rate model be

\[
\hat{h}(\mathbf{t}) = \sum_{i=1}^{3} \lambda_i p_i t^{\delta_i - 1} ; \quad t > 0 \quad \ldots (2.3.1)
\]

Here, the parameters \( \lambda_i \) and \( \delta_i \) \((i = 1, 2, 3)\) are such that \( \lambda > 0 \) and \( \delta_i < 1 \), \( \delta_2 = 1 \) and \( \delta_3 > 1 \) for the states \( S_1, S_2 \) and \( S_3 \) respectively. The arbitrarily fixed duration corresponding to the three states \( S_1, S_2 \) and \( S_3 \) are \((0 \text{-} 0.8], (0.8 \text{-} 2.0] \) and \((2.0 \text{-} 3.2] \). Also let \( p_i \left( 0 < p_i < 1; \sum_{i=1}^{3} p_i = 1 \Rightarrow p_3 = (1 - p_1 - p_2) \right) \) be the probability associated to the state \( S_i \).

(b) The cumulative hazard function is

\[
H(t) = \sum_{i=1}^{3} \lambda_i (t/\delta_i)^{\delta_i} ; \quad t > 0 \quad \ldots (2.3.2)
\]

(c) For the Bayesian analysis of the reliability characteristics, we treat \( \lambda \) and \( (p_1, p_2) \) as random variables with their respective prior p.d.f.s as

\[
g(\lambda) = \frac{\beta^\alpha}{\Gamma(\beta)} \lambda^{\beta-1} \exp(-\alpha \lambda) , \quad (\lambda, \alpha, \beta) > 0 \quad \ldots (2.3.3)
\]

and

\[
h(p_1, p_2) = \frac{\Gamma(a_1 + a_2 + a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} p_1^{a_1-1} p_2^{a_2-1} (1 - p_1 - p_2)^{a_3-1} ; (p_1, p_2) > 0, p_1 + p_2 < 1, (a_1, a_2, a_3) > 0 \quad \ldots (2.3.4)
\]
(d) $\delta_1$ is taken as known constant.

(e) $\text{SELF}$ is considered in its standard form, i.e.,

$$1. \left( \frac{0}{0} \right) \left( \frac{0}{0} \right)^2$$

... (2.3.5)

Here, $\hat{0}$ is an estimated value of the parameter $0$.

### 2.4 CHARACTERISATION OF THE LIFETIME DISTRIBUTION AND ITS RELIABILITY CHARACTERISTICS

On using (2.3.1) and (2.3.2) in a well known relationship

$$f(t) = h(t) \exp \left\{ -H(t) \right\}$$

one gets,

$$f(t) = \left( \sum_{i=1}^{3} \lambda_p t^{\delta_i - 1} \right) \exp \left\{ - \left( \lambda \sum_{i=1}^{3} \frac{p_i t^{\delta_i}}{\delta_i} \right) \right\} ; \ t > 0$$

... (2.4.1)

The reliability function is

$$R(t) = \exp \left\{ - \left( \lambda \sum_{i=1}^{3} \frac{p_i t^{\delta_i}}{\delta_i} \right) \right\}$$

... (2.4.2)

MTSF is

$$\text{MTSF} = \int_{0}^{\infty} \exp \left\{ - \left( \lambda \sum_{i=1}^{3} \frac{p_i t^{\delta_i}}{\delta_i} \right) \right\} dt$$

... (2.4.3)

### 2.5 M.L.E.s FOR PARAMETERS AND OTHER RELIABILITY CHARACTERISTICS

For the sample information $t$ from the lifetime distribution in (2.4.1), the likelihood function will be

$$L(t|\lambda, p_i) = \lambda^n \left( \prod_{i=1}^{n} \sum_{j=1}^{i} p_j t^{\delta_j - 1} \right) \exp \left\{ - \left( \lambda \sum_{j=1}^{n} \frac{p_j t^{\delta_j}}{\delta_j} \right) \right\}$$

... (2.5.1)
Thus, \( \log L_1 (t|\lambda, p_1) = n\log\lambda + \sum_{j=1}^{n} \log \left( \frac{\sum_{i=1}^{3} p_i t_j^{\delta_i - 1}}{\sum_{i=1}^{3} p_i t_j^{\delta_i}} \right) - \lambda \sum_{j=1}^{n} \frac{p_1 t_j^{\delta_1}}{\delta_1} + \frac{p_2 t_j^{\delta_2}}{\delta_2} + \frac{(1 - p_1 - p_2) t_j^{\delta_3 - 1}}{\delta_3} \)

\[
= n\log\lambda + \sum_{j=1}^{n} \log \left[ t_j^{\delta_3 - 1} + p_1 \left( t_j^{\delta_1 - 1} - t_j^{\delta_3 - 1} \right) + p_2 \left( t_j^{\delta_2 - 1} - t_j^{\delta_3 - 1} \right) \right] - \lambda \sum_{j=1}^{n} \frac{t_j^{\delta_1}}{\delta_1} + \frac{t_j^{\delta_2}}{\delta_2} + \frac{(1 - p_1 - p_2) t_j^{\delta_3 - 1}}{\delta_3}
\]

... (2.5.2)

To find the simultaneous M.L.E.s of \( \lambda, p_1 \) and \( p_2 \), we consider

\[
\frac{\partial \log L_1 (t|\lambda, p_1)}{\partial \lambda} = 0 \quad \text{which gives}
\]

\[
\left( \frac{n}{\lambda} - \sum_{j=1}^{n} \left( \frac{t_j^{\delta_3}}{\delta_3} + p_1 \left( \frac{t_j^{\delta_1}}{\delta_1} - \frac{t_j^{\delta_3}}{\delta_3} \right) + p_2 \left( \frac{t_j^{\delta_2}}{\delta_2} - \frac{t_j^{\delta_3}}{\delta_3} \right) \right) \right) = 0
\]

\[
\Rightarrow \lambda = \frac{n}{\sum_{j=1}^{n} \left( \frac{t_j^{\delta_3}}{\delta_3} + p_1 \left( \frac{t_j^{\delta_1}}{\delta_1} - \frac{t_j^{\delta_3}}{\delta_3} \right) + p_2 \left( \frac{t_j^{\delta_2}}{\delta_2} - \frac{t_j^{\delta_3}}{\delta_3} \right) \right) + \sum_{j=1}^{n} \frac{t_j^{\delta_1}}{\delta_1} - \frac{t_j^{\delta_3}}{\delta_3} + \sum_{j=1}^{n} \frac{t_j^{\delta_2}}{\delta_2} - \frac{t_j^{\delta_3}}{\delta_3}}
\]

... (2.5.3)

Similarly, \( \frac{\partial \log L_1 (t|\lambda, p_1)}{\partial p_1} = 0 \Rightarrow \)

\[
\sum_{i=1}^{n} \left( t_i^{\delta_3 - 1} + p_i \left( t_i^{\delta_1 - 1} - t_i^{\delta_3 - 1} \right) + p_2 \left( t_i^{\delta_2 - 1} - t_i^{\delta_3 - 1} \right) \right) - \lambda \left( \sum_{i=1}^{n} \frac{t_i^{\delta_1}}{\delta_1} + \sum_{i=1}^{n} \frac{t_i^{\delta_2}}{\delta_2} - \frac{t_i^{\delta_3}}{\delta_3} \right) = 0
\]

... (2.5.4)

And \( \frac{\partial \log L_1 (t|\lambda, p_1)}{\partial p_2} = 0 \) provides
\[
\sum_i \frac{1}{\bar{i}^1} \left[ \frac{\bar{t}_1^{i1}}{\bar{t}_1^{i1}} - \frac{\bar{t}_1^{i3}}{\bar{t}_1^{i3}} \right] \bar{p}_2 \left( \frac{\bar{t}_1^{i1}}{\bar{t}_1^{i1}} - \frac{\bar{t}_1^{i3}}{\bar{t}_1^{i3}} \right) = \lambda \left[ \frac{\bar{t}_2^{i2}}{\bar{t}_2^{i2}} - \frac{\bar{t}_2^{i3}}{\bar{t}_2^{i3}} \right] \left[ \frac{\bar{t}_3^{i2}}{\bar{t}_3^{i2}} - \frac{\bar{t}_3^{i3}}{\bar{t}_3^{i3}} \right] = 0 \quad \ldots \ (2.5.5)
\]

Equations in (2.5.3), (2.5.4) and (2.5.5) can be solved simultaneously by using suitable iterative procedure to get M.L.E.s for \( \lambda, p_1 \) and \( p_2 \). Finally, on using the invariance property of M.L.E.s one gets-
\[
\hat{p}_3 = 1 - \hat{p}_1 - \hat{p}_2 \quad \ldots \ (2.5.6)
\]
\[
\hat{h}(t) = \frac{\lambda}{3} \sum_{i=1}^{3} \hat{p}_i t^i - 1 \quad \ldots \ (2.5.7)
\]
\[
\hat{R}(t) = \exp \left[ -\left( \frac{1}{\delta_i} \hat{h}(t) \right) \right] \quad \ldots \ (2.5.8)
\]

and
\[
MTSF = \frac{\hat{R}}{t} \exp \left[ -\left( \frac{3}{\delta_i} \hat{h}(t) \right) \right] \quad \ldots \ (2.5.9)
\]

Further, by using general theory of M.L.E. [Rao, 1973], the asymptotic distribution of
\[
\begin{pmatrix}
\hat{\lambda} - \lambda \\
\hat{p}_1 - p_1 \\
\hat{p}_2 - p_2 \\
\hat{p}_3 - p_3
\end{pmatrix}
\]
is \( N(0, \Sigma^{-1}) \)

Where \( \Sigma \) is the Fisher’s information matrix having elements:
\[
\Sigma_{11} = E \left( -\frac{\partial^2 \log L}{\partial \lambda^2} \right), \Sigma_{22} = E \left( -\frac{\partial^2 \log L}{\partial p_1^2} \right), \Sigma_{33} = E \left( -\frac{\partial^2 \log L}{\partial p_2^2} \right), \Sigma_{44} = E \left( -\frac{\partial^2 \log L}{\partial p_3^2} \right).
\]
\[
\Sigma_{12} = E \left( -\frac{\partial^2 \log L}{\partial \lambda \partial p_1} \right), \Sigma_{13} = E \left( -\frac{\partial^2 \log L}{\partial \lambda \partial p_2} \right), \Sigma_{14} = E \left( -\frac{\partial^2 \log L}{\partial \lambda \partial p_3} \right),
\]
\[
\Sigma_{23} = E \left( -\frac{\partial^2 \log L}{\partial p_1 \partial p_2} \right), \Sigma_{24} = E \left( -\frac{\partial^2 \log L}{\partial p_1 \partial p_3} \right), \Sigma_{34} = E \left( -\frac{\partial^2 \log L}{\partial p_2 \partial p_3} \right).
\]

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Here, the exact values of various elements of $\Sigma$ are very difficult to obtain. However, following recommendations in [Cohen, 1965], the approximated values of these elements can be easily obtained as:

$$\Sigma_{11} = \left( \frac{\partial^2 \log \lambda}{\partial^2 \lambda} \right)_{\lambda, \lambda}, \Sigma_{12} = \Sigma_{21} = \left( \frac{\partial^2 \log \lambda}{\partial \lambda \partial \rho} \right)_{\lambda, \lambda} \rho$$

etc.

The elements of $\Sigma$ are:

$$\Sigma_{11} = \frac{n}{\bar{\lambda}^2}$$

$$\Sigma_{22} = \frac{n}{\bar{\lambda}^2} \left[ \sum_{i=1}^{n} \left( \frac{t_j^{\delta_2-1} - t_j^{\delta_1-1}}{t_j^{\delta_2-1} + \bar{\rho}_1 \left( t_j^{\delta_1-1} - t_j^{\delta_2-1} \right) + \bar{\rho}_2 \left( t_j^{\delta_2-1} - t_j^{\delta_1-1} \right)} \right)^2 \right]$$

$$\Sigma_{33} = \frac{n}{\bar{\lambda}^2} \left[ \sum_{i=1}^{n} \left( \frac{t_j^{\delta_3-1}}{t_j^{\delta_2-1} + \bar{\rho}_1 \left( t_j^{\delta_1-1} - t_j^{\delta_2-1} \right) + \bar{\rho}_2 \left( t_j^{\delta_2-1} - t_j^{\delta_1-1} \right)} \right)^2 \right]$$

$$\Sigma_{44} = \frac{n}{\bar{\lambda}^2} \left[ \sum_{i=1}^{n} \left( \frac{t_j^{2(\delta_3-1)}}{t_j^{\delta_2-1} + \bar{\rho}_1 \left( t_j^{\delta_1-1} - t_j^{\delta_2-1} \right) + \bar{\rho}_2 \left( t_j^{\delta_2-1} - t_j^{\delta_1-1} \right)} \right)^2 \right]$$

$$\Sigma_{12} = \Sigma_{21} = \frac{\Sigma_{11}}{\delta_1} \frac{\Sigma_{22}}{\delta_3}$$

$$\Sigma_{13} = \Sigma_{31} = \frac{\Sigma_{11}}{\delta_2} \frac{\Sigma_{22}}{\delta_3}$$

$$\Sigma_{14} = \Sigma_{41} = \frac{\Sigma_{11}}{\delta_2} \frac{\Sigma_{22}}{\delta_3}$$
\begin{align*}
\Sigma_{23} &= \Sigma_{24} = \sum_j \frac{\hat{\rho}_1(t_j^{\delta-1} - t_j^{\delta+1})}{\hat{\rho}_1(t_j^{\delta+1} - t_j^{\delta-1}) + \hat{\rho}_2(t_j^{\delta+1} - t_j^{\delta-1})^2} \\
\Sigma_{34} &= \Sigma_{44} = \sum_j \frac{t_j^{\delta-1} - t_j^{\delta+1}}{t_j^{\delta+1} - t_j^{\delta-1} + \hat{\rho}_1(t_j^{\delta-1} - t_j^{\delta+1}) + \hat{\rho}_2(t_j^{\delta+1} - t_j^{\delta-1})^2} \\
\end{align*}

**Lemma 1:** Let \( \hat{R}(t) = g(\hat{\lambda}, \hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3) \), then the asymptotic distribution of \( \hat{R}(t) \) for fixed \( t \) is \( N \left( R(t), G \Sigma^{-1} G \right) \) where \( G = (\partial R(t)/\partial \lambda, \partial R(t)/\partial \rho_1, \partial R(t)/\partial \rho_2, \partial R(t)/\partial \rho_3) \) and \( \Sigma^{-1} \) as defined above.

**Lemma 2:** Let \( \hat{h}(t) = h(\hat{\lambda}, \hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3) \), then the asymptotic distribution of \( \hat{h}(t) \), for fixed \( t \), is \( N \left( h(t), H \Sigma^{-1} H \right) \) where \( H = (\partial h(t)/\partial \lambda, \partial h(t)/\partial \rho_1, \partial h(t)/\partial \rho_2, \partial h(t)/\partial \rho_3) \) and \( \Sigma^{-1} \) is the same as above.

**Note:** The proofs for lemmas 1 and 2 can be seen in Rao, [1973].

### 2.6 ESTIMATES IN BAYESIAN SET-UP

In view of (2.5.1) and prior p.d.f.s in (2.3.3) and (2.3.4), the joint posterior p.d.f. of \( \lambda \) and \( (p_1, p_2) \) becomes

\[
\pi(\lambda, p_1, p_2 | t) = \frac{\prod_{i=1}^{\hat{n}} g(\lambda) h(p_1, p_2) d\lambda dp_1 dp_2}{\int \int \prod_{i=1}^{\hat{n}} g(\lambda) h(p_1, p_2) d\lambda dp_1 dp_2} \\
\]

\[
\lambda^{n+\beta-1} \exp \left[ -\lambda \left( \alpha + \sum_{j=1}^{\hat{n}} \delta_j \right) \right] \left( \prod_{j=1}^{\hat{n}} p_1^{\delta_j} p_2^{\delta_j-1} (1-p_1-p_2)^{\delta_j-1} \right) \\
\int \int \lambda^{n+\beta-1} \exp \left[ -\lambda \left( \alpha + \sum_{j=1}^{\hat{n}} \delta_j \right) \right] \left( \prod_{j=1}^{\hat{n}} p_1^{\delta_j} p_2^{\delta_j-1} (1-p_1-p_2)^{\delta_j-1} \right) d\lambda dp_1 dp_2 \\
\lambda > 0; (p_1, p_2) > 0, p_1 + p_2 < 1; (\alpha, \beta, \alpha_1, \alpha_2, \alpha_3, \alpha_4) > 0
\]
\[ \lambda^{n+1} \exp \left[ - \left( \lambda \left( \alpha + \sum_{j=1}^{n} \sum_{i=1}^{3} p_{ij} \delta_{i} \right) \right) \right] \left( \prod_{j=1}^{n} \sum_{i=1}^{3} p_{ij} \delta_{i} \right)^{1} p_{1}^{m-1} p_{2}^{3m-1} (1 - p_{1} - p_{2})^{3m-1} \]  

where,

\[ \phi(1, p_{1}) = \int_{0}^{1} \int_{0}^{1} (\alpha + \sum_{j=1}^{n} \sum_{i=1}^{3} p_{ij} \delta_{i}) \left( \prod_{j=1}^{n} \sum_{i=1}^{3} p_{ij} \delta_{i} \right)^{1} p_{1}^{m-1} p_{2}^{3m-1} (1 - p_{1} - p_{2})^{3m-1} \, dp_{2} \, dp_{1} \]

Therefore, on using (2.6.1), the Bayes estimate of \( \lambda \) with SELF becomes:

\[ \lambda^{*} = \text{E}(\lambda | \lambda) \]

\[ = \int_{0}^{1} \int_{0}^{1} \lambda \pi(\lambda, p_{1}, p_{2} | t) \, d\lambda \, dp_{2} \, dp_{1} \]

\[ = \frac{1}{\Gamma(n + \beta)} \int_{0}^{1} \int_{0}^{1} \lambda^{n+1} \exp \left[ - \left( \lambda \left( \alpha + \sum_{j=1}^{n} \sum_{i=1}^{3} p_{ij} \delta_{i} \right) \right) \right] \left( \prod_{j=1}^{n} \sum_{i=1}^{3} p_{ij} \delta_{i} \right)^{1} p_{1}^{m-1} p_{2}^{3m-1} (1 - p_{1} - p_{2})^{3m-1} \, dp_{2} \, dp_{1} \]

\[ = \frac{(n + \beta)}{\phi(1, p_{1})} \int_{0}^{1} \int_{0}^{1} \lambda^{n+1} \exp \left[ - \left( \lambda \left( \alpha + \sum_{j=1}^{n} \sum_{i=1}^{3} p_{ij} \delta_{i} \right) \right) \right] \left( \prod_{j=1}^{n} \sum_{i=1}^{3} p_{ij} \delta_{i} \right)^{1} p_{1}^{m-1} p_{2}^{3m-1} (1 - p_{1} - p_{2})^{3m-1} \, dp_{2} \, dp_{1} \]

Similarly, the Bayes estimate of \( p_{i} \) (i = 1, 2) under SELF will be:

\[ p_{i}^{*} = \text{E}(p_{i} | t) \]

\[ = \int_{0}^{1} \int_{0}^{1} p_{i} \pi(\lambda, p_{1}, p_{2} | t) \, d\lambda \, dp_{2} \, dp_{1} \]

\[ = \frac{1}{\phi(1, p_{1})} \int_{0}^{1} \lambda^{n+1} \exp \left[ - \left( \lambda \left( \alpha + \sum_{j=1}^{n} \sum_{i=1}^{3} p_{ij} \delta_{i} \right) \right) \right] \left( \prod_{j=1}^{n} \sum_{i=1}^{3} p_{ij} \delta_{i} \right)^{1} p_{1}^{m-1} p_{2}^{3m-1} (1 - p_{1} - p_{2})^{3m-1} \, dp_{2} \, dp_{1} \]
\[
\frac{1}{\phi(t_*, p_*)} \int_{0}^{\infty} \int_{0}^{\infty} p_j^{\alpha - 1} \left( \sum_{j=1}^{3} \frac{p_j \delta_j}{\delta_i} \right)^{\alpha/(\alpha + \beta)} \left( \prod_{j=1}^{3} p_j \delta_j^{-1} \right)^{\beta} \prod_{i=1}^{n} \left( 1 - p_i \right)^{p_i^{\delta_i}} dp_1 dp_2 dp_3 
\]

The Bayes estimate of \( p_3 \) becomes
\[
p_3^* = p_1^* - p_2^* \tag{2.6.4}
\]
and the Bayes estimates of \( h(t) \), \( R(t) \) and MTSF comes out to be
\[
\begin{align*}
\hat{h}(t) &= \frac{1}{\phi(t_*, p_*)} \int_{0}^{\infty} \int_{0}^{\infty} h(t) \left( \sum_{j=1}^{3} \frac{p_j \delta_j}{\delta_i} \right)^{\alpha/(\alpha + \beta)} \left( \prod_{j=1}^{3} p_j \delta_j^{-1} \right)^{\beta} \prod_{i=1}^{n} \left( 1 - p_i \right)^{p_i^{\delta_i}} dp_1 dp_2 dp_3 \\
&= \frac{1}{\Gamma(n + \beta)} \frac{1}{\phi(t_*, p_*)} \int_{0}^{\infty} \int_{0}^{\infty} \left( \sum_{j=1}^{3} \frac{p_j \delta_j}{\delta_i} \right)^{\alpha \gamma/(\alpha + \beta)} \exp \left[ -\gamma \left( \sum_{j=1}^{3} \frac{p_j \delta_j}{\delta_i} \right) \right] \left[ \prod_{j=1}^{3} p_j \delta_j^{-1} \right] \prod_{i=1}^{n} \left( 1 - p_i \right)^{p_i^{\delta_i}} dp_1 dp_2 dp_3 \\
&= \frac{(n+\beta)^{1/\gamma} \int_{0}^{\infty} \int_{0}^{\infty} \left( \sum_{j=1}^{3} \frac{p_j \delta_j}{\delta_i} \right)^{\alpha \gamma / (\alpha + \beta)} \exp \left[ -\gamma \left( \sum_{j=1}^{3} \frac{p_j \delta_j}{\delta_i} \right) \right] \left[ \prod_{j=1}^{3} p_j \delta_j^{-1} \right] \prod_{i=1}^{n} \left( 1 - p_i \right)^{p_i^{\delta_i}} dp_1 dp_2 dp_3 \\
\end{align*}
\]

\[
\hat{R}(t) = \frac{1}{\phi(t_*, p_*)} \int_{0}^{\infty} \int_{0}^{\infty} R(t) \left( \sum_{j=1}^{3} \frac{p_j \delta_j}{\delta_i} \right)^{\alpha/(\alpha + \beta)} \left( \prod_{j=1}^{3} p_j \delta_j^{-1} \right)^{\beta} \prod_{i=1}^{n} \left( 1 - p_i \right)^{p_i^{\delta_i}} dp_1 dp_2 dp_3 \\
&= \frac{1}{\phi(t_*, p_*)} \int_{0}^{\infty} \int_{0}^{\infty} \left( \sum_{j=1}^{3} \frac{p_j \delta_j}{\delta_i} \right)^{\alpha \gamma / (\alpha + \beta)} \exp \left[ -\gamma \left( \sum_{j=1}^{3} \frac{p_j \delta_j}{\delta_i} \right) \right] \left[ \prod_{j=1}^{3} p_j \delta_j^{-1} \right] \prod_{i=1}^{n} \left( 1 - p_i \right)^{p_i^{\delta_i}} dp_1 dp_2 dp_3 \\
&= \frac{(n+\beta)^{1/\gamma} \int_{0}^{\infty} \int_{0}^{\infty} \left( \sum_{j=1}^{3} \frac{p_j \delta_j}{\delta_i} \right)^{\alpha \gamma / (\alpha + \beta)} \exp \left[ -\gamma \left( \sum_{j=1}^{3} \frac{p_j \delta_j}{\delta_i} \right) \right] \left[ \prod_{j=1}^{3} p_j \delta_j^{-1} \right] \prod_{i=1}^{n} \left( 1 - p_i \right)^{p_i^{\delta_i}} dp_1 dp_2 dp_3 \\
\]

and

Chapter-2
\[ MTSF^* = \int_0^{1-p_1} \int_0^{1-p_2} \int_0^{1-p_3} \exp \left[ \lambda \left( \frac{1}{\delta_i} \sum_{i=1}^{n} \frac{p_i \delta_i}{\delta_i} \right) \right] \exp \left[ -\left( \alpha + \sum_{i=1}^{n} \frac{\sum_{j=1}^{3} p_i \delta_{ij}}{\delta_i} \right) \right] \right] \frac{1}{\Gamma(n+\beta)} \phi(p_1) p_1^{n-1} (1-p_1)^{\alpha-1} d\lambda dp_2 dp_3 \\
= \frac{1}{\phi(p_1)} \int_0^{1-p_1} \int_0^{1-p_2} \left( \alpha + \sum_{j=1}^{3} \frac{p_i \delta_{ij}}{\delta_i} \right) \left( \prod_{j=1}^{3} p_i \delta_{ij} \right)^{-\alpha-\beta} \right] \frac{n}{\prod_{j=1}^{3} p_i \delta_{ij}} p_1^{n-1} p_2^{a-1} (1-p_1-p_2)^{a-1} d\lambda dp_2 dp_3 \\
= \frac{(n+\beta)}{\phi(p_1)} \int_0^{1-p_1} \left( \alpha + \sum_{j=1}^{3} \frac{p_i \delta_{ij}}{\delta_i} + \sum_{j=1}^{3} \frac{p_i \delta_{ij}}{\delta_i} \right) \left( \prod_{j=1}^{3} p_i \delta_{ij} \right)^{-\alpha-\beta} \right] \frac{n}{\prod_{j=1}^{3} p_i \delta_{ij}} p_1^{n-1} p_2^{a-1} (1-p_1-p_2)^{a-1} dp_2 dp_3 \]

\[ \cdots (2.6.7) \]

2.7 AN EXAMPLE

For analyzing variations resulting in the classical and Bayesian estimates of various reliability characteristics, we consider the following data set:

1. Three sets of random samples, each of size 30, have been generated from the lifetime distribution in (2.4.1) for three sets of parametric values, as listed in Table-2.1, corresponding to the states 1, 2 and 3 respectively. Using the simulated sample information and the corresponding expressions in section- 2.5, M.L.E.s of the involved parameters and reliability characteristics \((\lambda, p_1, p_2, p_3, \text{ and } MTSF)\) in states 1, 2 and 3 have been obtained and listed in Table-2.1. The uncertainties in estimating the parameters are given in the form of variance-covariance matrices I, II and III corresponding to the states 1, 2 and 3 respectively.

2. Again, for getting the estimates in the Bayesian setup, three sets of random samples, each of size 5, were generated from (2.4.1). This
simulated sample information along with the relevant data set in Table-2.1 has been used to obtain Bayes estimates of the involved parameters and the same too are listed in Table-2.1.

3. Using the expressions in (2.3.1), (2.5.7) and (2.6.5) and the data set discussed above, the curves for $h(t)$, $\hat{h}(t)$ and $h^*(t)$ for varying $t$ have been plotted in Fig. 2.1. A comparison of these curves reveals a consistent uniform behavior of hazard rates in all the three states. Similarly, on using (2.4.2), (2.5.8) and (2.6.6), the curves for $R(t)$, $\hat{R}(t)$, $R^*(t)$ for varying $t$ have been plotted in Fig. 2.1. Here, in the present data set-up, it is observed that $\hat{h}(t)<h(t)<h^*(t)$ and $R^*(t)<R(t)<\hat{R}(t)$

4. The variances $V(\hat{h}(t))$ and $V(\hat{R}(t))$ are listed in Table-2.2 for analyzing the large sample properties of the corresponding classical estimates.

5. Posterior variances $V(h^*(t))$ and $V(R^*(t))$ are given in Table-2.3 for analyzing behavior of the corresponding estimates in the Bayesian set-up.

6. The trends in Fig. 2.1 and Fig. 2.2 clearly reveal that in the present data set, Bayes estimates are observed to be less consistent as compared with M.L.E.s in the corresponding situation.
2.8 CONCLUSION

Over-all notable feature of the results may be summarized as –

1. With observed sample information, the M.L.E.s of the parameters \( \lambda, p_1, p_2 \) and \( p_3 \) varies in terms of their large sample properties and varying duration of the three states.

2. Even experimental data can be used to update the respective Bayesian estimates of \( \lambda, p_1, p_2 \) and \( p_3 \).

3. \( h(t) \) and \( R(t) \) and their classical and Bayesian estimates also vary with arbitrarily fixed durations of the three states. Thus, for obvious reasons, the estimates obtained may also be analyzed with the changing pattern of these durations.
Table 2.1: True and estimated values of the parameters and MTSF

<table>
<thead>
<tr>
<th>States</th>
<th>True Values</th>
<th>MLE'S</th>
<th>Bayes Estimates</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>λ</td>
<td>0.0667</td>
<td>0.063406</td>
</tr>
<tr>
<td></td>
<td>p_1</td>
<td>0.75</td>
<td>0.764201</td>
</tr>
<tr>
<td>State1</td>
<td>p_2</td>
<td>0.15</td>
<td>0.149989</td>
</tr>
<tr>
<td></td>
<td>p_3</td>
<td>0.10</td>
<td>0.08581</td>
</tr>
<tr>
<td></td>
<td>MTSF</td>
<td>17.751</td>
<td>19.775</td>
</tr>
<tr>
<td></td>
<td>λ</td>
<td>0.0667</td>
<td>0.061409</td>
</tr>
<tr>
<td></td>
<td>p_1</td>
<td>0.10</td>
<td>0.113384</td>
</tr>
<tr>
<td>State2</td>
<td>p_2</td>
<td>0.75</td>
<td>0.74999</td>
</tr>
<tr>
<td></td>
<td>p_3</td>
<td>0.15</td>
<td>0.136626</td>
</tr>
<tr>
<td></td>
<td>MTSF</td>
<td>11.911</td>
<td>13.108</td>
</tr>
<tr>
<td></td>
<td>λ</td>
<td>0.0667</td>
<td>0.055253</td>
</tr>
<tr>
<td></td>
<td>p_1</td>
<td>0.15</td>
<td>0.134355</td>
</tr>
<tr>
<td>State3</td>
<td>p_2</td>
<td>0.10</td>
<td>0.09955</td>
</tr>
<tr>
<td></td>
<td>p_3</td>
<td>0.75</td>
<td>0.766095</td>
</tr>
<tr>
<td></td>
<td>MTSF</td>
<td>7.823</td>
<td>7.879</td>
</tr>
</tbody>
</table>

VARIANCE – CO-VARIANCE MATRICES:

Matrix I:

\[
\begin{pmatrix}
0 & 0.0000 & -2.084E-05 & 3.156E-05 \\
5.090E-05 & 1.054076 & 0.924873 & 0.263990 \\
-2.084E-05 & 0.924873 & 0.811504 & -0.231637 \\
3.156E-05 & 0.263990 & -0.231637 & 0.067179 \\
\end{pmatrix}
\]

Matrix II:

\[
\begin{pmatrix}
0 & -3.011E-05 & 7.093E-05 & 2.435E-05 \\
-3.011E-05 & 0.356631 & 0.312871 & 0.095170 \\
7.093E-05 & -0.312871 & 0.274465 & -0.083528 \\
3.435E-05 & 0.095170 & 0.083528 & 0.030054 \\
\end{pmatrix}
\]

Matrix III:

\[
\begin{pmatrix}
0 & 7.593E-05 & 0.000148 & 2.419E-05 \\
-7.593E-05 & 0.685488 & 0.612422 & 0.171340 \\
0.000148 & -0.612422 & 0.547103 & 0.153196 \\
2.419E-05 & 0.171340 & -0.153196 & 0.066627 \\
\end{pmatrix}
\]
Table: 2.2- Asymptotic Variances of $\hat{h}(t)$ and $\hat{R}(t)$ for varying $t$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>2</th>
<th>.4</th>
<th>.6</th>
<th>.8</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V(h(t))$</td>
<td>1.01E-03</td>
<td>3.50E-04</td>
<td>1.74E-04</td>
<td>1.02E-04</td>
<td>2.69E-05</td>
<td>2.04E-05</td>
<td>1.66E-05</td>
<td>1.43E-05</td>
</tr>
<tr>
<td>$V(R(t))$</td>
<td>2.30E-04</td>
<td>3.74E-04</td>
<td>4.80E-04</td>
<td>5.59E-04</td>
<td>2.34E-04</td>
<td>2.54E-04</td>
<td>2.71E-04</td>
<td>2.85E-04</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$t$</th>
<th>1.8</th>
<th>2.0</th>
<th>2.2</th>
<th>2.4</th>
<th>2.6</th>
<th>2.8</th>
<th>3.0</th>
<th>3.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V(h(t))$</td>
<td>1.29E-05</td>
<td>1.20E-05</td>
<td>3.23E-05</td>
<td>3.35E-05</td>
<td>3.50E-05</td>
<td>3.67E-05</td>
<td>3.85E-05</td>
<td>4.05E-05</td>
</tr>
<tr>
<td>$V(R(t))$</td>
<td>2.97E-04</td>
<td>3.07E-04</td>
<td>5.75E-04</td>
<td>5.86E-04</td>
<td>5.96E-04</td>
<td>6.84E-04</td>
<td>6.13E-04</td>
<td>6.21E-04</td>
</tr>
</tbody>
</table>

Table: 2.3- Posterior Variances of $h^*(t)$ and $R^*(t)$ for varying $t$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V(h^*(t))$</td>
<td>0.057</td>
<td>0.025</td>
<td>0.013</td>
<td>0.0066</td>
<td>0.0018</td>
<td>0.0016</td>
<td>0.0019</td>
<td>0.0023</td>
</tr>
<tr>
<td>$V(R^*(t))$</td>
<td>0.0067</td>
<td>0.0121</td>
<td>0.0163</td>
<td>0.0193</td>
<td>0.0079</td>
<td>0.0082</td>
<td>0.0082</td>
<td>0.0080</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$t$</th>
<th>1.8</th>
<th>2.0</th>
<th>2.2</th>
<th>2.4</th>
<th>2.6</th>
<th>2.8</th>
<th>3.0</th>
<th>3.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V(h^*(t))$</td>
<td>0.0029</td>
<td>0.0038</td>
<td>0.0029</td>
<td>0.0030</td>
<td>0.0032</td>
<td>0.0033</td>
<td>0.0033</td>
<td>0.0034</td>
</tr>
<tr>
<td>$V(R^*(t))$</td>
<td>0.0078</td>
<td>0.0075</td>
<td>0.0041</td>
<td>0.0051</td>
<td>0.0061</td>
<td>0.0070</td>
<td>0.0080</td>
<td>0.0088</td>
</tr>
</tbody>
</table>
Fig-2.1: Actual and Estimated Bathtub Shaped Probabilistic Hazard-rate Plots

Fig. 2.2: Actual and Estimated Reliability Plots