CHAPTER -2

NON LINEAR PLANAR OSCILLATION

OF A SATELLITE

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2.1 INTRODUCTION

The problem is studied as follows: "To discuss the chaos in non-planar oscillation of a satellite under the influence of magnetic torque in an elliptic orbit, when orbital plane of the satellite coincides with the equatorial plane of the central body".

Magnetic disturbance torque results from the interaction between the spacecraft's residual magnetic field and the geomagnetic field. The primary sources of magnetic disturbance torques are

(i) spacecraft magnetic moments,
(ii) eddy currents and
(iii) hysteresis.

Of these, the spacecraft's magnetic moment is usually the dominant source of disturbance torques. The spacecraft is usually designed of material selected to make disturbances from the other source negligible. Bastow [5] and Droll and Iuler [52] provide a survey of the problems associated with minimizing the magnetic disturbances in spacecraft designs and development.

The instantaneous magnetic disturbance torque, \( \bar{N}_{\text{mag}} \) in N.m, due to the spacecraft effective magnetic moment \( \bar{m} \) (in A.m\(^2\)) is given by

\[
\bar{N}_{\text{mag}} = \bar{m} \times \bar{B},
\]

...(2.1.1)
where $\mathbf{B}$ is the geocentric magnetic flux density (in Wb/m$^2$) and $\mathbf{m}$ is the sum of the individual magnetic moments caused by permanent and induced magnetism and the spacecraft generated current loops [Wertz, [213]].

The magnetic torque is applicable below synchronous orbit less than 35,000 km. For a 40,000 pole-cm electromagnet at an altitude of 550 km the magnetic torque is approximately 0.001 Nm. This means that the magnetic torque perturbation parameter $\varepsilon$ is in general very small ($\varepsilon<<1$).

Taking the magnetic torque to be of the order of eccentricity, the Hamiltonian function, the hyperbolic equilibrium solution and the double asymptotic solutions corresponding to the unperturbed Hamiltonian $H_0$ has been derived.

Let us consider $2\pi$ - periodic Hamiltonian system with one degree of freedom. Hamiltonian function is assumed to be analytic with respect to its arguments and depends on small parameter:

$$H = H(x, t, \varepsilon) = H_0(x) + \varepsilon H_1(x, t) + 0(\varepsilon^2),$$

$$x = (q, p) \quad \ldots \quad (2.1.2).$$

We assume that the unperturbed system ($\varepsilon = 0$) possesses hyperbolic equilibrium $x_0=0$ and and let $\bar{x}(t)$ be double asymptotic solution to $x_0$ i.e. $\lim_{t \to \pm \infty} \bar{x}(t) = x_0$ as $t \to \pm \infty$. In the extended phase space $(x, t)$, we have two asymptotic surfaces $o^0_u, o^0_s$ formed by solutions tending asymptotically to $x_0$ as $t \to \pm \infty$ respectively. In the unperturbed system they are doubly (coincide). For sufficiently small $\varepsilon$, there exists hyperbolic $2\pi$-periodic
solution \( x_0(t) \). In general, its asymptotic surfaces \( \omega_u^0, \omega_s^0 \) do not coincide and cross transversely. Points belonging to both of the surfaces are called homoclinic.
2.2 EQUATION OF MOTION

Consider a rigid satellite S moving in an elliptic orbit around the Earth E such that the orbital plane coincides with the equatorial plane of the Earth (Fig. 1.). The satellite is assumed to be a triaxial body with principal moments of inertia \( A < B < C \) at its centre of mass and \( C \) is the moment of inertia about the spin axis which is perpendicular to the orbital plane. These principal axes are taken as the co-ordinate axes \( x, y, z \); the \( z \)-axis being perpendicular to the orbital plane.

Let \( \vec{r} \) be the radius vector of the center of mass of the satellite, \( \theta \) the angle that the long axis of the satellite make with a fixed line \( EF \) lying in the orbital plane and \( \delta / 2 \) the angle between the radius vector and the long axis. The magnetic torque is calculated in the \( 'l, b, n' \) system of coordinates, the system for which the plane of the spacecraft orbit is the equatorial plane of the coordinate system. The \( l \) axis is parallel to the line from the centre of the Earth to the ascending node of the spacecraft orbit, the \( n \)-axis is perpendicular to the orbital plane and the \( b \) axis is such that for the unit vectors along the axes, \( \hat{b} = \hat{n} \times \hat{l} \). The \( l, b, n \) system would be inertial if the spacecraft orbit were fixed in inertial space. It may be noted that the \( z \)-axis coincides with the \( n \)-axis of \( l, b, n \) system.

Euler's dynamical equation of motion of the satellite about the \( z \)-axis is given by

\[
C \frac{d\omega_z}{dt} + (B - A)\omega_y \omega_z = G_z + N_z, \tag{2.2}
\]

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FIG. 1. Satellite planar oscillation in elliptic orbit with magnetic perturbation
where, \( G_z = z\)-component of the gravitational torque,
\[ N_z = z\)-component of the magnetic torque,
\[ \omega_1, \omega_2, \omega_3 = \text{angular velocities about the principal axes at the centre of mass of the satellite.}

Here, \( \omega_1 = \omega_2 = 0 \) and \( \omega_3 = \frac{d\theta}{dt} \). So, the Euler’s dynamical equation of motion becomes
\[ C \frac{d^2\theta}{dt^2} = G_z + N_z \quad \text{...(2.2.1)} \]

where,
\[ G_z = -\frac{3\mu}{2r^3} (B - A) \sin(\delta) \quad \text{(Bhatnagar and Bhardwaj [22])}, \]
\[ \mu = G(m_e + m_s) = Gm_e, \]
\[ m_e = \text{mass of the Earth}, \]
\[ m_s = \text{mass of the satellite}. \]

In equation (2.1.1)
\[ \overline{B(r)} = \frac{a^3 H_0}{r^3} \left( 3(m_e \hat{r} - \hat{m}) \right) \quad \text{...(2.2.2)} \]

Since equatorial plane of the earth coincides with the orbital plane of the satellite, so \( i = 0 \).

Therefore,
\[ \hat{m} = \begin{pmatrix} \sin(\theta_m') \cos(\Omega - \alpha_m) \\ -\sin(\theta_m') \sin(\Omega - \alpha_m) \\ \cos(\theta_m') \end{pmatrix} , \]
\[
\hat{r} = \begin{pmatrix} \cos(v) \\ \sin(v) \\ 0 \end{pmatrix},
\]

\[
\alpha_m = \alpha_{G_0} + \frac{d\alpha_{G}}{dt} t + \phi_m',
\]

... (2.2.3)

\[a^3 H_0 = \text{total dipole strength},\]

\[\theta'_m = \text{coelevation of the dipole},\]

\[\phi'_m = \text{east longitude of the dipole},\]

\[\alpha_{G_0} = \text{right ascension of the Greenwich meridian at some reference time},\]

\[\frac{d\alpha_{G}}{dt} = \text{average rotation rate of the earth},\]

\[t = \text{time since reference},\]

\[m = \text{magnet strength of the spacecraft},\]

\[v = \text{true anomaly measured from the ascending node}.\]

\[\Omega = \text{right ascension of ascending node},\]

\[i = \text{orbital inclination}.\]

All these quantities are constants for a particular year. For example, in 1978

\[a^3 H_0 = 7.943 \times 10^{15} \text{ Wb.m},\]

\[\theta'_m = 168.6^\circ,\]

\[\phi'_m = 109.3^\circ,\]

\[\alpha_{G_0} = 98.8279^\circ \text{ at } 0^\text{h} \text{ UT, December 31,}\]
\[
\frac{d\alpha_G}{dt} = 360.9856469^0/\text{day},
\]
\[
\Omega = \Omega_0 + \frac{d\Omega}{dt}t
\]

where,
\[
\frac{d\Omega}{dt} = -2.06474 \times 10^{-14} \frac{a^{-7/2} \cos i}{(1-e^2)^2}
\]

=0.9856^0/\text{day}.

Therefore,
\[
\bar{m} \bar{r} = \sin(\theta_m') \cos(\Omega - \alpha_m') \cos v - \sin(\theta_m') \sin(\Omega - \alpha_m') \sin(v)
\]

Substituting this into the equation (2.2.2) and simplifying yields
\[
B_i = \frac{a^3 H_0}{r^3} \left[ \sin(\theta_m') \cos(\Omega - \alpha_m') (3 \cos^2 v - 1) - \frac{3 \sin 2v}{2} [\sin(\theta_m') \sin(\Omega - \alpha_m')] \right]
\]
\[
B_h = \frac{a^3 H_0}{r^3} \left[ \frac{3}{2} \sin 2v \sin(\theta_m') \cos(\Omega - \alpha_m') - (3 \sin^2 v - 1)(\sin(\theta_m') \sin(\Omega - \alpha_m')) \right]
\]
\[
B_n = -\frac{a^3 H_0}{r^3} [\cos(\theta_m')]
\]

Negative sign in the orbit normal component $B_n$ assures the northward direction of the field lines.

The components of $\bar{m}$ in $(l, b, n)$ system are
\[
m_i = m \sin(\theta_m') \cos(\Omega - \alpha_m'),
\]
\[
m_h = -m \sin(\theta_m') \sin(\Omega - \alpha_m'),
\]
\[
m_n = m \cos(\theta_m').
\]

From the equation (2.1.1), the $z$-component of $N$, is
\[
N_z = m [m_i B_n - B_h m_h]
\]
\[
\frac{d^2 \theta}{dt^2} = -\frac{\mu}{2r^3} n^2 \sin \delta + \frac{\varepsilon \mu}{2r^3} \sin 2(\Omega - \alpha_m + \nu)
\]

where

\[
\Omega = \Omega_0 + \frac{d \Omega}{dt} \quad \Omega_i = \Omega_0 + \beta_2 \nu, \quad \beta_2 = \frac{1}{\Omega_i} \frac{d \Omega}{dt},
\]

\[
\alpha_m = \alpha_1 + \beta_1 \nu, \quad \alpha_1 = \alpha_{g0} + \phi', \quad \beta_1 = \frac{1}{\Omega_i} \frac{d \alpha_i}{dt},
\]

\[
n^2 = \frac{3(B - A)}{C}, \quad \varepsilon = \frac{3a^3 m H_0}{\mu C} \sin^2 \theta',
\]

\(\varepsilon\) is the parameter due to magnetic torque, \(\Omega_i\) is angular velocity of the moon.

Since the orbit of the rigid satellite is elliptic, therefore

\[r^2 \dot{v} = h \quad \text{and} \quad \frac{l}{r} = 1 + e \cos \nu.\]

Also from the Figure(1), \(\theta = v + \frac{\delta}{2}\)

Taking true anomaly \(v\) as the independent variable, then

\[
\frac{d \theta}{dt} = \frac{dv}{dt} + \frac{1}{2} \frac{d \delta}{dt}
\]

or

\[
\frac{d \theta}{dt} = \frac{h}{r^2} + \frac{1}{2} \frac{d \delta}{dv} \frac{dv}{dt}
\]

or

\[
\frac{d \theta}{dt} = \frac{h}{l^2} (1 + e \cos v)^2 \left[ 1 + \frac{1}{2} \frac{d \delta}{dv} \right]
\]

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Therefore,

\[ \frac{d^2 \theta}{dt^2} = \frac{d}{dt} \left( \frac{h}{l^2} (1 + e \cos v)^2 \left[ 1 + \frac{1}{2} \frac{d \delta}{dv} \right] \right) \]

or

\[ \frac{d^2 \theta}{dt^2} = -\frac{2e h}{l^2} (1 + e \cos v) \sin v \frac{h}{l^2} (1 + e \cos v)^2 \left[ 1 + \frac{1}{2} \frac{d \delta}{dv} \right] \]

\[ + \frac{h}{2l^2} (1 + e \cos v)^2 \frac{h}{l^2} (1 + e \cos v)^2 \frac{d^2 \delta}{dv^2} \]

or

\[ \frac{d^2 \theta}{dt^2} = \frac{\mu}{r^3} \left[ -2e \sin v \sin v \frac{d \delta}{dv} + \frac{1}{2} (1 + e \cos v) \frac{d^2 \delta}{dv^2} \right] \]

Substituting the value of \( \frac{d^2 \theta}{dt^2} \), equation (2.2.4) becomes

\[ -2e \sin v \sin v \frac{d \delta}{dv} + \frac{1}{2} (1 + e \cos v) \frac{d^2 \delta}{dv^2} = -\frac{n^2}{2} \sin \delta + \frac{\varepsilon}{2} \sin 2(\Omega - \alpha_m + v), \]

or

\[ (1 + e \cos v) \frac{d^2 \delta}{dv^2} - 2e \sin v \frac{d \delta}{dv} - 4e \sin v + n^2 \sin \delta = \varepsilon \sin 2(\Omega - \alpha_m + v) \]

or

\[ (1 + e \cos v) \frac{d^2 \delta}{dv^2} - 2e \sin v \frac{d \delta}{dv} - 4e \sin v + n^2 \sin \delta \]

\[ = \varepsilon \sin 2(\Omega_0 + \frac{d \Omega}{dt} t - \alpha_{G0} - \frac{d \alpha_G}{dt} t - \phi_m' + v), \]

or

\[ (1 + e \cos v) \frac{d^2 \delta}{dv^2} - 2e \sin v \frac{d \delta}{dv} - 4e \sin v + n^2 \sin \delta \]

\[ = \varepsilon \sin 2(\Omega_0 - \alpha_{G0} - \phi_m' + \frac{1}{\Omega} \left( \frac{d \Omega}{dt} - \frac{d \alpha_G}{dt} \right) v + v) \]

or

\[ (1 + e \cos v) \frac{d^2 \delta}{dv^2} - 2e \sin v \frac{d \delta}{dv} - 4e \sin v + n^2 \sin \delta \]

\[ = \varepsilon \sin 2 \left( (\Omega_0 - \alpha_{G0} - \phi_m') + \frac{1}{\Omega} \left( \frac{d \Omega}{dt} - \frac{d \alpha_G}{dt} \right) v + v \right) \]

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Taking $\sigma = q$ as the generalized coordinate, the above equation becomes

$$(1 + e \cos v) \frac{d^2 q}{dv^2} - 2e \sin v \frac{dq}{dv} - 4e \sin v + n^2 \sin q = \varepsilon \sin(a_i + bv) \quad ..(2.2.5)$$

where

$$a_i = 2(\Omega_0 - \alpha_{\text{in}} - \phi'_m),$$

$$b = 2 \left[ \frac{1}{\Omega_i} \left( \frac{d\Omega}{dt} - \frac{d\alpha_{\text{in}}}{dt} \right) \right]$$
2.3 HAMILTON'S EQUATION

Equation (2.2.5) is equivalent to the Hamilton's equation

\[
\frac{dq}{dv} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dv} = -\frac{\partial H}{\partial q},
\]

... (2.3.1)

where \( H \) is the Hamiltonian function.

Multiplying both sides by \((1+e \cos v)\) and rearranging the terms, equation (2.2.5) becomes

\[
\frac{d}{dv} \left[ (1+e \cos v)^2 \frac{dq}{dv} + 2(1+e \cos v)^2 \right] + n^2 \sin q(1+e \cos v) - \varepsilon \sin(a_i + bv)(1+e \cos v) = 0
\]

which is of the form

\[
\frac{dp}{dv} + \frac{\partial H}{\partial q} = 0
\]

where

\[
\frac{\partial H}{\partial q} = n^2 \sin q(1+e \cos v) - \varepsilon \sin(a_i + bv)(1+e \cos v)
\]

and

\[
\frac{\partial H}{\partial p} = \frac{dq}{dv} = \frac{p}{(1+e \cos v)^2} - 2
\]

Also,

\[
dH = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq
\]

\[
= \left( \frac{p}{(1+e \cos v)^2} - 2 \right) dp + \left( n^2 \sin q(1+e \cos v) - \varepsilon \sin(a_i + bv)(1+e \cos v) \right) dq.
\]

Therefore,

\[
H = \frac{p^2}{2(1+e \cos v)^2} - 2p - n^2 \cos q(1+e \cos v) - \varepsilon \sin(a_i + bv)(1+e \cos v) q.
\]
Taking $\varepsilon$ of the order of $\varepsilon$, $\varepsilon = \varepsilon_1$

$H$ can be written as

$$H = H_o + cH_1 + \ldots$$

$$H_o = \frac{p^2}{2} - 2p - n^2 \cos q,$$

$$H_1 = -p^2 \cos v - n^2 \cos q \cos v - \varepsilon_1 \sin (a_1 + bv)q,$$

where $p$ is the generalized momenta.

Therefore,

$$H = \frac{p^2}{2} - 2p - n^2 \cos q + \varepsilon_1 (-p^2 \cos v - n^2 \cos q \cos v - \varepsilon_1 \sin (a_1 + bv)q)$$

is the required Hamiltonian equation.
2.4 EQUILIBRIUM AND DOUBLE ASYMPTOTIC SOLUTION

Equilibrium solution corresponding to $H_0$ is given by

$$\frac{dq}{dv} = 0 = \frac{dp}{dv}.$$

Using equation (2.3.1)

$$\frac{dq}{dv} = p - 2 = 0$$

and

$$\frac{dp}{dv} = -n^2 \sin q = 0$$

which gives the hyperbolic equilibrium solution

$p(v) = 2, \ q(v) = 0, \Pi, \ldots.$

Thus in the phase space $(q,p)$: $(0,2), (\Pi,2)$ are the equilibrium points.

Also,

$$\frac{\partial^3 H_0}{\partial p^2} = 1$$

$$\frac{\partial^2 H_0}{\partial p \partial q} = 0$$

$$\frac{\partial^3 H_0}{\partial q^2} = n^2 \cos q$$

In general, for the equilibrium point $(q,p)$, if

$$\left( \frac{\partial^2 H_0}{\partial p \partial q} \right)^2 - \left( \frac{\partial^2 H_0}{\partial p^2} \right) \left( \frac{\partial^2 H_0}{\partial q^2} \right) < 0, \text{ point is stable.}$$

$$\left( \frac{\partial^2 H_0}{\partial p \partial q} \right)^2 - \left( \frac{\partial^2 H_0}{\partial p^2} \right) \left( \frac{\partial^2 H_0}{\partial q^2} \right) > 0, \text{ point is unstable.} \quad \ldots(2.4.1)$$
Case I: Equilibrium solution at (0,2):

Now at (0,2) the equation (2.2.1), gives

\[ 0^2 - 1 \cdot n^2 = -n^2 < 0. \]

Hence (0,2) is a stable conjugate point.

Case II: Equilibrium solution (π,2):

Now at (π, 2) the equation (2.4.1), gives

\[ 0^2 - 1 \cdot (-n^2) = n^2 < 0. \]

Hence (π, 2) is unstable conjugate point.

Thus the hyperbolic equilibrium solution is given by

\[ q(v) = π, \quad p(v) = 2. \]

Now we determine the unperturbed double asymptotic solution.

Again from the equation (2.3.1)

\[ \frac{dq}{dv} = \frac{\partial H_0}{\partial p} \cdot \frac{dp}{dv} = -\frac{\partial H_0}{\partial q}, \]

we have

\[ \frac{dq}{dv} = p - 2 \quad \text{and} \quad \frac{dp}{dv} = -n^2 \sin q. \]

Therefore,

\[ \frac{dp}{dq} = \frac{dp}{dv} \cdot \frac{dv}{dq} = \frac{-n^2 \sin q}{p - 2} \]

or

\[ (p - 2)dp = -n^2 \sin q \, dq \]

Integrating, we get

\[ \frac{p^2}{2} - 2p = n^2 \cos q + A, \]
where $A$ is the constant of integration, which can be determined by the initial conditions at $q=\pi$, $p=2$.

Thus $A=n^2-2$. So,

$$\frac{p^2}{2} - 2p = n^2 \cos q + n^2 - 2$$

or

$$p^2 - 4p - 2(n^2 \cos q + n^2 - 2) = 0$$

Solving the above equation,

$$p - 2 = \pm 2n \cos (q/2)$$

Therefore,

$$\frac{dq}{\cos (q/2)} = \pm 2ndv$$

On integrating and then solving, we get

$$\sin (q/2) = \frac{e^{nv} - e^{-nv}}{e^{nv} + e^{-nv}}.$$ 

And hence,

$$\cos (q/2) = \frac{2}{e^{nv} + e^{-nv}}.$$ 

Therefore, unperturbed double asymptotic solutions are given by

$$p^\pm(v) = 2 \pm \frac{2n}{\cosh nv},$$

$$\sin (q^\pm(v)) = \pm \frac{2 \sinh (nv)}{\cosh^2 (nv)},$$

$$\cos (q^\pm(v)) = \frac{2}{\cosh^2 (nv)} - 1.$$
2.5 CONCLUSION

Using Euler's Dynamical Equations of Motion, the equation of motion of a satellite in an elliptic orbit under the influence of a magnetic torque has been derived. It is also observed that in the phase space, [0, 2] and [π, 2] are the unperturbed equilibrium solutions. Further, it is calculated that the equilibrium point [0, 2] is stable, whereas the equilibrium point [π, 2] is unstable.