CHAPTER 2

STRUCTURE OF
MULTIRATE SYSTEMS
2. STRUCTURE OF MULTIRATE SYSTEMS

This chapter gives introduction to multirate system structures and error criterion. A block diagram representation for multirate systems is proposed. A multirate system can be represented in different forms few of them are discussed here. Multirate system analysis can be done using LTI operators. Analysis of \( \ell^2(z) \) error criterion for multirate systems is also considered in this chapter, which is a natural extension and has many properties of the standard chebyshev criterion for LTI filter design. The chebyshev criterion is important for many reasons like it has an energy representation, it is intimately linked to response of system to tones and it has become common specification for filter design.

2.1 MULTIRATE SYSTEM STRUCTURE

A multirate system is formally defined as an operator, \( \mathbf{M} \), on \( \ell^2(z) \) which is characterized by

\[
\mathbf{S}^L \mathbf{M} = \mathbf{M} \mathbf{S}^M
\]

(2.1)

Here \( \mathbf{M} \) is linear and bounded, \( \mathbf{S} \) is the shift operator defined by

\[
[S(x)](n) = x(n-1)
\]

(2.2)

and \( \mathbf{M} \) and \( L \) are integers. For derivation of structure for multirate systems few definitions are considered.

**Definition 2.1**: The \( L \)-polyphase decomposition of \( x \) is a length \( L \) vector

\[
X_L(z) = \begin{bmatrix}
X_{L,0}(z) \\
\vdots \\
X_{L,L-1}(z)
\end{bmatrix}
\]
where \( X_{L,k}(z) = \sum_{n=-\infty}^{\infty} x(Ln + k) z^{-n} \). Note that

\[
X(z) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-(L-1)} \end{bmatrix} X_L(z^L)
\]

The map \( x \rightarrow X_L \) is denoted \( P_L \). When apparent from context, \( x_{L,i} \) is abbreviated to \( x_i \), \( x_{L,i} \) (or its z-transform) is called the \( i \)th L-polyphase component. Note that the standard convention of using \( X \) for the z-transform of \( x \) is used in the definition. Two important properties of \( P_L \) are

\[
P_L^{-1} = P_L^* \tag{2.3}
\]

\[
P_L S_L = S P_L \tag{2.4}
\]

(2.3) shows that \( P_L \) is unitary, (2.4) is useful in converting powers of \( S \) to a single shift operation, referred to as lifting in the control systems literature [15] and also appears in the mathematical literature in the form of abstract shift operators [16]. An abstract version of the structure theorem for multirate systems can be seen in [16]. Other mathematical material related to this structure theory can be seen in [17] [18]. The interpretations given in references are useful especially when multirate systems are thought of in terms of parallel processing.

**Definition 2.2:** The down-sampling-by-L operator, \( D_L \), and the up-sampling-by-L operator, \( U_L \), are defined for a sequence \( x \) by:

\[
[D_L(x)](n) = x(Ln)
\]

\[
[u_L(x)](n) = \begin{cases} x(\frac{n}{L}) & \text{if } n \mod L = 0 \\ 0 & \text{otherwise} \end{cases}
\]

Some additional useful properties of down-sampling, up-sampling and the shift operator are
\[ S^{-1} = S^\dagger \quad (2.5) \]
\[ D_L S^L = S D_L \quad (2.6) \]
\[ U_L S = S^L U_L \quad (2.7) \]
\[ D_L^\dagger = U_L \quad (2.8) \]

These properties can be seen in [19][20].

**Definition 2.3**: For a set of filters with impulse responses, \( \{ t_{i,k} \}, i=0,...,L-1, \ k=0,...,M-1, \) and \( x \in \ell^2(Z)^M, \ y \in \ell^2(Z)^L, \) define a LTI-matrix operator, \( y = T(x) \) by \( Y(z) = T(z)X(z), \) where \( T(z) \) is a \( L \times M \) matrix with \( (i, k) \)-th element \( T_{i,k}(z). \) Operators having properties (2.5) can now be characterized [16][21][22][23].

**Theorem 2.1**: For every operator \( M \) having properties (2.1), \( P_L^\dagger M P_M^\dagger = T \) for some linear time-invariant \( T; \) i.e., \( M = P_L^\dagger TP_M. \)

**Proof**: \( P_L^\dagger M P_M^\dagger \) is linear time-invariant since

\[ P_L^\dagger M P_M^\dagger S = P_L^\dagger M S^M P_M^\dagger = P_L^\dagger S^L M P_M^\dagger = S P_L^\dagger M P_M^\dagger \]
using (2.3) and (2.4). Thus, the result follows from the structure theory of linear time-invariant systems [17][24][25].

These results are now expressed in a more standard form. Fig. 2.1(a) shows an implementation of the operator \( P_M. \) Fig. 2.1(b) shows an implementation of \( P_L^\dagger. \) In the figure, the blocks with an up-arrow and down-arrow are \( U_L \) and \( D_M, \) respectively. The entire implementation for a multirate system is shown in Fig. 2.2, this structure will be referred to as the matrix form of the multirate system. In the figure, the big arrow (\( \Rightarrow \)) indicates a vector output, and \( T \) is the \( z \)-domain representation of the operator \( T. \)
Fig. 2.1 (a) Implementation of $P_M$. (b) Implementation of $P_L^t$. 

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The system in Fig. 2.2 is now rearranged to a more manageable form. Consider one row of the matrix $T$ shown in Fig. 2.3(a). Using the standard identities \[5\] Fig. 2.3(a) is rearranged and Fig. 2.3(b) is obtained. The resulting rearrangement of the entire matrix form, shown in Fig. 2.4, is called the commutator form of the multirate system.

\[Fig. 2.3 (a) One row of the matrix  \hspace{1cm} (b) Rearrangement of the row\]
The commutator matrix of the system is defined as $G = \left[ G_0 \ldots G_{L-1} \right]^T$. The responses $\{G_i\}$, $i = 0, \ldots, L-1$ are related to $\{T_{i,k}\}$ by

$$
G_{i,k}(z) = \begin{cases} 
T_{i,0}(z) & k = 0 \\
zT_{i,M-1}(z) & k = 1, \ldots, M - 1
\end{cases}
$$

(2.9)

Here $G_{i,k}$ is the $j$th M-polyphase component of $G_k$. A more explicit expression for the $G_i$ is

$$
G_i(z) = \sum_{k=0}^{M-1} z^k T_{i,k}(z^M)
$$

(2.10)

Note that this is similar to the polyphase decomposition except the components are advanced instead of delayed. The commutator form is so-named because of the periodic alternation (switching) between different filters, $G_i$. That is, the output sample $n$ is computed by the filter $G_{n \mod L}$. The commutator form is useful in many applications.

### 2.2 ERROR CRITERION

Desirable characteristics of a general error criterion for multirate systems are:

(a) incorporation of worst case error from all sources, (b) consistency with the standard Chebyshev error criterion for LTI filter design, and (c) applicability to a general class of
inputs. In order to understand (A), consider the problem of designing a decimate-by-two system. Suppose that an ideal system \( M_{\text{ideal}} = D_2 \mathcal{H}_{\text{ideal}} \) and an approximating system \( M' = D_2 \mathcal{H}' \) are specified (where \( \mathcal{H}_{\text{ideal}} \) and \( \mathcal{H}' \) are convolution operators). The difference \( M_{\text{error}} = M - M_{\text{ideal}} = D_2 \mathcal{H}_{\text{error}} \), where \( \mathcal{H}_{\text{error}} = \mathcal{H}' - \mathcal{H}_{\text{ideal}} \), gives a system producing an error signal. For this system an ideal filter would be low-pass with pass-band \([0, f_p]\), \(0 \leq f_p \leq 1/4\), and stop-band \([f_s, 1/2]\), \(f_s \leq f_s \leq 1/4\). An input of \( a \delta(f - f_0) + b \delta(f - (f_0 + 1/2)) \), \(0 \leq f_0 \leq f_0\) with \( f_0 \) fixed would test the error due to aliasing as well as error due to approximation of the passband. For this case an output of \([a \mathcal{H}_{\text{error}}(f_0) + b \mathcal{H}_{\text{error}}(f_0 + 1/2)] \delta(f - 2f_0)\) is produced. The term \( a \mathcal{H}_{\text{error}}(f_0)\) represents error in the filter response and \( b \mathcal{H}_{\text{error}}(f_0 + 1/2)\) represents aliasing error. A natural criterion that arises is to minimize the worst possible error over a set where \( a \) and \( b \) are constrained. This would lead to an error criterion on the matrix \( A = [\mathcal{H}_{\text{error}}(f_0) \mathcal{H}_{\text{error}}(f_0 + 1/2)] \). The error criterion can be completely specified by considering the error criterion characteristics (B) and (C).

The Chebyshev error criterion arises from dealing with the general class of signals \( \ell^2(Z) \) [26], i.e., the error criterion is the operator norm of the system. Applying this analysis gives \( \max_{f_0} (|\mathcal{H}_{\text{error}}(f_0)|^2 + |\mathcal{H}_{\text{error}}(f_0 + 1/2)|^2)^{1/2} \) as the error criterion, i.e., the maximum two norm of the matrix \( A \) as \( f_0 \) varies. This frequency domain criterion is the standard \( L^\infty \) error criterion [25]. Consistency with the Chebyshev criterion is very useful. This property allows standard specifications for LTI systems to be interpreted in terms of multirate systems. Also, the performance of multirate systems can be compared (when appropriate) directly with the performance of LTI systems.
A model-matching problem, Fig. 2.5 for general multirate systems is now considered. An approximating multirate system, $M$, is to be designed which imitates an ideal multirate system, $M_{\text{ideal}}$. The approximating system is assumed to depend on a parameter $h$ and is indicated by writing $M(h)$.

For a nonzero input $x$, the relative error is

$$\left\| \frac{y}{x} \right\|_2 = \left\| \frac{[M(h) - M_{\text{ideal}}](x)}{\|x\|_2} \right\|_2$$

(2.11)

Where $\|x\|_2 = \sqrt{\sum_{n=-\infty}^{\infty} |x(n)|^2}$. In order to have a good approximating system, maximum relative error, $e(h)$, over all nonzero inputs is considered:

$$e(h) = \sup_{x \neq 0} \frac{\left\| [M(h) - M_{\text{ideal}}](x) \right\|_2}{\|x\|_2} \quad (2.12)$$

The maximum relative error is the operator norm of $M(h) - M_{\text{ideal}}$. The model-matching problem is to minimize $e(h)$ over all $h$, i.e., find

$$\hat{h} = \arg \inf_{h} e(h)$$

(2.13)
In other words, the best operator-norm approximation to $M_{\text{ideal}}$ over the class of operators $\{M(h)\}$ is desired.

For the multirate problem, Fig. 2.5 can be concretely represented as Fig. 2.6.

where, $T$ is the matrix of the approximating system and $T_{\text{ideal}}$ is the matrix of the ideal system, both $T$ and $T_{\text{ideal}}$ are $L \times M$ matrices.

![Fig. 2.6 Model-matching for multirate systems](image)

Substituting in the matrix form of the multirate system, the error function becomes

$$e(h) = \left\| P_{L}^{+} (T - T_{\text{ideal}}) P_{M} \right\|_{2} = \left\| T - T_{\text{ideal}} \right\|_{2}$$  \hspace{1cm} (2.14)

since $P_{M}$ and $P_{L}^{+}$ are unitary. The norm in (2.14) can be rewritten using a standard theorem on convolution operators (and multiplication operators) [17][24][25] as

$$e(h) = \text{ess sup}_{0 \leq f \leq d} \left\| T(f) - T_{\text{ideal}}(f) \right\|_{2}$$  \hspace{1cm} (2.15)

The error criterion shown in (2.14), the operator norm is commonly used in control theory [17][25]. Several excellent general references in this area are [27][28][29][30]. Some interesting signal processing related applications can be seen in [17][31][32][33][34]. Applying some more transformations an equivalent form of the problem can be found. Using (2.9), the matrix of the system, $T$ can be written as
where the $G_i$ are the $z$-domain responses of the filters in the commutator form. The modulation representation [35] (or alias-component form [5]) is given by the following definition.

**Definition 2.4**: The $M$-modulation representation, $X_{mod}$, of $x$ is

$$
X_{mod}(f) = \begin{bmatrix}
X(f) \\
X(f + \frac{1}{M}) \\
\vdots \\
X(f + \frac{M-1}{M})
\end{bmatrix} \tag{2.17}
$$

where $0 \leq f \leq 1/M$. A valuable property of the modulation representation is the relation between it and the polyphase form. Let $x$ be a sequence, $X_{mod}$ the corresponding $M$-modulation form, and $X_M$ the corresponding $M$-polyphase form. Define $W = e^{-2\pi i/M}$. Let $F_M = [W^kn]$, $k = 0, ..., M-1$, $n = 0, ..., M-1$ be the $M \times M$ DFT matrix. Also, let $S(f) = \text{diag}(1, e^{-2\pi i f}, ..., e^{-2\pi i (M-1)f})$. Then

$$
X_{mod}(f) = F_MS(f)X_M(Mf) \tag{2.18}
$$

Let $\pi = [e_0 \ e_{M-1} \ ... \ e_1]$, and $D(z) = \text{diag}(1, z^{-1}, ..., z^{-1})$ where $e_i$ is the $i$th column of $I_M$ (the $M \times M$ identity matrix). Combining (2.16) and (2.18) gives

$$
T(Mf) = \frac{1}{\sqrt{M}} G_{mod}(f) \left(\frac{1}{\sqrt{M}} F_M\right)^\pi S(f)^\pi \Pi D(Mf) \tag{2.19}
$$

where
\begin{equation}
G_{\text{mod}}(f) = \begin{bmatrix}
G_0(f) & G_0(f + \frac{1}{M}) & \ldots & G_0(f + \frac{M-1}{M}) \\
\vdots & \ddots & \ddots & \vdots \\
G_{l-1}(f) & G_{l-1}(f + \frac{1}{M}) & \ldots & G_{l-1}(f + \frac{M-1}{M})
\end{bmatrix}
\tag{2.20}
\end{equation}

\(G_{\text{mod}}\) is modulation commutator matrix and

\begin{equation}
\text{ess sup}_{0 \leq f \leq 1} \left\| T(f) \right\|_2 = \text{ess sup}_{0 \leq f \leq 1} \left\| T(Mf) \right\|_2 \\
= \frac{1}{\sqrt{M}} \text{ess sup}_{0 \leq f \leq 1} \left\| G_{\text{mod}}(f) \left( \frac{1}{\sqrt{M}} F^H_w S(f)^H \right) \right\|_2 \\
= \frac{1}{\sqrt{M}} \text{ess sup}_{0 \leq f \leq 1} \left\| G_{\text{mod}}(f) \right\|_2
\tag{2.21}
\end{equation}

since every term except \(G_{\text{mod}}(f)\) is unitary.

Now the model-matching problem can be expressed in terms of the commutator system. Let

\begin{equation}
G_{\text{error}}(f) = G_{\text{mod}}(f) - G_{\text{ideal}}(f)
\tag{2.22}
\end{equation}

\begin{equation}
N(f) = \frac{1}{\sqrt{M}} \left\| G_{\text{error}}(f) \right\|_2
\tag{2.23}
\end{equation}

and

\begin{equation}
e(g) = \text{ess sup}_{0 \leq f \leq 1} N(f)
\tag{2.24}
\end{equation}

where \(g\) is the impulse response of the matrix LTI system represented by \(G\), and \(G_{\text{ideal}}(f)\) is the modulation commutator matrix of the ideal system \(M_{\text{ideal}}\). Then the model matching problem (2.13) becomes

\begin{equation}
\hat{g} = \arg \inf_{g} e(g)
\tag{2.25}
\end{equation}

Here \(h\) is assumed as a function of \(g\), and hence the function \(e(h(g))\) is abbreviated to \(e(g)\). The function \(N(f)\) (2.23) is the norm of the frequency response of the LTI system in
the center of Fig. 2.6 (see (2.19)). This normed frequency response measures the error for a vector input at a fixed frequency to the center LTI system in Fig. 2.6. Alternatively, the vector input \( x \) to the center system can be transformed to a scalar input \( x = P_M^r x \) to the entire system. This observation gives signals that approach the worst case error for the system-they are inputs to the model-matching system, \( x \), whose transform \( P_M x \) is concentrated at a single frequency, \( f \). It is straightforward to check that this occurs when \( x \) is concentrated at \( f, f + \frac{1}{M}, \ldots, f + \frac{M-1}{M} \). This fact also implies that worst case signals on the output will be concentrated at \( f, f + \frac{1}{L}, \ldots, f + \frac{L-1}{L} \) since LTI systems are multiplication operators in the Fourier domain. Using an extension of earlier discussion, the error measure (2.24) can also be interpreted in terms of tones. Consider the commutator form of the model-matching problem as shown in Fig. 2.7. Use an input of the form \( X(f) = \sum_{m=-M}^{M-1} c_m \delta(f - (f_0 + \frac{m}{M})) \), \( -\frac{1}{2M} \leq f_0 \leq \frac{1}{2M} \) to the system.

The operation \( P_L^t \) has no effect on the norm of the output, so consider the output of the down-sampler, \( y \).

![Fig. 2.7 Commutator form of the model-matching problem](image)

The components of \( y \) are \( y_i = \left[ \sum_{m=0}^{M-1} c_m g_i^{\text{error}} \left( f_0 + \frac{m}{M} \right) \right] \delta(f - Mf_0) \). Consequently,
If $|c|_2 = 1$, then the worst case magnitude of (2.26) is $\|G_{\text{mod}}(f_0)\|_2$, finding the maximum of this magnitude over $f_0$ gives the same error criterion as (2.24).

2.3 CONCLUSIONS

An introduction to multirate systems is given in this chapter and error criterion for multirate systems is also considered. Different structures of multirate systems are proposed. $\ell^2(z)$ error criterion for evaluating multirate systems is proposed. This error criterion has many properties of the standard chebyshev criterion. The chebyshev criterion is considered, as it has become common specification for filter design.

A model-matching scenario is proposed for multirate systems, which helps in evaluating error. The proposed error criterion is applicable to a general class of inputs, and considers worst case error from all sources.