CHAPTER -1

GENERAL INTRODUCTION

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2.1 : DEFINITIONS
1.1 INTRODUCTION

"Invention, it must be humbly admitted, does not consist in creating out of voice, but out of chaos." :- Mary Shelley [196]

"It turns out that an eerie type of chaos can lurk just behind a facade of order - and yet, deep inside the chaos lurks an even eerier type of order" :- Douglas Hostadter (Ref. by Gleick[67] )

"Lo! thy dread empire, Chaos! is restored; dies before thy uncreating word: thy hand, great Anarch! lets the curtain fall; and universal darkness buries al". :- Alexander Pope[183]

"Chaos is found in greatest abundance wherever order is being sought. Chaos always defeats order because it is better organized".

:- Ly Tin Wheedle (Ref. by Pratchet [184])

Millay [151], Septima Clark (Ref by Mcfadden [142]) have a great belief in the fact that whenever there is chaos, it creates wonderful thinking. He consider chaos as a gift.

In 16th century, people tend to think of the Solar System as a paradigm of order and regularity. We imagine the planets fixed in their orbits around the Sun for all time - an orderly, predictable, unchanging, majestic clockwork that never needs rewinding. We can accept unforeseen changes in our everyday lives and even come to terms with natural and man-made disasters. We still have faith in the immutability of the orbits
of the planets and satellites. What grounds do we have for such beliefs? Is the behavior of the Solar System completely predictable, or could the planets ever collide?

These are the questions that many astronomers have attempted to answer, but it is only in this decade that a better understanding of the problem and a possible solution has emerged.

The key to this progress is the study of CHAOS, even simple, deterministic equations can give complicated unpredictable solutions. Chaos has revealed that our Solar System is not the paragon of predictability that we once imagined.

"If we find the answer to that (the universe), it would be the ultimate triumph of human reason—for then we would know the mind of God". :-Stephen Hawking [90]

WHAT IS CHAOS?

Dictionary meaning of chaos is “turmoil, turbulence, primordial abyss, and undesired randomness”. Scientist defined chaos as effectively unpredictable long time behavior arising in a deterministic dynamical system because of sensitivity to initial conditions. Chaos also refers to the question of whether or not it is possible to make good long-term predictions about how a system will act. A chaotic system can actually develop in a way that appears very smooth and ordered.
The order and chaos in man's initial theories about the earth and the solar system, is the geocentric theory, credited to Ptolemy [187]. Roman governor of Egypt, stated that all of the heavenly bodies revolved around the earth. Galileo([65], [66]), and Copernicus (Ref. by S K Ghosh [193]) were wise to explain away minor perturbations and outright conflicts with geocentricity by the use of orbital "epicycles" and other geometrical tricks, careful not to step on powerful ecclesiastical toes. After all, God had dominion over an ordered world, one not filled with unpredictability and random chance. Galileo himself was shunned and even supposedly imprisoned because he publicly espoused Copernicus' heliocentric theory. Though astronomy was already a developed science in the Middle East, the western world only began to seriously and scientifically explore the universe during the first part of the 16th century. Later century, Johannes Kepler (Ref. by Cooper [46]), assembled thirty years of observational data by Tycho Brahe [29], formulated his three laws of planetary motion. These laws were intended to describe the motion of one body as it revolved around another. He theorized that with further mathematical manipulations and calculations, the future motion of two or more planets and one star, or one planet and two stars, etc., could be predicted as accurately as one planet and one star.
In the 17th century Isaac Newton [167] showed that if two bodies attract each other with force that was proportional to the square of the distance between them, then the resulting motion of body relative to the other would be a precise mathematical curve called a conic section (that is, a circle, ellipse, parabola or hyperbola). Although Newton was not the first person to suggest that the inverse square law of force was responsible for the motion of the planets, his great triumph was to provide a mathematical proof of the consequences of such a law.

He showed that a planet moving under the effects of the Sun's gravity would describe an elliptical path, and that the period of this motion would depend only on the average distance from the Sun. In mathematical terms, he could show that the "two-body problem" was integrable, in other words, that it was possible to obtain a complete, practical solution to the problem using relatively simple mathematical equations.

But the Solar System is not composed of just two bodies. It is the Sun's gravitational field dominates the motion of planets and that, to a good first approximation, each an elliptical orbit around the Sun. The planets also influence each other's motion, however, all according to the inverse square law. And we can detect these effects although they are small. For example, the basic ellipse of the Earth's orbit is not fixed in space: it gradually rotates or precesses, at a current rate of 0.3 degrees per century due to perturbations by the other planets, most notably Jupiter.
When Newton published his great laws of gravitation and mechanics in the seventeenth century, the proving ground for these laws was the solar system. Newton showed that the clockwork like repeating motions of the planets, and all other members of the solar system, were explained and predicted by simple quantitative laws. His laws were shown to be the basis of all motion and for a long time were thought to be the basis of all physical phenomena.

The advent of the relativity theory in the twentieth century did not really change this view, at least for planets and stars. It was realized that Newton's laws were just a special case of more general laws. The Newtonian form of the laws only applies to objects which move at speeds much less than that of light. This is certainly true for planets, and so the clockwork nature and predictability of the solar system was still intact. This was nowhere more obvious than the prediction of the existence of the planet Neptune by Adams and Leverrier [117] from the observed disturbances of the planet Uranus. This calculation was a triumph of the Newtonian philosophy.

What was not realized until late in the nineteenth century was that there were cases where the uncertainties in the calculation could grow so rapidly that the future motion of the system was, in principle, completely unpredictable and chaotic. Sometimes some mechanical systems will approach states where there is more than one future state and which one is
chosen is a matter of pure chance. A simple example is a pendulum with a magnet hanging on it swinging over three or more other magnets. Such a simple system can be shown to be completely chaotic and it is not possible to calculate how the pendulum will be moving even a few minutes into the future.

Nonetheless, it has been assumed that chaotic systems are special and probably involve delicate balances of forces, but the stars and planets remain predictable. It has come as somewhat of a shock to find that this last bastion of predictability has fallen. Sussman and Wisdom [208], have made elaborate and accurate calculations of the motion of the planet Pluto over a period of 845 million years, which is about one sixth of the age of the solar system. Pluto is our strangest planet, with a very eccentric and highly tilted orbit. Their calculations show that the motion of Pluto is not infinitely calculable but is, in fact, chaotic! Even more surprising is that the chaotic nature of its motion requires only 200 million years to become evident. This means that the motion of Pluto cannot be predicted more than 200 million years into the future or back into the past. This is a very short time compared with the age of the solar system.

A further consequence is that the chaotic motion of Pluto will induce chaotic motion in all the other planets and so the whole solar system is ultimately chaotic and unpredictable in long time spans. The Newtonian philosophy is well and truly dead.
French mathematician Pierre Simon de Laplace ([108], [109]) tried to solve the problem of the Solar System's stability by making some simplifying assumptions about the nature of the gravitational interactions of the planets. He had a fundamental belief that once you had determined the laws governing the Universe, it was just a matter of solving the equations, with the appropriate starting conditions, to discover its past and future behavior and that "nothing would be uncertain". Laplace showed that his simplified system was integrable and that there were long-term periodicities (typically, tens of thousands of years) in the movement of the orbits of the planets: he thought he had achieved the elusive analytical solution. Unfortunately, the very terms that Laplace had neglected in his theory were those that could provide possible sources of chaos. So Laplace's proof of stability has to be discounted. For example, the motion of the ball in a spinning roulette wheel is, in principle, a deterministic system. Although the ball and wheel are subject to known forces, trying to predict the final outcome is unlikely to be a rewarding experience.

"Physicists like to think that all you have to do is say, these are the conditions, now what happens next?" -Richard P. Feynman (Ref. By Donahue [51])

For thousands of years humans have noted that small causes could have large effects and that it was hard to predict anything for certain. What had caused a stir among scientists was that in some systems small changes
of initial conditions could lead to predictions so different that prediction itself becomes useless. At the end of the 19th century, French mathematician, Jacques Hadamard ([79], [80]), proved a theorem on the sensitive dependence on initial conditions about the frictionless motion of a point on a surface or the geodesic flow on a surface of negative curvature. All this was about billiard balls and why you can't predict what three of them will do when they careened off each other on the table. French physicist Pierre Duhem [53], understood the significance of Hadamard's theorem. He published a paper in 1906 that made it quite plain that prediction was "forever unusable" because of the necessarily present uncertain initial conditions in Hadamard's theorem. These papers went unnoticed or rather unnoted by the man who was recognized as the Father of Chaos theory, Henri Poincare (1854-1912).

WHAT IS CHAOS THEORY?

"Chaos theory describes complex motion and the dynamics of sensitive systems. Chaotic systems are mathematically deterministic but nearly impossible to predict. Chaos is more evident in long-term systems than in short-term systems. Behavior in chaotic systems is aperiodic, meaning that no variable describing the state of the system undergoes a regular repetition of values. A chaotic system can actually evolve in a way that appears to be smooth and ordered, however. Chaos refers to the issue
of whether or not it is possible to make accurate long-term predictions of any system if the initial conditions are known to an accurate degree”.

"However, if we do discover a complete theory, it should in time be understandable in broad principle by everyone, not just a few scientists. Then we shall all, philosophers, scientists, and just ordinary people, be able to take part in the discussion of the question of why it is that we and the universe exist". -Stephen Hawking[90].

Poincaré[182] published *SCIENCE ET METHODE*, that contained one sentence concerning the idea of chance being the determining factor in dynamic systems because of some factor in the beginning that we didn't know about. All of these ideas went unnoticed because quantum mechanics had disrupted the whole physics world of ideas; and because there were no tools such as ergodic theorems about the mathematics of measure; and because there were no computers to simulate what these theorems prove.

We now know that, except for special cases, the general motion of many \(n\) bodies interacting through gravity, the "n-body problem", is not integrable. A simpler task is to attempt to solve the three-body problem. Henri Poincaré ([174] to [182]) tackled this problem in some depth. It is clear from his writings that he was aware of the unpredictability of some solutions of the equations of motion. He did not solve the three-body problem; in fact, he proved that a simple, general solution did not exist. Poincaré had found results that upset the accepted view of a purely
deterministic universe that had reigned since Sir Isaac Newton lined out linear mathematics. In his 1890 paper, he showed that Newton's laws did not provide a solution to the "three-body problem", in other words, how one deals with predictions about the earth, moon and sun. He had found that small differences in the initial conditions produce very great ones in the final phenomena, and the situation defied prediction. Poincaré's discoveries were dismissed in lieu of Newton's linear model, one was to just ignore the small changes that cropped up. The three-body problem was what Poincare had to interpret with a two-body system of mathematics. Why was it a problem? He was trying to discover order in a system where none could be discerned.

Poincaré's negative answer caused positive consequences in the creation of chaos theory. Poincaré was the first to appreciate the complicated behavior that could result from the gravitational interaction of just three bodies. He made a statement "A fully deterministic system does not necessarily imply explicit prediction on the evolution of a dynamical system".

About eighty years later, Edward Lorenz [121], using Poincaré's mathematics, described a simple mathematical model of a weather system that was made up of three linked nonlinear differential equations that showed rates of change in temperature and wind speed. Some surprising results showed complex behavior from supposedly simple equations; also
the behavior of the system of equations was sensitively dependent on the initial conditions of the mathematical model. He spelled out the implications of his discovery, saying it implied that if there were any errors in observing the initial state of the system, and this is inevitable in any real system, prediction as to a future state of the system was impossible. Lorenz labeled these systems that exhibited sensitive dependence on initial conditions as having the "butterfly effect": this unique name came from the proposition that a butterfly flapping its wings in Hong Kong can effect the course of a tornado in Texas.

During 1970-71, interest in turbulence, strange attractors and sensitive dependence on initial conditions arose in the world of physics. Li and Yorke [118], coined the word chaos to refer to the mathematical problem in chaos theory that described a time evolution with sensitive dependence on initial conditions. Robert May [141], a mathematician-biologist whose research was well read, used the word and the theory, making them the word famous.

The nature side of chaos entails all the physical sciences. The knowledge side of chaos deals with the human sciences. Chaos may manifest itself in either form or function or in both. Chaos studies the interdependence of things in a far-from-equilibrium state. Every open nonlinear dissipative system has some relationship to another open system and their operations will intersect, overlap and converge. If the systems are
sensitive to the initial conditions, in other words, we don’t know exactly in
detail every little piece of information, and then we have a potentially
chaotic system. Not all systems will be chaotic, but those where a lack of
infinite detail is unknown, then these systems have an indeterminate
quality about them. We can’t tell what’s going to happen next. They are
unpredictable. If these systems are perturbed either internally or externally
they will display chaotic behavior and this behavior will be amplified
microscopically and macroscopically. Further research in non-linear
dynamical systems that displayed a sensitive dependence on initial
conditions came from Ilya Prigogine [185], a Nobel-prize winning
chemist, who first began work with far-from-equilibrium systems in
thermodynamic research. His research in non-linear dissipative structures
led to the concept of equilibrium and far-from equilibrium to categorize the
state of a system, that led to systemic behavior different from what we
expected by the customary interpretation of the Second Law of
Thermodynamics. Phenomena of bifurcation and self-organization emerge
from systems in equilibrium if there was disruption or interference. This
disruption or interference became the next step to Chaos Theory; it became
Chaos/Complexity Theory. Prigogine talked about his theory as if he were
Aristotle: a far-from-equilibrium system can go ‘from being to becoming’. These ‘becoming’ phenomena showed order coming out of chaos in
systems, chemical systems, and living systems.
From Lorenz simulation, a theory was proposed ‘catastrophe theory’, or a mathematical description of how a chaos system bifurcates or branches. Out of these bifurcations came pattern, coherence, stable dynamic structures, networks, coupling, synchronization and synergy. From the study of complex adaptive systems used by Poincaré, Lorenz and Prigogine, Norman Packard [168] and Chris Langton [107] developed theories about the ‘edge of chaos’ in their research with cellular automata. The energy flowing through the system, and the fluctuations, cause endless change which may either dampen or amplify the effects. In a phase transition of chaotic flux, (when a system changes from one state to another), it may completely reorganize the whole system in an unpredictable manner.

Two scientists, physicist Mitchell Feigenbaum [57] and computer scientist Oscar Lanford [106], came up with a picture of chaos in hydrodynamics using Renormalization ideas. They were studying nonlinear systems and their transformations. Feigenbaum found the constants or ratios that are responsible for the phase transition state when order turns to chaos. These Feigenbaum numbers helped to predict the onset of turbulence (chaos) in systems. Since then, chaos theory or Nonlinear Science has taken the scientific world by a storm, and applications in the real world began. Optics, economics, electronics, chemistry, biology, and psychology used this new analytic tool. Fractal geometry is being used to
graphically show change and evolution in technology, sociology, economics, psychotherapy, medicine, psychology, astronomy, evolutionary theory, and the metaphorical application is spreading to art, humanities, philosophy and theology. Strange attractors were showing up in biology, statistics, psychology and economics and in every field of endeavor.

Gontikakis, Efthymiopoulos, Anastasiadi [74], consider the possibility of particles being injected at the interior of a reconnecting current sheet (RCS), and study their orbits by dynamical systems methods. Despite the presence of a strong electric field, a 'mirror' trapping effect persists, to a certain extent, for orbits with appropriate initial conditions within the sheet. The mirror effect is stronger for electrons than for protons. Three types of orbits are distinguished:

(i) chaotic orbits leading to escape by stochastic acceleration,
(ii) regular orbits leading to escape along the field lines of the reconnecting magnetic component, and
(iii) mirror-type regular orbits that are trapped in the sheet, making mirror oscillations.

Dynamically, the latter orbits lie on a set of invariant KAM theory that occupies a considerable amount of the phase space of the motion of the particles. They observed the phenomenon of 'stickiness', namely chaotic orbits that remain trapped in the sheet for a considerable time.
WHAT IS COMPLEXITY?

Complexity or the edge of chaos yielded self-organizing, self-maintaining dynamic structure that occurred spontaneously in a far-from-equilibrium system. Complexity had no agreed upon definition, but it could manifest itself in our everyday lives.

Complexity: “In fact, of all the terms that form the lingua franca of chaos theory and the general theory of systems, bifurcation may turn out to be the most important, first because it aptly describes the single most important kind of experience shared by nearly all people in today’s world, and second because it accurately describes the single most decisive event shaping the future of contemporary societies.” Erwin Laszlo [113]

Bifurcation once meant splitting into two or more forks. In chaos theory it means: when a complex dynamical chaotic system becomes unstable in its environment because of perturbations, disturbances or ‘stress’, an attractor draws the trajectories of the stress, and at the point of phase transition, the system bifurcates and it is propelled either to a new order through self-organization or to disintegration. The phase transition of a system at the edge of chaos began with the studies of John Von Neumann [212] and Steve Wolfram ([219], [220]), on cellular automata. Their research revealed the edge of chaos was the place where the parallel processing of the whole system was maximized. The system performed at its greatest potential and was able to carry out the most complex
computations. At the bifurcation stage, the system was in a virtual area where choices are made—the system could choose whatever attractor was most compelling, could jump from one attractor to another—but it was here at this stage that forward futuristic choices were made: this was deep chaos. The system self-organized itself to a higher level of complexity or it disintegrated. The phase transition stage may be called the transcurrent stage, the place where transitory events happen. Transcurrent is a philosophical term meaning that there is an effect on the system as a whole produced from the inside of the system having a transitory effect; and, a scientific term in that it is a nonperiodic signal of sudden pulse or impulse. Per Bak, Chao Tang and Kurt Wiesenfeld [3], reckons nature abiding on the edge of chaos or what they call ‘self-organized criticality’.

After the bifurcation, the system may settle into a new dynamic regime of a set of more complex and chaotic attractors, thus becoming an even more complex system that it was initially. Three kinds of bifurcations happen:

1. Subtle, the transition is smooth.
2. Catastrophic, the transition is abrupt and the result of excessive perturbation.
3. Explosive, the transition is sudden and has discontinuous factors that wrench the system out of one order and into another.
Chaos/Complexity in our daily life encounter with the traffic flow, weather changes, population dynamics, organizational behavior, shifts in public opinion, urban development and decay, cardiological arrhythmias, epidemics, cell differentiation, immunology, decision-making, the fracture structures, and turbulence. Chaos theory uses ideas as chaos in dynamical systems, biological systems, turbulence, quantised systems, global affairs, economics, the arms race, and celestial systems.

Cambel [34], identifies chaos as "inherent in both the complexity in nature and complexity in knowledge."

1. Complexity can occur in natural and man-made systems, as well as in social structures and human beings.

2. Complex dynamical systems may be very large or very small, indeed, in some complex systems, large and small components live cooperatively.

3. The system is neither completely deterministic nor completely random, and exhibits both characteristics.

4. The causes and effects of the events that the system experiences are not proportional.

5. The different parts of complex systems are linked and affect one another in a synergistic manner.
6. There is positive and negative feedback. The level of complexity depends on the character of the system, its environment, and the nature of the interactions between them.

WHERE IT'S ALL GOING

If we lived in a completely deterministic world there would be no surprises and no decision making because an event would be caused by certain conditions that could lead to no other outcome. Nor could we consider living in a completely random world for there would be, as Cambel says, "no rational way of reaching a well-reasoned decision". What kind of answers do we get when we recognize that a system is indeed unstable and that it is indeed an example of chaos at work.

Peng, Petrov and Showalter [173], studied the usefulness of chaos control in chemical processing and combustion. Freeman [61], studied the brain functions and concluded that indeed chaos "affords an opportunity to exploit further the manifestations of brain activities". Bergé, Pomeau, and Vidal [9], assert that chaos theory has "great predictive power" that allows an understanding of the overall behavior of a system. Kauffman [97], uses the self-organization end of chaos to assert that nature itself is spontaneous.

Wentworth d'Arcy Thompson [209], used order to compose the working of nature using the transformations of coordinates to compare species of animals, as an example comparing one form of a fish, with
another could be shown on a coordinate map and used to show how they differ and how they were alike. The same kind of transformation coordinate map could compare chimpanzee skulls to human skulls.

Cramer [48], claimed that by overcoming the objections to mysticism and scientism, the "theory of fundamental complexity is valid" with the interaction of order and disorder as a necessity in nature.

"In nature, then, forms are not independent and arbitrary, they are interrelated in a regular way...And even organs arising to serve new functions develop according to the principle of transformation. At the branch points where something new emerges, disruptions of order are in fact necessary; abrupt phase changes occur. Indeed, the interplay of order and chaos constitutes the creative potential of nature". :-Cramer.

Henri Poincaré first noticed the idea that many simple nonlinear deterministic systems can behave in an apparently unpredictable and chaotic manner. The importance of chaos was not fully appreciated until the widespread availability of digital computers for numerical simulations and the demonstration of chaos in various physical systems. This realization has had broad implications for many fields of science, and it have been only within the past decade when the field has undergone explosive growth. The ideas of chaos have been very fruitful in diverse disciplines as biology, economics, chemistry, engineering, fluid mechanics, physics, etc.
French mathematician Georges Julia [95] studied the chaotic orbits in complex analytical systems, but Benoit Mandelbrot ([130] - [137]), gave some rules for computation. His work on noise interference problems revealed distinct ratios between order and disorder on any scale he used. The seemingly chaotic behavior of noise displayed a fractal structure. Mandelbrot recognized a self-similar pattern that the fractals formed. He then cross-linked this new geometrical idea with hundreds of examples, from cotton prices to the regularity of the flooding of the Nile River. Mandelbrot believed that fractals were found nearly everywhere in nature, at places such as coastlines, mountains, clouds, aggregates, and galaxy clusters.

"I coined fractal from the Latin adjective fractus. The corresponding Latin verb frangere means "to break": to create irregular fragments. It is therefore sensible-and how appropriate for our needs!-that, in addition to "fragmented", fractus should also mean "irregular," both meanings being preserved in fragment.":- Benoit Mandelbrot [133].

"It's an experience like no other experience I can describe, the best thing that can happen to a scientist, realizing that something that's happened in his or her mind exactly corresponds to something that happens in nature. It's startling every time it occurs. One is surprised
that a construct of one's own mind can actually be realized in the honest-to-goodness world out there. A great shock, and a great, great joy."

:-Leo Kadanoff (as Ref. by Gleick [67])

The first fractals described are dated at the end of the XIXth century. Cantor's dust is probably the most ancient known fractal figure. Mathematician behind this was Georg Cantor ([35], [36]). The next classical figure is about 40 years younger than the Cantor set, introduced by the great mathematician Waelaw Sierpinski [200]. Helhe von Koch Koch Snowflake [103], a Swedish mathematician introduced Koch curve with the properties of fractals. Fundamental work was done by Hausdorff [89], then developed by Besicovitch [10]. The Hausdorff- Besicovitch dimension has played, later on, a major role in the domain of fractal. Fractals can be used in Computer graphics for data compression. It would take huge amounts of data to store the exact data needed to construct the cratered surface of a moon. This would be memory, if all we wanted was a realistic lunar landscape for a science fiction film. The answer to this is fractal forgeries; these mimic the desired forms without worrying about the precise details, they also require very little storage space because of easy compression.
What is a relation between fractals and chaos theory?

There is a strong link between fractals and chaos. Fractal geometry is the geometry, which describes the chaotic systems we find in nature. Fractals are a language, a way to describe geometry. Both fractals and deterministic chaos are tools used to model many different systems. Deterministic chaos, as opposed to just chaos, implies that a specific chaotic system is being used to model a phenomenon. While used for similar things, fractals and chaos are still very different. Whereas chaos is unpredictable and sensitive, fractals are self-similar at different scales. In fact, many fractals aren't unpredictable at all (for instance, the Sierpinski Triangle is very predictable and orderly). However, many fractals are chaotic and some chaotic phenomena resemble fractals. In short, the definitions overlap quite a bit, but aren't synonymous by any definition.

During the 1970's Michele Henon [92], discovered a very simple iterated mapping that show a chaotic attractor, now called Henon's attractor, that allowed him to make a direct connection between deterministic chaos and fractals. The Henon attractor is self-similar.

Goldberger et al. [69] state that physiology may prove to be one of the richest laboratories for the study of fractals and chaos as well as other types of nonlinear dynamics. A good example is the study of heart rate time series. Conventional wisdom states that the heart displays 'normal' periodic rhythms that become more erratic in response to stress or age. However
recent evidence suggests just the opposite: physiological processes behave more erratically (chaotically) when they are healthy and young. Normal variation in heart rate is 'ragged' and irregular, suggesting that mechanisms controlling heart rate are intrinsically chaotic. Such a mechanism might offer greater flexibility in coping with emergencies and changing environments. Basic ideas of chaotic dynamics in population biology are summarized by Schaffer and Kot [195]. Lipsitz and Goldberger [119] found a loss of complexity in heart rate variation with age. Based on this result, they defined aging as a progressive loss of complexity in the dynamics of all physiological systems. Sugihara [205]-[207], using a different analytical approach, found that prediction-decay and nonlinearity models are good predictors of human health. Healthy patients have a steeper heart rate decay curve, and have greater nonlinearity in their heart rhythms and examined the question of whether natural population cycles are deterministic or purely stochastic. They state that populations are embedded in a dynamic web of other species and environmental forces, implying that irregularities in population cycles (which have traditionally been 'smoothed' prior to modelling) may provide important information regarding their dynamics. For pure additive noise, the correlation of adjacent values was independent of the prediction interval, but for chaotic trends correlations decline as the prediction interval increased. They found that measles epidemics display chaotic properties, but that chickenpox
epidemic patterns are best modeled as noise superimposed on a strong annual cycle. Ellner and Turchin [55] have argued that it is potentially misleading to make a strict distinction between chaotic and stochastic dynamics. Using an approach of non-linear time-series modelling and estimation of Lyapunov exponents (Godfray and Grenfell [68]), they demonstrated that ecological populations vary from noise-dominated, stable dynamics to weakly chaotic ones. However, Sugihara claims that their approach is fundamentally flawed, and offers an alternative method based on locally weighted maps. Hastings et al. [88] summarize the various methods available for detecting deterministic chaos in biological time series.

Muzzio, Wachlin, Carpintero, ([159], [158]), studied Regular and Chaotic Motion in a Restricted Three--Body Problem and globular clusters of Astrophysical Interest. A large percentage of the stellar orbits turned out to be chaotic, contrary to what happens in the usual restricted three--body problem of celestial mechanics where most of the orbits are regular. Using different "start spaces" they investigated the orbit families of the cluster stars. They confirmed the presence of chaotic orbits, particularly in the outer parts of the cluster and their relevance for the structure of the cluster and discussed Spatial Structure of Regular and Chaotic Orbits in Self-Consistent Models of Galactic Satellites.
Carpintero, Muzzio, Vergne, Wachlin [37], investigated the orbits of the stars that make up galactic satellites and found that many of those orbits are chaotic. The Lyapunov times that were obtained showed that the time scales of the chaotic processes are shorter than, or comparable to, other time scales characteristic of galactic satellites. Kouprianov, Shevchenko ([104], [105], [198]), discussed the chaotic behavior in the rotational motion of planetary satellites, modeled as a triaxial rigid body. The Lyapunov characteristic exponents (LCEs) are used as indicators of the degree of chaos of the motion. The theory developed for the planar case is most probably still applicable in the case of spatial rotation, if the dynamical asymmetry of the satellite is sufficiently small or/and the orbital eccentricity is relatively large (but, for the dynamical model to be valid, not too large). Chirikov approach ([39]-[42]), Shevchenko, I. I. [197], suggested a simple method for estimating the maximum Lyapunov characteristic exponent (MLCE) of motion in a chaotic layer in the neighborhood of nonlinear resonance separatrix of a Hamiltonian system under an asymmetric periodic perturbation. Mel'nikov, Shevchenko [147], consider the problem of calculating the Lyapunov time (the characteristic time of predictable dynamics) of chaotic motion in the vicinity of separatrices of orbital resonances in satellite systems. They studied the dynamics in the vicinity of separatrices of the resonance multiplets corresponding to the 3:1 commensurability of mean motions of
Uranian satellites (Umbriel and Miranda) and the multiplets corresponding to the 2 : 1 commensurability of mean motions of Saturnian satellites (Mimas and Tethys). These chaotic regimes have most probably contributed much to the long-term orbital evolution of the two satellite systems. They obtained Analytical estimates of the Lyapunov time based on the separatrix map theory.

**According to Winter, Mourão and Yokoyama [216],** the satellite system of Saturn is the only one known to have coorbital satellites. Dione has Helene librating around its L₄ point. Tethys has Calipso and Telesto around its L₄ and L₅ points, respectively. In these intrincated satellite system it is well known that there is a mean motion resonance 2:1 between Enceladus and Dione and between Mimas and Tethys. They examined the possible existence of coorbital satellites for Mimas and Enceladus. Their results show a clear instability for all particles coorbital with Mimas. On the other hand Enceladus presented a long-term stability. The particles with initial small amplitude of libration remain stable for over 10⁷ orbital periods. Therefore, there is a possibility of the Cassini mission to find satellites coorbital with Enceladus and no chance of find coorbitals with Mimas.

**Morbidelli ([152], [153], [154]),** discussed the Chaotic Structure of the Asteroid belt (Jupiter's Trojan asteroids) in the early Solar System. The chaotic structure of the asteroid belt is explored, taking into account first
only the perturbations provided by the 4 giant planets, and then including also the effects of the inner planets. They found that both the inner belt (a<2.5 AU) and the outer part of the main belt (a>2.8 AU) are mostly chaotic. In the outer part of the belt chaos is due to the presence of numerous mean motion resonances with Jupiter and three-body resonances Jupiter-Saturn-asteroid. In the inner belt, chaos is generated by mean motion resonances with Mars and three-body resonances Mars-Jupiter-asteroid. Due to the chaoticity of the belt, asteroids tend to slowly migrate in eccentricity. This phenomenon of "chaotic diffusion" allows many bodies in the inner belt to become Mars-crossers. The number of asteroids leaking out from the inner belt is large enough to keep the population of Mars-crossing asteroids in steady state, despite of the short dynamical lifetime of the later. They speculated that chaotic diffusion could have substantially eroded the high-eccentricity part of the asteroid belt and provided the impactors responsible for the late heavy bombardment phase of the early Solar System.

In 1981, the Voyager 2 spacecraft took the first pictures of Saturn's moon Hyperion. Its shape is approximately that of an ellipsoid with three unequal axes of lengths 380, 290 and 230 kilometers, respectively. It was also found that Hyperion's orbit is not in resonance with its rotation period about Saturn. Hyperion seemed to be completing a rotation every 13 days. All other known, large body satellites are approximately spherical and have
approximately equal orbital and rotational periods. Wisdom, Peale and Mrignard [218] developed a model of Hyperion's dynamics, based on Euler's equations for the motion of a rigid body. They studied the tumbling of Hyperion by observing Poincaré sections generated by setting two of the Euler angles and their derivatives equal to zero so that only one angle and its derivative remains. The system can be reduced to two first order differential equations which then can be used to generate a Poincaré section corresponding to a given eccentricity and given principle moments of inertia. A Poincaré section is a plot of the behavior of only one variable of the system and its rate of change. It can be provide a qualitative image of all the possible states of the system. If chaos is observed in this Poincaré section, it is predicted that the Poincaré sections corresponding to the other two angles will also exhibit chaos. It can be predicted by observing different Poincaré sections of the reduced equation of motion when a rotating, orbiting satellite will rotate chaotically, and when it will rotate in a stable, predictable manner. It was found that the Poincaré section, which in theory corresponds to the possible rotation states of Hyperion, does, in fact, exhibit a large chaotic zone. Later observations, made by the astronomer James Klavetter ([101], [102]), of the brightness of Hyperion, gave evidence that the dynamical model of Wisdom, Peale and Mrignard [218] was accurate.
Henri Poincaré realised that in the Solar System, chaos and order, stability and instability were closely connected with a phenomenon called "Resonance". Resonance pervades the Solar System. It happens when any two periods have a simple numerical ratio. The most fundamental period for an object in the Solar System is its orbital period. This is the time it takes to complete one orbit and depends only on its distance from the central object. For example, the Jovian satellite Io has an orbital period of 1.769 days, nearly half that of the next satellite Europa - with a period 3.551 days. They are said to be in a 2:1 orbit-orbit resonance. This particular resonance has important consequences because the perturbations, resulting from Europa's gravity, force the orbit of Io to become more elongated, or eccentric. As Io moves closer to Jupiter and then further away in the course of a orbit, it experiences significant tidal stresses resulting in the active volcanoes that Voyager observed.

In the Saturnian system, resonant pairs of satellites include Mimas and Tethys, Enceladus and Dione, and Titan and Hyperion. Resonances are curiously absent from the Uranian satellite system, although there have been recent explanations that invoke the effects of chaos. Among the planets, Jupiter, with a period of 11.86 years, and Saturn, with a period of 29.46 years, are close to a 5:2 resonance, yet the only true orbit-orbit resonance between planets is a curious 3:2 resonance between Neptune and Pluto. Cohen and Hubbard [44] discovered this, not by observation, but by
a "numerical integration" of the system. When the equations of motion of a system cannot be solved by mathematics, a possible alternative is to solve them using a digital computer. Although such a numerical integration provides less insight than a mathematical solution, it is one of the most powerful tools in modern dynamical astronomy.

The most likely reason that resonance is so common in satellite systems is due to the effects of tides. As a satellite raises a tide on a planet, there is an exchange of angular momentum between the bodies, resulting in a change in the orbit of the satellite and in the spin of the planet. Consequently, the orbits of the natural satellites today may bear little resemblance to their original ones; this is certainly true in the case of the Moon. As a satellite evolves, its orbital period changes and it may encounter a resonance with another satellite. In certain circumstances, the satellites become locked in a resonance and continue to evolve tidally, maintaining the resonant configuration. The planets do raise tides on the Sun but these are not so important because of the greater relative distances involved. This may explain the lack of orbit-orbit resonances in the planetary system.

The breakthrough came when Jack Wisdom [217], developed a new numerical method for studying the motion of asteroids at resonance. This was the mechanism of removal that astronomers had been seeking. Wisdom showed that there was an extensive chaotic zone at the 3:1 resonance which...
matched the observed width of the gap. This discovery was to solve another problem in Solar System dynamics. Most people think that meteorites are fragments of asteroids that eventually collide with the Earth. We can measure the "age" of meteorites by finding out how long they have been exposed to the cosmic rays in the Solar System. Members of one particular class of meteorite, have very short exposure ages of only a few million years. George Wetherill, of the Carnegie Institution in Washington DC, had shown that these meteorites had to come from the vicinity of the 3:1 resonance, but he lacked a mechanism. Wisdom provided the mechanism and carried out numerical integrations to show chaotic orbits of objects at the 3:1 resonances could become eccentric enough for them to start crossing the Earth's orbit.

The whole problem of where objects in Earth-crossing orbits come from is more than an abstract academic question. The impact of a large asteroid on the Earth would be one of the worst natural disasters that our planet could face. The 3:1 gap is not entirely clear of asteroids. At least two asteroids in the gap, Alinda and Quetzalcoatl, are actually in resonance with Jupiter, and 1989 AC, the asteroid that will pass within just 0.011 astronomical units of the Earth in 2004, is also probably in resonance with Jupiter at the gap. It is important for dynamicists studying the Solar System to understand where such objects come from and how they evolve. This requires knowledge of chaos.
The two periods involved in a resonance relation do not have to be orbital periods. Another common form of resonance in the Solar System is spin-orbit resonance, where the period of spin (the time it takes the orbit to rotate once about its axis) has a simple numerical relationship with its orbital period. For example, Mercury is locked in a 3:2 spin-orbit resonance. A more obvious example is our own Moon, which is in synchronous rotation because of the 1:1 spin-orbit resonance that forces it to keep the same face towards the Earth. The far side of the Moon was completely hidden from us until the era of spaceflights. Most natural satellites in the Solar System are in synchronous spin states although this was not their original state: they have evolved into such configurations because of tidal effects. A simple theory allows us to predict the timescales for evolution into the synchronous state. The timescale depends on the mass of the satellite and its distance from the central object.

Prior to the Voyager encounters with Saturn, people had wondered whether or not the satellite Hyperion was in synchronous rotation. After all, it is small and one of the most distant of the Saturnian satellites. Observations from Voyager 2 revealed an irregularly shaped object shaped rather like a hamburger, or a potato. Measurements of Hyperion's rotation made by Voyagers 1 and 2 suggest a spin period of 13 days, compared with an orbital period of 21 days, so Hyperion does not appear to be in an obvious spin-orbit resonance.
Wisdom, Mignard, Peale [218], published a classic paper in which they showed that the simple theory worked out for satellite rotations does not apply to Hyperion, because it is distinctly nonspherical. Hyperion's rotation is certainly not synchronous, but neither is it regular; it is chaotic. Hyperion is "attitude unstable", which means that its spin axis is not fixed and the satellite is tumbling in space as well as rotating chaotically. In normal circumstances, the satellite orbit would become more circular and eventually the chaotic behavior would disappear, but, ironically, tiny Hyperion is locked in an apparently stable 4:3 orbit-orbit resonance with the massive satellite of Saturn, Titan. This forces Hyperion's orbit to be eccentric rather than circular, so the chaos persists, resulting in a satellite with a chaotic spin but a regular orbit.

So chaotic motion does exist in the Solar System in a variety of forms. But are the orbits of the planets chaotic? The answer to this question is likely to come from long-term integrations of the planetary system using a new generation of digital computers. In this decade, there have been a number of separate efforts to investigate the motions of Jupiter, Saturn, Uranus, Neptune and Pluto. The inner planets are notoriously difficult to include in such integrations because very small time-steps are needed to follow them accurately. The Digital Orrery numerically integrates orbits in the Solar System. It is attached to a computer workstation at the Massachusetts Institute of Technology and can
study the motions of 10 gravitationally interacting bodies at 60 times the speed of a VAX minicomputer.

Sussman and Wisdom [208] produced integrations using the Orrery which revealed that Pluto’s orbit shows the telltale signs of chaos, due in part to its peculiar resonance with Neptune. This does not mean, however, that the resonance is unstable or that Pluto and Neptune could ever collide, even though their orbits intersect. Recent work suggests that this chaos arises from resonances within resonances; these can limit the extent of Pluto’s wandering and preserve the main resonance with Neptune. If Pluto’s orbit is chaotic, then technically the whole Solar System is chaotic, because each planet, even one as small as Pluto, affects the others to some extent through gravitational interactions. But we now realise that although chaos means that some orbits are unpredictable, it does not necessarily mean that planets will collide – chaotic motion can still be bounded.

Jacques Laskar ([110], [111], [112]) published the results of his numerical integration of the Solar System over 200 million years. These were not the full equations of motion, but rather averaged equations along the lines of those used by Laplace. Unlike Laplace, however, Laskar’s equations had some 150,000 terms. Laskar’s work showed that the Earth’s orbit (as well as the orbits of all the inner planets) is chaotic and that an error as small as 15 meters in measuring the position of the Earth today would make it impossible to predict where the Earth would be in its orbit in just over 100
million years' time. Laskar's results still have to be confirmed by integrating the full equations of motion, but this will have to wait until the next generation of supercomputers arrives. Meanwhile, we can take comfort from the fact that his work does not imply that orbital catastrophe awaits our planet, only that its future path is unpredictable. It seems likely that the Solar System is chaotic but nevertheless confined, although we have yet to prove it. More than 300 years after the publication of Newton's Principia, we are still struggling to understand the full implications of his square law of gravity. We have begun to view our system of chaos in a light that is revealing the true intricacies of its majestic clockwork.

Our understanding of the Solar System has been revolutionized over the past decade by the finding that the orbits of the planets are inherently chaotic. According to Murray, Holman [156], in extreme cases, chaotic motions can change the relative positions of the planets around stars, and even eject a planet from a system. Moreover, the spin axis of a planet-Earth's spin axis regulates our seasons—may evolve chaotically, with adverse effects on the climates of otherwise biologically interesting planets. Some of the recently discovered extra solar planetary systems contain multiple planets, and it is likely that some of these are chaotic as well. Quillen, Thorndike, Stephen ([189], [190]), studied Structure in the Eridani Dusty Disk Caused by Mean Motion Resonances with a 0.3 Eccentricity Planet at Periastron. Using a simple Hamiltonian system for
first- and second-order resonances, Quillen, Alice C.[191], explored how the capture probability depends on the order of the resonance, drift rate and initial particle eccentricity.

According to Hahn, Ward [86], the depletion of the solar nebula's gas causes several secular resonances to sweep across the Kuiper Belt and excite substantial eccentricities and inclinations among mass less bodies in a Kuiper Belt. Jiang, Yeh [94], have investigated the effect of protostellar discs on the resonance capture. Their results show that the gaseous drag of a protostellar disc can trap Kuiper Belt objects into the 3 : 2 resonance rather easily. In addition, no objects are captured into the 2 : 1 resonance in their simulation. Nagasawa, Ida[163], have investigated excitations of orbital eccentricities and inclinations of Kuiper belt objects (KBOs) caused by the sweeping secular resonances during the primitive solar nebula depletion. Since nebula gravitational potential rotates the longitudes of perihelia and ascending nodes, the nebula depletion leads to migration and secular resonances. In the outer (classical) Kuiper belt (the region beyond 42 AU), inclinations and eccentricities are respectively distributed up to 0.6 (radian) and 0.2, and their root mean squares are about 0.2 (radian) and 0.1. These large values are not explained by present planetary perturbations alone. They have investigated the sweeping secular resonances in the Kuiper belt with both direct orbital integration and the analytical method and have found that the sweeping secular resonances can account for the
eccentricity and inclination in the outer belt. Inclinations of objects in the outer belt are excited to the observational level if the residual nebula with about 0.1% of the density in the minimum mass nebula model is depleted in a timescale of $10^7$-$10^8$ yr. For inclination excitation, Jovian perturbations and nebula potential are the most important, and Neptunian perturbations do not play an important role during the residual nebula depletion, although Neptune with more than one-fifth of its present mass is needed for enough eccentricity excitation. If further observation of the KBOs at semi major axis $>\sim 50$ AU confirms our model, it would give important clues about Neptune's formation and the depletion of the solar nebula. Nagasawa, Ida, Lin [162], have applied the sweeping secular resonances to the extra solar planetary system. They analyzed the evolution of planetary eccentricities of Upsilon Andromedae and HD 168443 system during protoplanetary disk depletion. They have found that the protoplanetary disk depletion changes the eccentricities of planets. On the other hand, the amplitudes of the periastron libration among the planets are not so changed. The origin of high eccentricities of the largest planet in Upsilon Andromedae system and the inner planet of HD 168443 can be accounted by the sweeping secular resonance.

In the solar system, there are many kinds of resonances, such as mean motion resonances, secular resonances, and spin-orbit resonances. According to Yoshikawa, Makoto [222], the mean motion resonances are
very important not only for the stabilization of the system but also for the orbital evolution of the bodies in the solar system. Guzzo, Massimiliano([77], [78]), numerically detected the web of three-planet resonances (i.e., resonances among mean anomalies, nodes and perihelia of three planets) with respect to the variation of the semi-major axis of Saturn and Jupiter, in a model including the planets from Jupiter to Neptune. The measure confirms the relevance of these resonances in the long-term evolution of the outer Solar System and provides a technique to identify some of the related coefficients.

Couetdic, Varadi, Moore, Haghhighipour[47], presented differential continuation method designed to study the locations of the resonant periodic orbits in the phase-parameter space of the three-body problem. They studied systems with a star, and two planets orbiting around the first object in a mean-motion resonance. The method is based on the use of the so-called Plücker coordinates instead of the classical Newton method. A variant of the same method is also used with great efficiency for the search of initial resonant periodic orbits for the continuation. This method enables the investigation of those systems through an extensive study of their phase-parameter space. Their results on the stable orbital configurations in the 2:1 resonance provide constraints on the existence of terrestrial planets in extra-solar planetary systems. The results of an extensive numerical study of the periodic orbits of planar, elliptic restricted three-body
planetary systems consisting of a star, an inner massive planet, and an outer mass less body in the external 1:2 mean-motion resonance has been presented. Using the method of differential continuation, the locations of the resonant periodic orbits of such systems are identified, and through an extensive study of their phase-parameter space it is found that the majority of the resonant periodic orbits are unstable. For certain values of the mass and the orbital eccentricity of the inner planet, however, stable periodic orbits can be found. They have also discussed the applicability of such studies to the 1:2 resonance of the extra solar planetary system GJ 876. Marzari, Scholl, Tricarico ([139], [140]), have discussed the stability of the 3:1 Resonance Locking in the 55 Cancri Planetary System. The most crowded extra solar planetary system discovered so far is the one around 55 Cancri with four planets. They have applied Laskar's frequency map analysis which yields a quantitative measure for the stability of the system.

Nauenberg [164], has discussed the stability and eccentricity for two planets in a 1:1 resonance, and their possible occurrence in extra solar planetary systems. According to Löger, Firneis[120], in Nonlinear Dynamics a small perturbation can drive a system into stability or instability; above all, this is true with periodic perturbations, caused e.g. by natural satellites. Tides raised by them have a reactive influence on their rotation. The position of the spin-axis is changed as well as the period of the rotation. Franklin, Fred, Soper [60], looked for clues to the amount
and direction of Saturn's migration relative to Jupiter in their current orbits, specifically, from the fact that the former lies barely sunward of their 5:2 mean motion resonance. Numerical simulations by them suggested that Saturn has moved in the same direction (presumably inward) as has Jupiter but that the pair has never passed through their more distant 3:1 resonance. Thus, their relative migration has been less than about 1.5 AU. Passage through 5:2 is consistent with their current orbits and could have contributed to their present eccentricities, e (and especially to their large periodic variations), but it probably cannot generate their e's if both, and particularly Jupiter's, were initially less than about 0.03. In fact, it is most likely that their initial e's were both close to the current values. Very long term capture into 5:2 for the Sun-Jupiter-Saturn case occurs with a remarkably high frequency. Marzari, Scholl[138], have investigated the possible role of secular resonances in the dynamical evolution of Trojans during the early phases of the Solar System. According to their studies, a significant population of planetesimals can be captured in Jupiter and Saturn Trojan orbits by the mass growth of the two planets. If they compare the implications of their model with the present Trojan populations, two severe problems arise:

(1) All the captured planetesimals have low inclinations while the observed Jupiter Trojans have significantly higher inclinations exceeding even 20°.
(2) No Trojan has been discovered near Saturn's Lagrangian points.

They have shown that the presence of secular resonances in the Trojan regions of both Jupiter and Saturn may explain the contradiction between their model for Trojan capture and observations. They have related the high inclinations of Jupiter Trojans to the $n_{16}$ secular resonance, even if this resonance is effective in pumping up inclinations for orbits with comparatively large libration amplitudes of about 60°. Papaloizou, Szuszkiewicz[171], investigated orbital resonances expected to arise when a system of two planets, with masses in the Earth mass range, undergoes convergent migration while embedded in a section of gaseous disc where the flow is laminar. They considered surface densities corresponding to 0.5-4 times that expected for a minimum mass solar nebula at 5.2 au. For the above mass range, the planets undergo type I migration. Using hydrodynamic simulations, they found that, when the configuration is such that convergent migration occurs, the planets can become locked in a first-order commensurability for which the period ratio is $(p + 1)/p$ with $p$ being an integer and migrate together maintaining it for many orbits. Slow convergent migration results in commensurabilities with small $p$ such as 1 or 2. Instead, when the convergent migration is relatively rapid as tends to occur for disparate masses, higher $p$ commensurabilities are realized such as 4:3, 5:4, 7:6 and 8:7. However, in these cases the dynamics is found to have a stochastic character with some
commensurabilities showing long-term instability with the consequence that several can be visited during the course of a simulation. Furthermore, the successful attainment of commensurabilities is also a sensitive function of initial conditions. When the convergent migration is slower, such as occurs in the equal-mass case, lower p commensurabilities such as 3:2 are obtained, which show much greater stability. Resonant capture leads to a rise in eccentricities that can be predicted using a simple analytic model that assumes the resonance is isolated. They found that, once the commensurability has been established, the system with an 8:7 commensurability is fully consistent with this prediction. Barkin, Ferrandiz[4] discussed Resonant Rotation of Two-layer Moon and Mercury.

Callegari, Ferraz-Mello, Michtchenko[33], discussed the Dynamics of Two Planets in the 3/2 Mean-motion Resonance in Application to the Planetary System of the Pulsar PSR B1257+12. Roig, Nesvorny, Ferraz-Mello [192], performed numerical simulations of the orbital evolution of both real and fictitious asteroids in the 2:1 resonance. Their models include gravitational perturbations by the major planets. Based on the dynamical lifetimes, we classify the observed resonant asteroids into three groups:

(i) the Zhongguos, which seems to be stable over the age of the Solar system;
the Griquas, with typical lifetimes in the resonance of the order of some 100 Myr; and

the strongly unstable asteroids, which escape from the resonance in a few 10 Myr or less.

Our simulations confirm that the Zhongguos may be primordial asteroids, located in the 2:1 resonance since the formation of the Solar system. The dynamics of the Zhongguos constitute a typical example of slow chaotic evolution confined to a small region of the resonance. On the other hand, an analysis of the size distribution of the Zhongguos reveals a rather steep distribution. Such a distribution would not be compatible with a long collisional history, rather suggesting that the Zhongguos are likely to be the outcomes of a recent breakup event. Thus, while dynamics points toward a primordial resonant origin, the size distribution rather points to a recent origin. A possible explanation is that the Zhongguos formed by the catering/fragmentation of a large resonant or near-resonant asteroid.

Beaugé, et. al.([6], [7], [8]) studied Planetary migration and extra solar planets in the 2/1 mean-motion resonance. They reviewed recent results on the dynamics of multiple-planet extra-solar systems, including main sequence stars and the pulsar PSR B1257+12 and, comparatively, our own Solar System. They discussed Resonances and stability of extra-solar planetary systems. Michtchenko and Ferraz-Mello([148], [150]), studied Resonant Structure of the Outer Solar System in the Neighborhood of the
Planets. Ferraz-Mello [58], discussed Tidal Acceleration, Rotation and Apses Alignment in Resonant Extra-Solar Planetary Systems. Michtchenko, Beaugé, Roig[149], have studied Planetary Migration and the Effects of Mean Motion Resonances on Jupiter's Trojan Asteroids.

According to Hamilton [87], Resonances are fundamental in orbital dynamics, and are responsible for a diverse set of Solar System phenomena including gaps, density waves, and bending waves in Saturn's rings; asteroids and satellites on tadpole and horseshoe orbits; gaps and enhancements in the asteroid belt and the delivery of meteoroids to Earth; tidal heating and volcanism on Io; the broad and dusty Saturnian E-ring; stable longitudes in the geopotential; the large vertical extent of the Jovian ring's diffuse halo; the high orbital eccentricities of Pluto and some Kuiper Belt objects; spin-locking of planetary satellites; and the dusty ring exterior to Earth's orbit. The Solar System did not form in its current resonant-rich state though, rather it evolved slowly into this state under the influence of drag forces including planetary tides, planetary migration, Poynting-Robertson drag, plasma drag, and nebular drag.

Dynamical characteristics of Kuiper Belt Objects (KBOs) in resonance with Neptune preserve a record of events that shaped the outer solar system. In particular, the spatial distribution of 2:1 resonant KBOs reflects Neptune's migration history. Numerical simulations by Murray, Chiang [157], revealed that a faster Neptunian migration causes more 2:1
objects to be captured into libration about one libration center than about
the other, with the result that up to three times more 2:1 KBOs might be
discovered at longitudes trailing Neptune than leading it. This phenomenon
is made possible by asymmetric libration, a hallmark of the 2:1 resonance
not shared by the 3:2 resonance. They explore the dynamics of asymmetric
libration analytically and provide physical, mechanistic explanations for
these previous numerical findings. In the restricted 3-body problem, the
perturbed, Neptune, accelerates both the KBO and the Sun, contributing
direct and indirect terms to the disturbing potential, respectively. The
direct and indirect terms act in opposition to one another, and the balance
between these perturbative effects leads to asymmetric libration. The
averaged indirect potential is zero for the 3:2 resonance and thus
asymmetric libration does not occur. They provide a blow-by-blow account
of the torques felt by a KBO near resonance to achieve a physical
understanding of this asymmetry in capture outcomes.

Hadjifotinou, Hadjidemetriou [83], constructed a symplectic mapping
for the study of the dynamical evolution of Edge worth-Kuiper belt objects
near the 2:3 mean motion resonance with Neptune. The mapping is six-
dimensional and is a good model for the Poincaré map of the 'real' system,
that is, the spatial elliptic restricted three-body problem at the 2:3
resonance, with the Sun and Neptune as primaries. The mapping model is
based on the averaged Hamiltonian, corrected by a semi analytic method so
that it has the basic topological properties of the phase space of the real system both qualitatively and quantitatively. They start with two dimensional motion and then they extend it to three dimensions. Both chaotic and regular motion is observed, depending on the objects' initial inclination and phase.

Sidlichovský [199] presented an adiabatic approximation for the non-planar, circular, restricted 3BP for the external resonance 4/7. It can be used as a model for resonant Kuiper belt objects. The Hamiltonian is truncated at the fourth order in eccentricities and inclinations, he left slow variables frozen and after solving the pendulum problem for fast variables he used the averaged effect of fast variables on slow variables. In this way he obtained the guiding trajectories for slow variables as contour lines of adiabatic invariant. He discussed the existence of a chaotic region which is formed by trajectories crossing a critical curve which corresponds to the separatrix of fast pendulum motion, where the assumption of sharp division between fast and slow frequencies is not correct and the adiabatic theory fails. Qualitatively it helps us to understand how the protective mechanism works as the interplay of mean motion and Kozai Lidov resonance.

Grimm, Robert E. [75], has studied a comparison of time domain electromagnetic and surface nuclear magnetic resonance sounding for subsurface water on Mars. Pan, Margaret et. al. [169] developed a framework based on energy kicks for the evolution of high-eccentricity
long-period orbits in the circular planar restricted three-body problem with Jacobi constant. They use this framework to explore mean motion resonances between the test particle and the massive bodies. This approach produces a pendulum-like equation describing the librations of resonance orbits about fixed points that correspond to periodic trajectories in the rotating frame. They found a striking analogy exists between these new fixed points and the Lagrangian points, as well as between librations around the fixed points and the well-known tadpole and horseshoe orbits; they call the new fixed points the "generalized Lagrangian points." Their approach gives a condition for the onset of chaos at large semi major axis.

Guillens, Vicira, Gomes [76], have searched for short-lived asteroids in the neighborhood of the 3:1 resonance. The borders of this resonance and the positions of the asteroids with respect to them are determined by using the mean elliptic planar restricted three-body problem and a method that invokes the concept of a "representative plane". The existence of a non-negligible number of asteroids in the 3:1 resonance neighborhood implies that a process must be continually acting to replenish this region. They suggest that these asteroids may have wandered to the resonance borders through small semi major axis displacements due to the Yarkovsky effect. This result also confirms that asteroids are continually feeding the chaotic 3:1 resonance region and consequently the Mars crossers and NEA population.
Kinoshita, Hiroshi[100] found the 1:1 secular resonance takes place only in a restricted problem, in which the mass less body is perturbed by the bodies whose motions are given of time. The alignment of the pericenters of the planets, which is found in an extra solar planetary system, is not a secular resonance. The alignment comes from the fact that the amplitude of one eigen frequency is larger than the sum of the amplitudes of other eigen frequencies. Zhou, Lehto, Sun, Zheng [226, 227] showed that the occurrence of apsidal secular resonance depends only on the mass ratio semi-major rate and eccentricity rate between the two planets. The dynamical stability the planets in apsidal secular resonance with and without disk-generated torque are studied with applications to the extrasolar planetary systems. Their numerical investigation confirms the existence of the 3:1 resonance and implies a complex orbital motion. Different stable motion types, with and without apsidal coronation, are found. Owing to the high eccentricities in this system, they apply a semi-analytical method based on a new expansion of the Hamiltonian of the planar three-body problem in the discussion. They analyze the occurrence of apsidal coronation in this MMR and its influence on the stability of the system.

Dvorak, Rudolf [54], shows that the effect of solar system chaos small bodies, such as comets and asteroids, is quite different from experienced by the major planets. It is more obvious in the motion of
large number of small bodies in the planetary region. Thus, the existence of
different groups of comets and asteroids is due to the different qualities of
the various resonances - mean motion resonances, secular resonances and
three-body resonances - but especially because of resonance overlap.
Moreover, he has also found chaotic motion in the motion of the planets
and appears to be present on even a larger level in extra-solar planetary
systems.

Melita, Brunini [145], presented a Comparative Study of Mean-
Motion Resonances in the Trans-Neptunian Region. They have also studied
how certain evolutionary models, related with the orbital expansion of the
outer planets during their formation stage, could result in resonant
populations with a noticeably different primordial number of members.
Trapping of planetesimals into coronation resonance with a proto-planet
may have played an important role in the accretion of m-km-sized bodies
during the Solar System formation. Mothé-Diniz, Gomes [155], studied
this process by comparing the numerical equilibrium points with the
analytical ones, obtained from Lagrange's equations. Using a variable
model for the drag force, a great number of numerical integrations is done
for groups of 100 planetesimals. These simulations allowed them to find
two new classes of resonances: secular libration and extended coronation.
They discussed the role that corotation resonance trapping may have played
in the accretion mechanism of a forming planetary system.
Papaloizou [170], studied Migration and Resonances in Extrasolar Planetary Systems. They discussed the orbital evolution of two planets locked in 2:1 commensurability through migration tidally induced by the disc using both analytic methods and numerical hydrodynamic simulations. Lee, Man Hoi[116], has shown that a diversity of 2:1 resonance configurations can be expected in extrasolar planetary systems, and their geometry can provide information about the origin of the resonances. Assembly during planet formation by the differential migration of planets due to planet-disk interaction is one scenario for the origin of mean-motion resonances in extrasolar planetary systems.

According to Lee and Peale ([114], [115], [172]), the 2:1 orbital resonances of the GJ 876 system can be easily established by the differential planet migration due to planet-nebula interaction. Significant eccentricity damping is required to produce the observed orbital eccentricities. The geometry of the GJ 876 resonance configuration differs from that of the Io-Europa pair, and this difference is due to the magnitudes of the eccentricities involved. They have shown that a large variation in the configuration of 2:1 and 3:1 resonances and, in particular, asymmetric librations can be expected among future discoveries.

Kinoshita, Hiroshi[99], showed that for a planetary system a secular resonance with a mixed secular term does not take place.

Nagasawa, Lin, Thommes [161], studied Dynamical Shake-up of Planetes
Systems. I. Embryo Trapping and Induced Collisions by the Sweeping Secular Resonance and Embryo-Disk Tidal Interaction. Psychoyos, Hadjidemetriou ([81], [82], [186]), discussed the dynamics of 2/1 Resonant Extrasolar Systems Application to HD82943 and GLIESE876. A complete study is made of the resonant motion of two planets revolving around a star, in the model of the general planar three body problem. Several families of symmetric periodic orbits are computed numerically, for the 2/1 resonance, and for the masses of some observed extrasolar planetary systems. In this way they obtain a global view of all the possible stable configurations of a system of two planets. These define the regions of the phase space where a resonant extrasolar system could be trapped, if it had followed in the past a migration process.

According to Érdi, Pál[56], the discovery of the resonant pair of planets orbiting the star GJ 876 there has been an increasing interest in the dynamics of resonant exoplanets. They described the main characteristics of the resonant system GJ 876, and study the phase space structure of the 2:1 resonance in two exoplanetary systems. In the case of the system GJ 876 they showed that the inner planet is situated in the stable region of the 2:1 resonance. They have also shown that in the case of the system HD 82943 the phase space structure around the 2:1 resonance is chaotic. Gabryszewski, Wlodarczyk [64], gave a report on a numerical experiment showing that even in one of the most chaotic regions of the Solar System -
the region of the giant planets, there are numerous narrow bands where mean motion resonances (MMRs) can stabilize orbits of small bodies in a time span comparable to their lifetimes (among Jupiter and Saturn).

Lykawka, Mukai, ([122], [123], [124]), have investigated basic properties and dynamical evolution of classical transneptunian objects (TNOs) around the 7:4 mean motion resonance with Neptune (a~43.7 AU), motivated by observational evidences that apparently present irregular features near this resonance. They have explored the dynamical long-term evolution in the scattered disk (but not its early formation) based on the computer simulations performed together with extra computations. Their results demonstrated that the scattered disk has been evolving continuously since early times until present. Nesvorný, Roig ([165], [166]), explored the stability of the 2:3 mean motion resonance with Neptune and compared to the observed resonant population. It is shown that orbits with small and moderate amplitudes of the resonant angle are stable over the age of the Solar System. Snellgrove [204], simulated the resonance trapping in the GJ876 system by embedding two protoplanets of the observed mass range in a disc cavity. Simulation shows that a disc with parameters expected for protoplanetary disc causes 2:1 resonance trapping and eccentricities of the observed range are obtained. He also simulated a 'Jupiter-Saturn' type configuration of two protoplanets, the outer protoplanet is less massive than the inner one. Simulation shows a 3 : 2 resonance trapping is obtained.
Subsequently the migration reverses and both protoplanets migrate outwards, still locked into resonance. This behaviour is explained as being caused by a torque imbalancence, and an increased mass flow across the protoplanet orbits. By considering Mars in a Keplerian and circular orbit,

**Yokoyama, et. al.** [221] have shown that once captured in the resonance, the inclination of the satellite reaches very high values. Here, the integrations are extended to much longer times and escape situations are analyzed. These escapes are due to the interaction of new additional resonances, which appear as the inclination starts to increase reaching some specific values. They have also included the effect of Mars' eccentricity in the process of the capture. The role played by this eccentricity becomes important, particularly when Phobos encounters a double resonance. **Gomes, Rodney et. al.** [73], studied the transfer process from the scattered disk to the high-perihelion scattered disk. They found that the Kozai mechanism (coupling between the argument of perihelion, eccentricity, and inclination), associated with a mean motion resonance (MMR), is the main responsible for raising both the perihelion distance and the inclination of SD objects. They have shown that bodies can be temporarily detached from the planetary region by dynamical interactions with the planets.

**Bois et. al.** [27], presented different orbital topologies for extrasolar planetary systems including a 2:1 mean motion resonance namely the
Gliese 876 HD 82943 and HD 160691 planetary systems as well as the pair Io-Europa in our solar system. Using a new technique called MEGNO, they have shown that the orbital parameters may allow the existence of a stability zone in the semi-major axes parameter space preserved by a mechanism of apsidal secular resonances. Moreover their global dynamics analysis in the 3-D orbital parameter space allowed them to make firm evidence new resources of the 2:1 orbital resonances: a stable orbital topology requires aligned or anti-aligned apsidal lines according to the characteristics of these different planetary systems. These configurations coupled with adequate relative positions of the planets on their orbits avoid close approaches. For instance in the HD 160691 system the apsidal lines are anti-aligned and the stabilizing mechanism avoids close approaches at their periapses (high outer eccentricity of the second planet helps). In the pair Io-Europa conjunctions occur when Io is near periapse and Europa near apoapse whereas conjunctions of the GJ 876 companions occur when both planets are near periapses. Cionco [43] studied the resonant dynamical friction of Orbital migrations in planetesimal discs. Van Paep, Hoover [211], studied Palaeodynamic sedimentary cycles on Mars in resonance with short similar cycles on Earth. Friedland [62], applied Dynamic autoresonance theory to the problem of thresholds on migration timescales for capture into resonances in the planar-restricted three-body problem. They explained small abundance of Kuiper Belt objects in the 2:1
resonance and define accurate bounds on the timescales involved in the early evolution of the solar system. Cuk et. al. ([49], [50]), described a new type of secular resonance for irregular satellites at higher inclinations (40-45°), which locks the satellite's pericenter with that of the planet.

Haghighipour [84], [85] has discussed Planetary Migration, Resonance Capture and Dynamical Friction in a Planetesimal Disk. Tsiganis, Morbidelli, Levison, Gomes[210] studied the migration of the four outer planets of the solar system, interacting with a planetesimals disk of 50 Earth masses, truncated at 30 AU. They assumed a more compact configuration (within 15 AU). In their simulations, a slow migration begins, due to particles that leak out of the disk and encounter Neptune. As the system slowly stretches, a pair of planets is forced to cross an orbital resonance. Then, the planetary orbits become eccentric, the inner part of the disk is destabilised. The mass flow, towards the planetary region, increases and the planets are extracted from the resonance. A second, fast, migration phase begins, typically accompanied by a phase of encounters among the planets. This instability is suppressed in 65% of the cases studied.

According to Funk, Pilat-Lohinger, Dvorak, Freistetter, Órdi[63], as the first extrasolar planet was discovered about 10 years ago, a major point of dynamical investigations was the determination of stable regions in extrasolar planetary systems where additional planets may exist. Using
numerical methods, they investigated the dynamical stability in known multiple planetary systems (HD74156, HD38529, HD168443, HD169830) with special interest on the region between the two known planets and on the mean motion resonances inside this region. As a dynamical model they took the restricted 4-body problem containing the host star, the two planets and massless test-planets. They used the Lie-integrator and additionally the Fast Lyapunov Indicators as a tool for detecting chaotic motion. They have also investigated the inner resonances with the outer planet and the outer resonances with the inner planet with test-planets located inside the resonances.

Holmes, Dermott, Gustafson, Grogan [93], discussed Resonant Structure in the Kuiper Disk: An Asymmetric Plutino Disk. They developed a dynamical model of the Kuiper disk, and run numerical integrations of particles originating from source bodies trapped in the 3:2 external mean motion resonance with Neptune to determine what percentage of particles remain in the resonance for a variety of particle and source body sizes. They found that a disk with a size-frequency distribution weighted toward large particles, which are more likely to remain in resonance, may have a stronger, more easily identifiable resonant signature than a disk composed of small particles. Flynn, Saha, Prasenjit [59], implemented Lie transform perturbation theory to second order for the planar spin-orbit problem. The perturbation parameter is the asphericity of the body, with the orbit...
eccentricity entering as an additional parameter. They studied first- and
second-order resonances for different values of these parameters. For
nearly spherical bodies such as Mercury and the Moon, first-order
perturbation theory is adequate, whereas for highly aspherical bodies such
as Hyperion, the spin is mostly chaotic, and perturbation theory is of
limited use. However, in between they identified a parameter range in
which second-order perturbation theory is useful and in which as-yet
unidentified objects may be in second-order resonances. Brassel, Lehto
[30], investigated the role of secular resonances on the motion of
terrestrial-planet trojans in tadpole motion for 5Myr in the inclination
range $\leq 90^\circ$.

**Chaos on the Large Scale**

One of the most interesting issues in the study of chaotic systems is
whether or not the presence of chaos may actually produce ordered
structures and patterns on a larger scale. It has been found that the
presence of chaos may actually be necessary for larger scale physical
patterns, such as mountains and galaxies, to arise. The presence of chaos in
physics is what gives the universe its "arrow of time", the irreversible flow
from the past to the future. For centuries mathematicians and physicists
have overlooked dynamical systems as being random and unpredictable.
The only systems that could be understood in the past were those that were
believed to be linear, but in actuality, we do not live in a linear world at
all. In this world linearity is incredibly scarce. The reason physicists didn't know about and study chaos earlier is because the computer is our "telescope" when studying chaos, and they didn't have computers or anything that could carry out extremely complex calculations in minimal time. Now, thanks to computers, we understand chaos a little bit more each and every day. The stock markets are said to be nonlinear, dynamic systems. Chaos theory is the mathematics of studying such nonlinear, dynamic systems. Does this mean that chaoticians can predict when stocks will rise and fall? Not quite; however, chaoticians have determined that the market prices are highly random, but with a trend. The stock market is accepted as a self-similar system in the sense that the individual parts are related to the whole. The applications of chaos theory are infinite; seemingly random systems produce patterns of spooky understandable irregularity. From the Mandelbrot set to turbulence to feedback and strange attractors; chaos appears to be everywhere. Breakthroughs have been made in the past in the area chaos theory, and, in order to achieve any more colossal accomplishments in the future, they must continue to be made. Understanding chaos is understanding life as we know it.

**Chaos in the Real World**

In the real world, there are very good examples of instability, disease, political unrest, family and community dysfunction. Disease is unstable because at any moment there could be an outbreak of some deadly
disease for which there is no cure. This would cause terror and chaos. Political unrest is very unstable because people can revolt, throw over the government and create a vast war. A war is another type of a chaotic system. Family and community dysfunction is also unstable because if you have a very tiny problem with a few people or a huge problem with many people, the outcome will be huge with many people involved and many people's lives in ruin. Chaos is also found in systems as complex as electric circuits, measles outbreaks, lasers, clashing gears, heart rhythms, electrical brain activity, circadian rhythms, fluids, animal populations, and chemical reactions, and in systems as simple as the pendulum. It also has been thought possibly to occur in the stock market.

**Where is chaos really applicable?**

One application, already mentioned several times, is the weather. The Earth is huge, yet something very small can start a chain of events that dramatically alters the weather. This, among other things, is the reason that the meteorologist seldom seems to be correct - without knowing the position of every molecule on the world, an impossible task, it is not possible to predict the weather with 100% accuracy. Another application of chaos theory, which many people would consider to be more important is in economics. Chaos theory tends to model most economic systems better than fractal models. This is most likely because of the nature of the economy -
one small change can set off a chain of events and start a depression, or a
time of prosperity.

Non-integrability of a dynamical system has been studied by Bolotin
and Mackay, [28], Sansaturio, Vigo, Ferrándiz,(194), Maciejewski, et.
al.([125] - [129]) analysed the integrability of a dynamical system
describing the rotational motion of a rigid satellite under the influence of
gravitational and magnetic fields. They proved that for a symmetric
satellite the system does not admit an additional real meromorphic first
integral except for one case when the value of the induced magnetic
moment along the symmetry axis is related to the principal moments of
inertia in a special way. The non-integrability of equations of motion can
be proved by means of the separatrices crossing method, using Melnikov's
method [146].

Pucacco[188], Meletlidou, Ichtiaroglou, Winterberg[143, 144] have studied Non-integrability of Hill's lunar problem. They have
shown that the Hamiltonian of Hill's problem does not possess an integral
of motion, analytically continued from the integrable two-body problem in
a rotating frame.

Attitude motion of satellite has been discussed by
Comisel, Ciobanu, Bunescu, Simunek, Chum, Thrulik[45], Kim, Choo
[98], Kang[96]. Zanardi, Quirelli, Kuga, ([223], [224]) studied
spacecraft attitude prediction by considering the influence of the residu
magnetic torques. Assuming basically the inclined dipole model for the Earth's magnetic field, an analytical averaging method is applied to obtain the mean residual torque along every orbital period. The orbit mean anomaly is utilized to compute the average components of residual torque in the spacecraft body frame reference system. The theory is developed accounting for orbit elements time variations, not restricted to circular orbits (it contains terms up to second order in eccentricity), giving rise to many curvature integrals solved analytically. They observed that the residual magnetic torque does not have component along the rotation axis. The inclusion of this torque on the rotational motion differential equations of spin stabilized spacecrafts yields the conditions to derive an analytical solution. Their solution shows that such residual torque does not affect the spin velocity magnitude, contributing only for the variations of the right ascension and declination of the rotation axis. This in turn produces a drift on the spin axis of the spacecraft. Numerical simulations performed with data of the spin stabilized SCD2 Brazilian satellite are shown, and compared with routine attitude determinations of the satellite control center. 

Abrashkin, Balakin, Belokonov, Voronov, Zaitsev, Ivanov, Sazonov, Semkin, ([1], [2]), determined Uncontrolled Attitude Motion of the Foton-12 Satellite and Quasi-Steady Microaccelerations onboard it. The determination was carried out by the onboard measurement data of the Earth's magnetic field strength vector. Results of in-flight tests of three

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modes of uncontrolled attitude motion of the Progress spacecraft are described by Bryukhanov, Tsvetkov, Belyaev, Babkin, Matveeva, [32]. The proposed modes of experiments related to microgravity are as follows:

1. triaxial gravitational orientation,
2. gravitational orientation of the rotating satellite, and
3. spin-up in the plane of the orbit around the axis of the maximum moment of inertia.

For the obtained motions the quasi-static component of microaccelerations was computed at a point onboard, where installation of experimental equipment is possible.

A semi-analytical approach to study the rotational motion of an artificial satellite, under the influence of torque due to the solar radiation pressure is proposed by Zanardi, Vilhena, Cabette, Garcia, [225], taking into account the influence of Earth's shadow. A mapping for the shadow function is proposed and a semi-analytical process is applied. When the satellite is totally illuminated or it is inside the penumbra, a known analytical solution is used to compute the satellite's attitude. A numerical simulation shows, when the penumbra region is included, the attenuation of the rotational motion during the transition from the shadow to the illuminate region and vice versa. Singh ([201], [202]), has analyzed the nonlinear planar oscillation of a satellite launched into an elliptic orbit under the influence of external forces of general nature such as...
accretion and tidal effects, etc. These forces are considered in the form of a frictional force and a periodic force.

Bhatnagar, Khan and Saha ([24], [25]), have studied non-linear oscillations of a satellite in an almost elliptic orbit around the Earth under the influence of solar radiation pressure Bhatnagar and Bhardwaj ([21], [22], [23]), Bhardwaj and Bhatnagar ([11], [12]), Bhardwaj and Tuli ([18], [19], [20]), have studied non-linear planar oscillations of a satellite in circular and elliptic orbits under the influence of third body torque. Bhardwaj and Kaur ([13] - [17]) have studied Melnikov’s function, resonance and chaos in non-linear planar oscillations of a satellite in an elliptic orbit under the influence of magnetic torque.

The title of the thesis is “Chaos in non-linear planar Oscillation of a satellite under the influence of magnetic torque in an elliptic orbit”. The study has been performed in six chapters.

Chapter one is an introductory one. This chapter has two sections. In section one, the evolution, historical background, resonance and chaotic phenomenon arising in various kinds of non-linear problems of dynamical systems and recent developments in the field of Chaos Theory has been discussed. All the reviews of concerned works are also discussed in this section. In section two some definitions have been given to make the thesis self-contained.
Chapter two. deals with the equations of the rotational motion of a satellite under the influence of a magnetic torque in an elliptic orbit, the Hamiltonian function and the equilibrium and double asymptotic solutions. The problem is studied as follows: "To discuss the chaos in non-planar oscillation of a satellite under the influence of magnetic torque in an elliptic orbit, when orbital plane of the satellite coincides with the equatorial plane of the central body". Consider a rigid satellite S moving in an elliptic orbit around the Earth E such that the orbital plane of the satellite coincides with the equatorial plane of the Earth. Let \( \vec{r} \) be the radius vector of the center of mass of the satellite, \( \theta \) the angle that the long axis of the satellite make with a fixed line EF lying in the orbital plane, \( \delta/2 \) the angle between the radius vector and the long axis, \( e \), the eccentricity of the orbit of a satellite and \( v \) the true anomaly measured from the ascending node. The satellite is assumed to be a triaxial body with principal moments of inertia \( A<B<C \) at its centre of mass and \( C \) is the moment of inertia about the spin axis which is perpendicular to the orbital plane. These principal axes act as co-ordinate axes \( x, y, z \); the \( z \)-axis being perpendicular to the orbital plane. The magnetic torque has been taken to be of the order of eccentricity. This chapter is divided into five sections. Section one is an introductory one. Section two contains the equation of motion. Section three deals with the Hamiltonian function. In section four...
we have found the equilibrium and double asymptotic solutions and section five draw the conclusion.

**Chapter three** is devoted to the non-integrability of the equations of motion. This chapter contains four sections. Section one is an introductory one. In section two, through Melnikov's method, it has shown that the equations of motion are non-integrable. Section three depicts the graphical representation of the Melnikov's function, and section four, draw the conclusions.

**Chapter four** discussed the non-resonant planar oscillation of the satellite. Since the magnetic torque parameter is very small so using BKM method (Bogoliubov-Krylov-Mitropolsky[26]), non-resonance case has been studied. This chapter is divided into three sections. Section one is an introductory one, section two deals with the non resonance case and in section three conclusions has been drawn. It has been observed that the main resonance occurs at \( n = b \) and \( n = 1 \) and the parametric resonance occurs at \( n = \frac{1}{2} \), where \( n \) is the frequency of the periodic force.

**Chapter five** discussed the resonance case. This chapter has four sections. Section one is an introductory one. Section two deals with the resonance case. Section three depicts the graphical representation of resonance curves and section four draws the conclusion. It is observed that the amplitude of the oscillation remains constant up to the second order of
approximation. The analysis regarding the stability of the stationary planar oscillation of a satellite near the resonance frequency shows that discontinuity occurs in the amplitude of the oscillation at a frequency of the external periodic force which is less than the frequency of the natural oscillation.

In **Chapter six** the width of the resonance zone calculated and the existence of the chaotic region predicted analytically. This chapter has four sections. Section one is an introductory one, section two determines the width of the resonance zone and predicts analytically the existence of the chaotic region. In section three, the existence of the chaotic region through surface of section method has been shown and the Poincare's surface of section has been plotted by varying the magnetic torque parameter, the mass distribution parameter and the eccentricity of the orbit of the satellite. It is found that some of the regular trajectories are captured into the chaotic zone, due to the elongated orbits, irregular mass distribution of the satellite and for some values of the magnetic torque parameter. Section four draws the conclusions.
1.2 DEFINITIONS

1.2.1 ATTRACTOR

An attractor is a set of points or a subspace in the phase space toward which a time history approaches after transients die out. For example, equilibrium points or fixed points in maps, limit cycles, or a torodial surface for quasi-periodic motions, are all classical attractors.

1.2.2 BIFURCATION

A bifurcation is a sudden change in a system that occurs when a parameter is slightly modified. Sometimes it refers to the specific situation in which a system is asymptotically periodic and the periodic behavior undergoes doubling. For example, a system may oscillate about two states. Then a parameter is increased and the system oscillates about four states, two of which are close to one of the original states and two of which are close to the other original state.

1.2.3 CENTRE MANIFOLD

In dynamical systems theory, the motion in the neighbourhood of an equilibrium point can be classified according to whether the eigen solutions are stable, unstable or oscillatory. The subspace
of the phase space which is spanned by the purely oscillatory solutions is sometimes called the center manifold.

1.2.4 CHAOS

Chaos is structured random behavior of a non-linear, complex, dynamical system. The behavior over long-time scales is

(1) unpredictable,

(2) seemingly random but not arbitrarily so,

(3) sensitive to initial conditions, and

(4) characterized by a strange attractor that is often a fractal.

Chaotic behavior is different from random behavior in that it is not completely random and the strange attractor governs its structure.

1.2.5 DETERMINISTIC DYNAMICAL SYSTEM

Deterministic dynamical system refers to a system whose equations of motion, parameters and initial conditions are known and are not stochastic or random. However, a deterministic system may have motions that appear random.

1.2.6 EQUILIBRIUM POINT OR FIXED POINT

In a continuous dynamical system, a point in phase space towards which a solution may approach as transient decay is called an Equilibrium point. In mechanical systems,
usually means a state of zero accelerations and velocity. For maps, equilibrium point may come in a finite set where the system visits each point in a sequential manner as the map or difference equation is iterated.

1.2.7 **HETEROCLINIC ORBITS**

Heteroclinic orbits is an orbit in a map or difference equation that occurs when stable and unstable orbits from different saddle point intersect.

1.2.8 **HOMOCLINIC ORBIT**

Homoclinic orbit is an orbit in a map that occurs when stable and unstable manifolds of the saddle point intersect.

1.2.9 **KAM THEOREM**

If we are given an integrable system with N degrees of freedom, then its trajectories in the 2N-dimensional phase space are constrained to lie on N-dimensional surfaces in the phase space. These N-dimensional surfaces are called KAM surfaces. If we perturb such a system by a weak perturbation which makes the system non-integrable, most KAM surfaces remain intact. However the perturbation induces resonance zones locally in the phase-space which make the system chaotic in the region of the chaotic zones. As the perturbation grows, these resonance zones
grow and destroy the KAM surfaces around them. Overlap occurs when two neighbouring resonance zones destroy all KAM surfaces between them, then chaos starts.

1.2.10 **KOZAI RESONANCE**

Kozai Resonance named after Y. Kozai is a type of secular resonance defined by the 1:1 commensurability of the secular precession rates of the perihelion and the orbit normal such that the argument of perihelion is stationary (or librates). This resonance, which generally requires significant orbital eccentricity and inclination, causes coupled oscillations of these two orbital elements (with little or no perturbation of the semi-major axis). It is likely a common, if intermittent, feature in the long term dynamics of many minor planets.

1.2.11 **LYAPUNOV EXPONENT**

Lyapunov exponents indicate the rate of divergence of nearby trajectories. Such motions deviate exponentially rapidly and the coefficient in the exponential is the *Lyapunov exponent*. In other words, in a system with at least one large Lyapunov exponent, there is extreme sensitivity to initially conditions, while in a system with small Lyapunov exponents, nearby situations initially
deviate more slowly. There is a Lyapunov exponent for each independent direction.

1.2.12 MANIFOLD

Manifold is a subspace of phase space in which solutions with initial conditions in the manifold stay in the manifold or subspace, under the action of differential or difference equations. Actually we mean a topological manifold here. A topological manifold of dimension $n$ is a topological space such that each point has a neighbourhood which is homeomorphic to the interior of a sphere in Euclidean space of dimension $n$. Such a manifold $M$ is a differentiable of order $r$ (i.e. has a differentiable structure of $C^r$).

1.2.13 MEAN MOTION RESONANCE

This is intuitively the most obvious type of resonance in a planetary system; it occurs when the orbital periods of two bodies are close to a ratio of small integers.

1.2.14 MELNIKOV FUNCTION

The theory of chaotic motions focuses on the saddle points of the Poincare map of continuous space flows. Near such points there are subspaces where trajectories are swept into the point, (stable manifold) and subspaces, where trajectories are kept away
from the point, (unstable manifold). The Melnikov's function provides a measure of the distance between the stable and unstable manifolds. Chaos is possible when the Melnikov function has a simple zero. (named after a Russian Mathematician Melnikov)

1.2.15 NONLINEAR

Nonlinear is an adjective that applies to a system to indicate that output responses are not linearly related to input changes. In a linear system, doubling an input doubles the change in the output. Linear systems are always exactly solvable. A nonlinear system may or may not be solvable. If it is solvable then it would be predictable.

1.2.16 NON-PERIODIC MOTION

Non-periodic motion is motion that does not repeat with time.

1.2.17 PARAMETERS

Parameters are numbers that allow one to vary the rules that govern a system.

1.2.18 PERIODIC BEHAVIOR

Periodic behavior is non-chaotic since the system returns to the original state after a certain time, at which point it repeats what
previously did. The *period* is the minimum time it takes for the motion to repeat.

### 1.2.19 PHASE SPACE

In mechanics, phase space is an abstract mathematical space whose co-ordinates are generalized co-ordinates and generalized momenta. In dynamical system, governed by a set of first-order evolution equations, the co-ordinates are the state variables or components of the state vector. When the motion is periodic, the phase plane orbit traces out a closed curve whereas chaotic motions have orbit which never close or repeat.

### 1.2.20 POINCARE SECTION MAP

Poincare section is the sequence of points in the phase space generated by the crossing of a continuous evolution trajectory through a generalized surface (or plane) in the phase space. For periodically forced second-order non-linear oscillator, a Poincare map can be obtained by stroboscopically observing the position and velocity, i.e. the \( (x(t_n), \dot{x}(t_n)) \) is called a Poincare map, \( t_n \) is selected according to some rule. When,

1. \( t_n = 1 \)-period, the orbit consists of one point;
2. \( t_n = 2 \)-period, the orbit consists of two point;
3. \( t_n = 3 \)-period chaos may occur.

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1.2.21 QUASI–PERIODIC

Quasi–periodic is a vibration motion consisting of two or more incommensurable frequencies. An orbit on a torus is quasi–periodic, if it is closed.

1.2.22 RENORMALIZATION

Renormalization is a mathematical theory in functional analysis in which properties of some mathematical set of equations at one scale can be related to those at another scale by a suitable change of variables. It is developed by the Nobel prize winner Physicist K. Wilson (Cornell University). It is used in the theory of quadratic maps to derive the Feigenbaum number.

1.2.23 RESONANCE

If the basic frequencies of a dynamical system are incommensurable, the phenomenon of resonance occurs.

1.2.24 SADDLE POINT

In the geometric theory of ordinary differential equation, an equilibrium point with real eigen value with at least one positive and one negative eigen value is called a Saddle point.
1.2.25 **SECULAR RESONANCE**

This is a commensurability of the frequencies of precession of the orientation of orbits, as described by the direction of perihelion (or periapse) and the direction of the orbit normal.

1.2.26 **SENSITIVITY TO INITIAL CONDITIONS**

*Sensitivity to initial conditions* means that small changes at a particular time lead to large differences later. The *Butterfly Effect* is another term for sensitivity to initial conditions. It originates from the unpredictability of weather: In certain situations, it is believed that a butterfly flapping its wings creates a small disturbance in one particular location that can eventually produce a large change in the weather throughout the world.

1.2.27 **SPIN-ORBIT RESONANCE**

This is a commensurability of the period of rotation of a satellite with the period of its orbital revolution; the "external driving" in this case is the gravitational tidal torque from the planet which is non-vanishing if the satellite is irregular in shape.

1.2.28 **STOCHASTIC PROCESS**

Stochastic process often refers to a type of chaotic motion found in conservative or non-dissipative systems.
1.2.29 **STRANGE ATTRACTOR**

Strange attractor refers to the attracting set in the phase space on which chaotic orbits move, an attractor that is not an equilibrium point nor a limit cycle, nor a quasi-periodic attractor.

It is an attractor in a phase space with fractal dimension. When in the neighbourhood of a fixed point we get similar structure within a structure we call it a strange attractor.

1.2.30 **SYSTEM**

A system is a set of objects that are governed by a precise set of rules. It is a very general concept. A *physical system* is one that occurs in nature. Some examples are a collection of billiard balls on a pool table, the weather, a fluid in a heated jar and the planets of the solar system. In the case of the weather and a fluid, the objects are gas and liquid molecules. A *mathematical system* is one described by one or more equations.

1.2.31 **TRANSIENT CHAOS**

Transient chaos is a term which is used in describing motion that looks chaotic during a finite time, that is, it appears to on a strange attractor, but eventually settles into a periodic or quasi-periodic motion.
1.2.32 **UNIVERSALITY**

*Universality* in chaos is the idea that very different systems can exhibit the same type of chaos. An example is the class of systems that follow the period-doubling route to chaos.

1.2.33 **UNPREDICTABILITY**

*Unpredictability* means that one cannot forecast in detail the evolution of the system.

1.2.34 **UNSTABLE**

*Unstable* means not stable. When a small change is implemented in a *stable* system, the system returns to its original state. An example is a flat bottom jar: If one tilts it slightly then the jar rocks back and forth and soon comes to rest in a vertical position. In an unstable system, a small change leads to a very different behavior. A pencil balancing on its tip is unstable because perturbing it by the slightest amount makes it falls. In many cases, a system is stable in certain directions and unstable in others.