CHAPTER -6

CHAOS

6.1: INTRODUCTION
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6.1 INTRODUCTION

Goldreich and Peale ([70], [71], [72]) obtained a pendulum like equation for each spin-orbit state by writing the equation of motion of the planar oscillation of a satellite in terms of resonance variables and eliminating the higher frequency non-resonant terms through averaging. The strength of each resonance depends on the orbital eccentricity and the principal moments of inertia through \( \frac{B - A}{C} \). Wisdom, Peale and Mignard [218], have studied the same spin-orbit coupling in the motion of Hyperion for those cases where averaging fails. Chirikov ([39]- [42]), predicts the presence of large chaotic zone through the resonance overlap criterion.

Chirikov's Criterion:

This criterion states that when the sum of two unperturbed half-widths equals the separation of resonance centres, large-scale chaos ensues. In the spin-orbit problem the two resonance with the largest widths are the p=1 and p=3/2 states. For these two states the resonance overlap criterion becomes

\[
R_0 \sqrt{\frac{1}{H(1, e)}} + n R_0 \sqrt{\frac{3/2}{H(3/2, e)}} = \frac{1}{2}
\]

Through Chirikov's criterion, we have estimated the half width of the chaotic separatrix.
It is known that most of the Hamiltonian system give regular and irregular trajectories. Henon and Heiles [91], have shown that the phase space is divided into two regions in which trajectories behave chaotically or quasi-periodically. One of the best methods to show whether a trajectory is chaotic or quasi-periodic is through the surface of section method. In the case of quasi-periodic trajectory the points are contained in smooth curves, while for the chaotic trajectories they appear to fill up the area in the phase space in random manner.

In our spin orbit problem, which is $2\pi$-periodic in dimensionless time, we have drawn surface of section by looking at the trajectories stroboscopically with a period $2\pi$. Through surface of section method, it is observed that by varying the magnetic torque parameter some of the regular trajectories are captured into the chaotic zone. The eccentricity of the orbit and the mass distribution parameter are also effective in changing the regular motion into irregular one.

The theory is applied to an Artificial satellite of Earth. For $e = 0.0549$ the mean eccentricity of Artificial satellite, the critical value of $n$ above which large-scale chaotic behaviour is expected is 0.347.
6.2 ESTIMATION OF RESONANCE WIDTH

The Equation (2.2.4), can be written as

\[
\frac{d^2 \theta}{dt^2} + \frac{\mu}{2r^3} n^2 \sin \delta - \frac{\varepsilon \mu}{2r^3} \sin 2(\Omega - \alpha_m + \nu) = 0 \quad (6.2.1)
\]

Taking

\[
n^2 = \frac{3(B-A)}{C}, \quad \Omega = \Omega_0 + \beta_2 \nu, \quad \beta_2 = \frac{1}{\Omega_1} \frac{d\Omega}{dt},
\]

\[
\alpha_m = \alpha_1 + \beta_1 \nu, \quad \alpha_1 = \alpha_{i0} + \phi_m', \quad \beta_1 = \frac{1}{\Omega_1} \frac{d\alpha_1}{dt}
\]

Also, \( \theta = \nu + \frac{\delta}{2} \Rightarrow \delta = 2(\theta - \nu) \)

Then the Equation (6.2.1) becomes

\[
\frac{d^2 \theta}{dt^2} + \frac{\mu}{2r^3} (w_0^2 \sin(2(\theta - \nu))) - \frac{\varepsilon \mu}{2r^3} (\sin(2(\Omega_0 - \alpha_1) + b\nu)) = 0 \quad (6.2.2)
\]

In equation (6.2.2), if the units are so chosen that the orbital period of the satellite is 2\( \pi \) and its semi-major axis is 1, then the dimensionless time is equal to the mean longitude or the true anomaly which is 2\( \pi \)-periodic and \( \mu = 1 \). As \( r \) and \( \nu \) are 2\( \pi \) periodic in time, using Fourier like Poisson-Series (Wisdom et al [218]), Equation (6.2.2), becomes

\[
\frac{d^2 \theta}{dt^2} + \frac{w_0^2}{2} \sum H\left(\frac{m}{2}, e\right) \sin(2\theta - mt) - \frac{\varepsilon}{2} \sum H\left(\frac{m}{b}, e\right) \sin\left(2(\Omega_b - \alpha_1) + \frac{mt}{b_1}\right) = 0 \quad (6.2.3)
\]

where \( b = 2b_1 \).
The coefficients \( H\left(\frac{m}{2}, e\right) \) are proportional to \( e^{\frac{|m|}{2}} \) and are tabulated by Cayley [38] and Goldreich and Peale [70]. When \( e \) is small,

\[
\begin{align*}
H\left(\frac{m}{2}, e\right) &= -\frac{e}{2} \text{ when } \frac{m}{2} = \frac{1}{2} \\
&= 1 \text{ when } \frac{m}{2} = 1 \\
&= \frac{7e}{2} \text{ when } \frac{m}{2} = \frac{3}{2}
\end{align*}
\]

The half integers \( \frac{m}{2} \) and \( \frac{m}{b} \) will be denoted by the symbol \( p \) and \( p_t \).

i.e.,

\[
\begin{align*}
H(p, e) &= -\frac{e}{2} \text{ when } p = \frac{1}{2} \\
&= 1 \text{ when } p = 1 \\
&= \frac{7e}{2} \text{ when } p = \frac{3}{2}
\end{align*}
\]

and

\[
\begin{align*}
H(p_t, e) &= -\frac{e}{2} \text{ when } p_t = \frac{1}{2} \\
&= 1 \text{ when } p_t = 1 \\
&= \frac{7e}{2} \text{ when } p_t = \frac{3}{2}
\end{align*}
\]

Resonances occur whenever one of the arguments of the sine or cosine functions is nearly stationary i.e., whenever \( \left| \frac{d\theta}{dt} - p \right| \ll \frac{1}{2} \). In such cases, it is often useful to rewrite the equation of motion in terms of the slowly varying resonance variable
\[ v_p = \theta - pt \Rightarrow \frac{d^2 v_p}{dt^2} = \frac{d^2 \theta}{dt^2} \]

\[ 2v_p = 2\theta - mt \]

Thus Equation (6.2.3), can be written as

\[ \frac{d^2 v_p}{dt^2} + \frac{w_0^2}{2} \sum_{n=0} H\left( p + \frac{n}{2}, e \right) \sin 2v_p - \frac{\varepsilon}{2} \sum_{n=0} H\left( \frac{p}{b} + \frac{n}{2}, e \right) \sin 2\left( \Omega_0 - \alpha_1 + \frac{2pt}{b} \right) = 0 \]

or

\[ \frac{d^2 v_p}{dt^2} + \frac{w_0^2}{2} \sum_{n=0} H\left( p + \frac{n}{2}, e \right) \sin 2v_p - \frac{\varepsilon}{2} \sum_{n=0} H\left( \frac{p}{b} + \frac{n}{2}, e \right) \sin 2\left( \Omega_0 - \alpha_1 + \frac{2pt}{b} \right) \]

\[ + \frac{w_0^2}{2} H(p, e) \sin 2v_p - \frac{\varepsilon}{2} H\left( \frac{p}{b}, e \right) \sin 2\left( \Omega_0 - \alpha_1 + \frac{2pt}{b} \right) = 0 \quad \ldots (6.2.4) \]

If \( w_0 \) is small enough the terms in the sum will oscillate rapidly compared to the much slower variation of \( v_p \), determined by the first two terms and consequently will give little net contribution to the motion. As a first approximation for small \( w_0 \), then, these high frequency terms may be removed by holding \( v_p \), fixed and averaging Equation (6.2.4) over an orbital period. The resulting equation is

\[ \frac{d^2 v_p}{dt^2} + \frac{w_0^2}{2} H(p, e) \sin 2v_p - \frac{\varepsilon}{2} H\left( \frac{p}{b}, e \right) \sin 2\left( \Omega_0 - \alpha_1 + \frac{2pt}{b} \right) = 0 \quad \ldots (6.2.5) \]

This is an equation of a pendulum perturbed by a force \( \frac{\varepsilon}{2} H\left( \frac{p}{b}, e \right) \sin 2\left( \Omega_0 - \alpha_1 + \frac{2pt}{b} \right) \).

The Equation (6.2.5) can be studied under the cases \( \varepsilon = 0 \) and \( \varepsilon \neq 0 \).
The case \( \varepsilon = 0 \) has been studied by Wisdom et al [218]. When \( \varepsilon \neq 0 \) the Equation (6.2.5), represents the equation of motion of disturbed pendulum given by

\[
\frac{d^2x_p}{dt^2} + f'(x_p) = m_p\phi'(t, e) \tag{6.2.6}
\]

where

\[
x_p = 2v_p,
\]

\[
f'(x_p) = k_{1p}^2 \sin x_p,
\]

\[
k_{1p}^2 = w_0^2 H(p, e),
\]

\[
m_p = \frac{k_{2p}^2}{b}, \quad k_{2p}^2 = cH\left(\frac{p}{b}, e\right),
\]

\[
\phi'(t, e) = \sin \frac{4p}{b}(t + a_3), \quad a_3 = \frac{(\Omega_0 - \alpha_1)b}{2p}
\]

The unperturbed part of Equation (6.2.6) is

\[
\frac{d^2x_p}{dt^2} + f'(x_p) = 0
\]

Multiplying by \( 2 \frac{dx_p}{dt} \), we get

\[
2 \frac{dx_p}{dt} \frac{d^2x_p}{dt^2} + 2k_{1p}^2 \sin x_p \frac{dx_p}{dt} = 0
\]

Integrating, we get

\[
\left(\frac{dx_p}{dt}\right)^2 = -2k_{1p}^2 \int \sin x_p dx_p + c_{1p}
\]

\[
= c_{1p} + 2k_{1p}^2 \cos x_p
\]
where $c_{1p}$ is constant of integration.

The motion to be real if $c_{1p} + 2k_{1p}^2 \geq 0$

There are three Categories of motion depending upon

$c_{1p} > 2k_{1p}^2$, $c_{1p} < 2k_{1p}^2$ and the intermediate case $c_{1p} = 2k_{1p}^2$

**CATEGORY-I:** $c_{1p} > 2k_{1p}^2$

As $c_{1p} > 2k_{1p}^2$. So \( \frac{dx_{1p}}{dt} \) never vanishes in this case, it is always either positive or negative and the pendulum is making complete revolution in one sense or the other.

In this case the unperturbed solution is

\[
x_{1p} = l_{1p} + c_{1p} \sin l_{1p} + O(c_{1p}^2),
\]

where, $l_{1p} = n_{1p} t + \varepsilon_{1}$, $c_{1p} = \frac{k_{1p}^2}{n_{1p}^2}$

and

\[
\frac{1}{n_{1p}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{dx_{1p}}{(c_{1p} + 2k_{1p}^2 \cos x_{1p})^{1/2}}.
\]

c_{1p}, \varepsilon_{1}$ are arbitrary constants and $l_{1p}$ is an argument The periodic portion of this series can be regarded as an oscillation about the mean state of motion which is revolution with a period \( \frac{2\pi}{n_{1p}} \). The half amplitude of the oscillation is evidently less than \( \pi \) and it decreases as increases. Here, we may observe that \( \frac{dx_{1p}}{dt} \neq 0 \) and the motion is said to be of the **Type-I** i.e. revolution.
In case of perturbed pendulum by making use of the theory of variation of parameters Brown and Shook [31], we have

\[ \frac{dC_{1p}}{dt} = \frac{m}{k_p} \frac{\partial x}{\partial l} \dot{\phi}, \]

\[ \frac{dl_p}{dt} = n - \frac{m}{k_p} \frac{\partial x}{\partial l} \dot{\phi} \]

where

\[ k_p = \frac{\partial}{\partial l} \left( n^2 \frac{\partial x}{\partial l} \right) \frac{\partial x}{\partial l} - n \frac{\partial^2 x}{\partial l^2} \frac{\partial x}{\partial C_1} \]

...(6.2.8)

Now, as

\[ c_{1p} = \frac{k_{1p}^2}{n_p^2} \Rightarrow n_p^2 = \frac{k_{1p}^2}{c_{1p}} \]

Therefore,

\[ \frac{\partial n_p}{\partial c_{1p}} = -\frac{n_p}{2c_{1p}}, \]

\[ \frac{\partial x_p}{\partial l_p} = 1 + c_{1p} \cos l_p, \quad \frac{\partial^2 x_p}{\partial l_p^2} = -c_{1p} \sin l_p \]

\[ \frac{\partial x_p}{\partial c_{1p}} = \sin l_p, \quad \frac{\partial^2 x_p}{\partial c_{1p} \partial l_p} = \cos l_p \]

Substituting the above values and writing \( k_p = k_{1p} \), Equation (6.2.8) becomes

\[ k_{1p} = -\frac{n_p}{2c_{1p}} - \frac{n_p c_{1p} \cos^2 l_p}{2} + n_p c_{1p} \]

\[ \approx -\frac{n_p}{2c_{1p}} \]
Substituting the value of \( k_{i\rho} \) in Equation (6.2.7), we get

\[
\frac{dc_{i\rho}}{dt} = -\frac{2m_{\rho}k_{i\rho}^2}{n_{\rho}^2} \sin \frac{4p}{b} (t + a_3)
\]

since, both \( m_{\rho} \) and \( k_{i\rho}^2 \) are small quantities, the term \( m_{\rho}k_{i\rho}^2 \) is of the third order and therefore rejected.

Thus,

\[
\frac{dc_{i\rho}}{dt} \equiv 0
\]

so, \( c_{i\rho} \) is a constant up to second order of approximation.

The second equation of Equation (6.2.7), gives

\[
\frac{dl_{\rho}}{dt} = n_{\rho} + \frac{2m_{\rho}c_{i\rho}}{n_{\rho}} \sin l_{\rho} \sin \frac{4p}{b} (t + a_3),
\]

\[
\frac{d^2l_{\rho}}{dt^2} = m_{\rho} \sin \frac{4p}{b} (t + a_3) - \frac{2m_{\rho}^2c_{i\rho}}{n_{\rho}^2} \sin l_{\rho} \sin \frac{4p}{b} (t + a_3) - \frac{4m_{\rho}^2c_{i\rho}}{n_{\rho}^2} \sin l_{\rho} \sin \frac{4p}{b} (t + a_3)
\]

\[
+ 2m_{\rho}c_{i\rho} \cos l_{\rho} \sin \frac{4p}{b} (t + a_3) + \frac{4m_{\rho}^2c_{i\rho}^2}{n_{\rho}^2} \cos l_{\rho} \sin l_{\rho} \sin \frac{4p}{b} (t + a_3)
\]

\[
+ \frac{8m_{\rho}c_{i\rho}p}{n_{\rho}b} \sin l_{\rho} \cos \frac{4p}{b} (t + a_3).
\]

In this equation on R.H.S. all terms except first one are of second or higher order. Rejecting, we get

\[
\frac{d^2l_{\rho}}{dt^2} \equiv m_{\rho} \sin \frac{4p}{b} (t + a_3).
\]

Also,
\[ l_\rho = n_\rho t + \varepsilon_1 \Rightarrow t = \frac{l_\rho - \varepsilon_1}{n_\rho}, \]

Therefore,
\[
\frac{d^2 l_\rho}{dt^2} \equiv m_\rho \sin \frac{4\rho}{b} \left( \frac{l_\rho - \varepsilon_1}{n_\rho} + a_3 \right)
\]
\[ \equiv 2m_\rho \sin \frac{2pl_\rho}{bn_\rho} \]

In the first approximation, taking \( n_\rho = n_{\rho 0} \), we get
\[
\frac{d^2 l_\rho}{dt^2} \equiv 2m_\rho \sin \frac{2pl_\rho}{bn_{\rho 0}}
\]

If we take \( \frac{2pl_\rho}{bn_{\rho 0}} = x_\rho \), then the above equation becomes
\[
\frac{d^2 x_\rho}{dt^2} + k_{3_\rho^2} \sin x_\rho = 0 , \quad \text{...(6.2.9)}
\]

where, \( k_{3_\rho^2} = \frac{4\rho}{bn_{\rho 0}}, \) \( m_\rho = -\varepsilon \)

or \( \left( \frac{dx_\rho}{dt} \right)^2 = c_{2_\rho} + 2k_{3_\rho^2} \cos x_\rho \)

where \( c_{2_\rho} \) is constant of integration.

Equation (6.2.9) describes the motion of pendulum, we get again three types of motion, Type I is that in which \( \frac{dx_\rho}{dt} \) is never zero, Type II is that in which \( \frac{dx_\rho}{dt} = 0 \), at 0 or \( \pi \),

For \textbf{Type-I}, our solution is
\[
x_p = N_p t + e_2 + \frac{k_3}{N_p^2} \sin(N_p t + e_2) + \frac{k_3^2}{8N_p^4} \sin 2(N_p t + e_2) + .......... \\
\]

and
\[
\frac{1}{N_p} = \frac{1}{2\pi} \int_0^{2\pi} \frac{dx_p}{(c_{2p} + 2k_3^2 \cos x_p)^{1/2}},
\]

where \( c_{2p}, \ e_2 \) are arbitrary constants.

In the first approximation, taking \( N_p = N_{p0} \), we get
\[
x_p = x_{p0} + \frac{k_3}{N_p} \sin x_{p0}
\]

where
\[
x_{p0} = N_{p0} t + e_2
\]

This is the case of \textbf{revolution}.

For the \textbf{Type II}, the solution is
\[
x_p = \lambda \sin(p't + \lambda_0)
\]

where
\[
p' = 2 \sqrt{\frac{ep}{bn_{p0}}}
\]

\( \lambda \) and \( \lambda_0 \) being arbitrary constants. This is the case of \textbf{libration}.

\textbf{TYPE III}, occurs when \( c_{2p} = 2k_3^2 = \frac{8ep}{bn_0} \)

The solution is
\[
x_p + \pi = 4 \tan^{-1} e^{\lambda_1'} + \alpha_0
\]

where \( \alpha_0 \) is an arbitrary constant and the other having a particular value.
When $t \to \pm \infty$, $x_p \to \pm \pi$, at both places, $\left( \frac{dx_p}{dt} \right) = 0$ and all higher derivatives of $x_p$ approach to zero. Near this point, while $x_p$ approaches to one of the limits $\pm \pi$, $t$ tends to become an indeterminate function of $x_p$. This is the case of \textit{infinite period separatrix} which is asymptotic forward and backward in time to the unstable equilibrium.

Thus the results of type I, type II and type III enable us to conclude that the magnetic torque perturbation plays a significant role. It may change a \textit{revolution} to \textit{libration} or to \textit{infinite period separatrix}.

\textbf{CATEGORY II: $c_{1p} < 2k_{1p}^2$}

In this case unperturbed solution is

$$x_p = c_{1p} \sin l_p + O(c_{1p}^3)$$

where

$$n_p = k_{1p}\left[ 1 - \frac{1}{16} c_{1p}^2 + \ldots \right],$$

$$l_p = n_p t + \varepsilon_1,$$

$c_{1p}$ and $\varepsilon_1$ are arbitrary constants.

In case of perturbed equation again, using the theory of variation of parameters, we get

$$\frac{dc_{1p}}{dt} \equiv \frac{m_p}{k_{1p}} \cos l_p \sin \frac{4p}{b} (t + a_3),$$

$$\frac{dl_p}{dt} \equiv k_{1p} - \frac{m_p}{k_{1p} c_{1p}} \sin l_p \sin \frac{4p}{b} (t + a_3),$$
\[
\frac{d^2 l_p}{dt^2} = -\frac{m_p}{c_{1p}} \cos l_p \sin l_p \frac{4 p}{b} (t + a_3) - \frac{4 p m_p}{b k_{1p} c_{1p}} \sin l_p \cos \frac{4 p}{b} (t + a_3)
\]

Also, \( x_p = c_{1p} \sin l_p, \quad l_p = n_p t + \varepsilon_1 \Rightarrow t = \frac{l_p - \varepsilon_1}{n_p} \)

Therefore,

\[
\frac{d^2 l_p}{dt^2} = -\frac{4 p m_p}{b k_{1p} c_{1p}} \sin l_p \cos \frac{4 p}{b} \left( \frac{l_p - \varepsilon_1}{n_p} + a_3 \right)
\]

In the first approximation of \( n_p = n_{p0}, \quad c_{1p} = c_{1p0} \), we get

\[
\frac{d^2 l_p}{dt^2} = -\frac{4 p m_p}{b k_{1p} c_{1p0}} \sin l_p \cos \frac{4 p}{b} \left( \frac{l_p - \varepsilon_1}{n_{p0}} + a_3 \right)
\]

\[\text{...(6.2.10)}\]

As a special case, let us assume that

\[
\frac{2 p}{b} \left( \frac{l_p - \varepsilon_1}{n_{p0}} + a_3 \right) = \frac{n_i \pi}{2}, \quad n_i \in I
\]

When \( n_i \) is odd, the Equation (6.2.10) becomes

\[
\frac{d^2 l_p}{dt^2} = -\frac{4 p m_p}{b k_{1p} c_{1p0}} \sin l_p
\]

or

\[
\frac{d^2 l_p}{dt^2} + k_{4p}^2 \sin l_p = 0, \quad k_{4p}^2 = -\frac{4 p m_p}{b k_{1p} c_{1p0}} > 0 \quad \text{as} \quad m < 0
\]

which is again the equation of pendulum. As in previous case this equation gives us revolution, libration and infinite period separatrix motion.

On the other hand, if \( n_i \) is even

Then,

\[
\frac{d^2 l_p}{dt^2} = -\frac{4 p m_p}{b k_{1p} c_{1p0}} \sin l_p = k_{4p}^2 \sin l_p
\]

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When \( I_p \) is small, the solution of above equation is given by
\[
I_p = e^{k_i x_p} + e^{-k_i x_p}
\]

**CATEGORY III:** \( c_{i_p} = 2k_{i_p}^{-2} \)

The unperturbed solution is
\[
x_p + \pi = 4\tan^{-1} e^{k_i x_p} + \alpha_0,
\]
where \( \alpha_0 \) is an arbitrary constant and the other having a specific value.

This is the case of infinite period separatrix which is asymptotic forward and backward in time to the unstable equilibrium. We are mainly concerned in this category of motion. In this category the nature of the unperturbed solution does not change by taking into account the magnetic torque.

Near the infinite period separatrix broadened by the high frequency term into narrow chaotic band (Chirikov – [40]), for small \( n \), the half width of the chaotic separatrix is given by
\[
\omega_p = \frac{I_p - I_p^s}{I_p^s} = 4\pi\varepsilon_i \lambda^3 e^{-\pi\lambda / 2},
\]
where \( \varepsilon_i \) is the ratio of the coefficient of the nearest perturbing high-frequency term to the coefficient of the perturbed term, and \( \lambda = \frac{\Omega}{\omega} \) is the ratio of the frequency difference between the resonant term and the nearest non resonant term (\( \Omega \)) to the frequency of small-amplitude liberations (\( \omega \)).
For the synchronous spin-orbit state perturbed by the third-body torque, we have

\[ \lambda = \frac{1}{n}, \]

\[ \epsilon_1 = \epsilon = \text{third body torque parameter}, \]

\[ \omega_1 = \frac{I_1 - I_1^s}{I_1^s} = \frac{4\pi \epsilon}{n^3} e^{-\frac{\pi}{2n^2}}. \]

Here, \( \omega_1 \) increases both with \( \epsilon \) and \( n \). An estimate of \( n \) at which the wide spread chaotic behaviour can be observed is given by using the Chirikov's overlap criterion. This criterion states that when the sum of two unperturbed half-widths equals the separation of resonance centers, large-scale chaos ensues. In the spin-orbit problem the two resonances with the largest widths are the \( p=1 \) and \( p=\frac{3}{2} \) states. For these two states the resonance overlap criterion becomes

\[ n R_0 \sqrt{|H(1, e)|} + n R_0 \sqrt{|H(3/2, e)|} = \frac{1}{2}, \]

or

\[ n R_0 = \frac{1}{2 + \sqrt{14\epsilon}}. \]

The theory is applied to an Artificial satellite of Earth. For \( e = 0.0549 \) the mean eccentricity of Artificial satellite, the critical value of \( n \) above which large-scale chaotic behaviour is expected is 0.347.
6.3 THE SPIN ORBIT PHASE SPACE

It is known that most of the Hamiltonian systems give regular and irregular trajectories. Henon and Heiles [91], have shown that the phase space is divided into two regions in which trajectories behave chaotically or quasi-periodically. One of the best methods to show whether a trajectory is chaotic or quasi-periodic is through the surface of section method.

In our spin orbit problem, which is $2\pi$-periodic in dimensionless time, we have drawn surface of section by looking at the trajectories stroboscopically with period $2\pi$. The section has been drawn with $\frac{dq}{dv}$ versus $v$ at every periapse passage. In the case of quasi-periodic trajectory the points are contained in smooth curves while for the chaotic trajectories they appear to fill up the area in the phase space in random manner. Since the orientation denoted by $q$ is equivalent to the orientation denoted by $\pi + q$, we have, therefore, restricted the interval from 0 to $\pi$.

In the case of non-resonant quasi-periodic trajectories successive points on the surface of section will trace a simple curve which covers all values of $q$. For small values of $n$ resonance states will be separated from non-resonant states by a narrow chaotic zone. All these possibilities are shown from Figure 41 to Figure 60 for various values of $\varepsilon$, $e$, $n$ and $a_1$.

For figures 41 to 56, $a_1=0.1153$, $b = 2.164264351851$, we have plotted two different figures for $n=0.1$ and $e=0.0549$ and for different values of $\varepsilon$.
i.e. for $\varepsilon=0$, 0.001, (Fig. 41), $\varepsilon=0.01$, 0.1, (Fig. 42), $\varepsilon=0.2$, 0.3, (Fig. 43), $\varepsilon=0.4$, 0.5, (Fig. 44), $\varepsilon=0.6$, 0.7, (Fig. 45), $\varepsilon=0.8$, 0.9, (Fig. 46). At $\varepsilon=0.001$, $c=0.0549$, for different values of $n$ i.e. for $n=0.1$, 0.2, (Fig. 47), $n=0.3$, 0.4, (Fig. 48), $n=0.5$, 0.6, (Fig. 49), $n=0.7$, $n=0.8$, (Fig. 50), $n=0.9$, 0.99, (Fig. 51). At $n=0.1$ and $\varepsilon=0.001$, for different values of $c$ i.e. for $c=0.0001$, 0.001, (fig. 52), $c=0.01$, 0.1, (fig. 53), $c=0.2$, 0.3 (Fig. 54), $c=0.4$, 0.5, (Fig. 55), $c=0.6$, 0.7, (Fig. 56).

For figures 57, 58, 59, we have plotted two different figures at $b = 2.164264351851, n=0.1$, $c=0.0549$ and $\varepsilon=0.001$ for different values of $a_1$ i.e. for $a_1=0.1153$, 0.1454, (Fig. 57), $a_1=0.1659$, 0.1849, (Fig. 58), $a_1=0.1947$, 0.2072, (Fig. 59).

For the Artificial Satellite case, $a_1=0.1153$, $b = 2.164264351851$, we have plotted two different figures for $\varepsilon = 0.0549$, $n=0.347$ and for different values of $\varepsilon$ i.e. $\varepsilon=0.0$, 0.5, (Fig. 60).

In each of these figures, we have plotted $\frac{dq}{dv}$ versus $q$, at every periapse passage. It may be observed that the chaotic separatrix surrounds each of the resonance states, and each of these chaotic zones is separated from others by non-resonant quasi-periodic rotation trajectories.

From Figures 41 to 46, we observed that as the magnetic torque is introduced (comparing with the Wisdom Case, $\varepsilon=0$) the regular curves starts disintegrating and this disintegration increases with the increase in $\varepsilon$. 

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From Figures 47 to 51, we observed that as $n$ increases, the regular curves disintegrate and this disintegration increases with the increase in $n$.

From Figures 52 to 56, we observed that as the eccentricity increases, the regular curves disintegrate and this disintegration increases with the increase in $e$.

From Figures 57 to 59, we observed that as $a_1$ increases, regular curves almost remain same.

In Figure 60, we have plotted for an Artificial Satellite with parameters $e = 0.0549$, $n = 0.347$, for $\varepsilon = 0.0$ and $\varepsilon = 0.5$. We observed that due to magnetic torque, the chaotic zone is further increased.

From Figures 41 to 56, we conclude that the regular curves start disintegrating due to magnetic torque (i.e. for different values of $\varepsilon$), the increase in eccentricity ($e$) and the irregular mass distribution of the satellite (i.e. for different values of $n$) and this disintegration increases with the increase in $\varepsilon$, $e$ and $n$. 
6.4 CONCLUSION

From these studies, we conclude that the magnetic torque plays a very significant role in changing the motion of revolution into libration or infinite period separatrix. The half width of the chaotic separatrices estimated by the Chrikov's criterion is not effected by the magnetic torque.

We further conclude, that in the spin–orbit phase space the regular curves start disintegrating due to magnetic torque, the increase in the eccentricity and the irregular mass distribution of the satellite and this disintegration increases with the increase in $e$, $n$ and $e$. 