CHAPTER 6

MULTIRATE FILTER

BANK DESIGN
6. MULTIRATE FILTER BANK DESIGN

In this chapter minimization of error of multirate filter banks is considered. The difference between the expected output and actual output of a multirate filter bank is called error. For minimization of error different techniques exist. In this chapter $H_o$ optimization technique is used for minimization of error. A two channel multirate system is considered for simplicity. Different examples are considered to prove the advantage of the proposed design. Different theorems are proved and used. A procedure is developed to design synthesis filters in a multirate filter bank.

6.1 INTRODUCTION

Consider a multirate filter bank as in Fig. 6.1. In this discrete-time system, $H_0(z)$, $H_1(z)$, $F_0(z)$ and $F_1(z)$ are transfer functions of linear time-invariant (LTI) filters, $\downarrow 2$ denotes the down-sampler (sub-sampler, decimator) by a factor of two, and $\uparrow 2$ denotes the up-sampler (expander, interpolator) by the same factor. This multirate filter bank is useful in sub-band coding [5][67][69] [70].

$H_0(z)$ and $H_1(z)$ are frequency selective analysis filters, $H_0(z)$ being low-pass and $H_1(z)$ high-pass. The input signal $x(k)$ is first processed by these analysis filters $H_0(z)$ and $H_1(z)$. The outputs of these filters can be sub-sampled without loss of information. These outputs are then quantized (not shown) and transmitted over a communication channel (also not shown).
The up-samplers and $F_0(z)$ and $F_1(z)$, synthesis filters are required to reconstruct $x(k)$, ideally with at most a time-delay error. The reconstructed signal is denoted $\hat{x}(k)$, so perfect reconstruction is said to be achieved [5] if for some integer $m \geq 0$

\[ \hat{x}(k) = x(k - m) \]

that is, the desired system from $x(k)$ to $\hat{x}(k)$ is LTI with transfer function $T_d(z) = z^{-m}$. The system from $x(k)$ to $\hat{x}(k)$ in general is time varying because of the presence of the down- and up-samplers. It is assumed that the analysis filters have already been designed for good coding of the input and the synthesis filters are to be designed to achieve as close to perfect reconstruction as possible.

Consider the error system in Fig. 6.2 to define the degree of closeness. $e(k)$ is the reconstruction error between the desired output, $x(k-m)$ and the actual output $\hat{x}(k)$. 

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**Fig. 6.1: Multirate filter bank**

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Let $\ell_2$ denote the Hilbert space of square-summable discrete-time signals defined over the time set $\mathbb{Z}$ of all integers. The norm on $\ell_2$ is

$$
\|x\|_2 = \left(\sum_k |x(k)|^2 \right)^{\frac{1}{2}}
$$

Then $J = \sup_{\|x\|_2} \|e\|_2$, the $\ell_2$-induced norm from $x(k)$ to $e(k)$, is the worst-case norm of $e(k)$ when $x(k)$ is normalized.

The quantity $J$ is considered as the performance measure of the multirate filter bank. A small value of $J$ means that the error $e(k)$ is small uniformly over all inputs $x(k)$, $J = 0$ means that perfect reconstruction has been achieved. The problem statement is as follows:
Given causal, stable, IIR analysis filters $H_0(z)$ and $H_1(z)$ and given a tolerable time delay $m$, design causal, stable IIR synthesis filters $F_0(z)$ and $F_1(z)$ to minimize $J$, hence the optimum performance measure is

$$J_{opt} = \inf_{H_0(z), F_1(z)} J = \inf_{H_0(z), F_1(z)} \sup_{[a], [b]} \|e\|_2$$

where the infimum is over all causal, stable, IIR synthesis filters. The idea is to regard $T_0(z)$ as a model ideally to be matched by the multirate filter bank. The design problem is therefore one of $\ell_2$-induced norm model matching. Observe that if $F_0(z) = F_1(z) = 0$, then $J$ equals the induced norm from $x(k)$ to $x(k-m)$, which equals 1. Thus $J_{opt} \leq 1$.

$T_0(z)$ depends on the time delay $m$, so does $J_{opt}$. The function $J_{opt}(m)$ can be shown to be a non-increasing function of $m$. A result of this chapter is that, under mild conditions on the analysis filters, $J_{opt}(m)$ converges to 0 as $m$ tends to $\infty$. In words: If large time delay is tolerable, synthesis filters can be designed to give reconstruction that is close to perfect. Two channel multirate filter banks are considered for simplicity of notation and block diagrams. The extension to the M-channel case is easily possible.

In [5][13][49][66][94][95][96] the constraint of perfect reconstruction is placed on the system, in [97][98] this is relaxed to a weaker constraint (nearly all pass system with small aliasing). [25] deals with $L_2$-induced norm model matching in continuous time. The optimal model-matching approach to multirate filter design can be seen in [45][89].
In [45][89] it is observed that one could equivalently convert the problem by blocking of signals to get an LTI error system. The above references deal with multirate filter design, whereas this chapter deals with multirate filter bank design.

Three examples are considered to illustrate some aspects of the design procedure to be developed. It turns out that the orders of the synthesis filters, \( F_0(z) \) and \( F_1(z) \), increase linearly with \( m \). If \( m \) is large for \( J_{opt}(m) \) to reach a pre-specified small value, high-order synthesis filters result, and in this sense the design problem is considered to be hard. The first example is easy, the second moderate, and the third hard.

**Example 6.1** : \( H_0(z) \) from a Butterworth filter

Digital IIR filters are traditionally obtained by transforming analog ones. Let \( G(s) \) be the third order analog Butterworth filter,

\[
G(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}
\]

let \( H_0(z) \) be the bilinear transformation of \( G(s) \) (without pre-warping). Then \( H_0(z) \) is a third-order low-pass filter. If \( H_1(z) = H_0(-z) \), then \( H_1(e^{j\omega}) = H_0(e^{-j\omega}) \) so \( H_1(z) \) is third-order high-pass.

The magnitude Bode plots of these two filters is shown in Fig. 6.3. The function \( J_{opt}(m) \) is graphed in Fig. 6.4 (solid curve).
Fig. 6.3: Magnitude Bode plots in dB versus $\omega/2\pi$: Example 6.1:
$|H_0|$ (solid) and $|H_1|$ (dot)

Fig. 6.4: $J_{opt}(m)$ versus $m$: Example 6.1 (solid), Example 6.2 (dash), Example 6.3 (dot).
$J_{\text{opt}}(m) = J_{\text{opt}}(m+1)$ when $m$ is even is a property to be observed. The graph decreases relatively rapidly, with, for example, $J_{\text{opt}}(9) = 0.01241$. This means that for the time delay $m = 9$, the $\ell_2$-norm of $e(k)$ is at most 1.241% of the $\ell_2$-norm of $x(k)$, for every $x(k)$. For this value of $m$, optimal synthesis filters are computed and their Bode plots are shown in Fig. 6.5.

![Magnitude Bode plots in dB versus $\omega/2\pi$](image)

**Fig. 6.5: Magnitude Bode plots in dB versus $\omega/2\pi$ : Example 6.1:**
- $|F_0|$ (solid) and $|F_1|$ (dot)

In general the Matlab program gives

$$\text{order } F_0 = \text{order } F_1 \leq 2(m + \text{order } H_0 + \text{order } H_1) - 1 \quad (6.1)$$

In this case the right-hand side equals $2(9 + 3 + 3) - 1 = 29$, the orders can be reduced with negligible error to 11, that is, the $\ell_2$-induced norm is still 0.01241. If the system in Fig. 6.2 is simulated with the input $x(k) = \cos(0.5k) + \cos(2.5k)$, the sum of a low-frequency sinusoid ($\omega = 0.5$) and a high-frequency sinusoid ($\omega = 2.5k$) then the result of the simulation will be as shown in Fig. 6.6.
It can be seen that the reconstruction error is very small relative to the size of the input. This input does not have finite \( \ell_2 \)-norm, since it doesn't converge to zero. It makes sense in this case to look instead at the rms value of the signal:

\[
\text{rms}(x) = \left( \lim_{N \to \infty} \frac{1}{2N+1} \sum_{k=-N}^{N} x(k)^2 \right)^{1/2}
\]

For this particular input, \( \text{rms}(x) = 1 \). The rms value of the error is computed to be 0.008409. It can be proved [99] that the \( \ell_2 \)-induced norm and the rms-induced norm are equal \( \sup \| e \|_2 = \sup_{\text{rms}(x)=1} \text{rms}(e) \), and thus \( \text{rms}(e) \leq J_{\text{opt}} \times \text{rms}(x) \). This inequality evidently holds for the present simulation (0.008409 < 0.01241 \times 1).

**Example 6.2:** \( H_0(z) \) an FIR filter

\( H_0(z) \) is assumed as a FIR linear-phase filter of order 19, it uses a Hamming window. \( H_1(z) \) is assumed as equal to \( H_0(-z) \). The magnitude Bode plots of these two
filters are shown in Fig. 6.7. The function $J_{op}(m)$ is graphed in Fig. 6.4 (dashed curve).

It can be seen that $J_{op}(m) = J_{op}(m + 1)$ when $m$ is even and $\geq 10$, and that $J_{op}(m) = 1$ (i.e., the reconstruction error is 100%) for $m \leq 9$. The graph lies to the right of that in Example 6.1. This means that for the same level of accuracy, a higher time delay is required in the second example. For the time delay $m = 21$, $J_{op} = 0.02969$. For this value of $m$, optimal synthesis filters were computed having order 77, which can be reduced with negligible error to 23.

The Bode plots are shown in Fig. 6.8 and the result of the simulation in Fig. 6.9. It can be seen that the reconstruction error is very small relative to the size of the input. The rms value of the error equals 0.01987.
Fig. 6.8: Magnitude Bode plots in dB versus $\omega / 2 \pi$: Example 6.2: $|F_0|$ (solid) and $|F_1|$ (dot)

Fig. 6.9: Example 6.2: $x(k)$ (solid) and $e(k)$ (dash) versus $k$. 
Example 6.3: \( H_0(z) \) from an elliptic filter

\( G(s) \) is assumed as fifth-order analog elliptic filter. \( H_0(z) \) and \( H_1(z) \) are obtained as in Example 6.1. Their magnitude Bode plots are shown in Fig. 6.10. Note the very narrow transition bands.

![Magnitude Bode plots in dB versus \( \omega/2\pi \)](image)

**Fig. 6.10:** Magnitude Bode plots in dB versus \( \omega/2\pi \)  Example 6.3:

\( |H_0| \) (solid) and \( |H_1| \) (dot)

The graph of \( J_{\text{opt}}(m) \) is shown in Fig. 6.4 (dotted curve). In this case, \( J_{\text{opt}}(m) = J_{\text{opt}}(m+1) \) for \( m \) odd. Evidently \( J_{\text{opt}}(m) \) converges much more slowly than in the preceding examples. To get less than 2% error requires \( m = 41 \), \( J_{\text{opt}}(41) = 0.01977 \). For this value of \( m \), the filter orders are initially 101, but can be reduced to 53 with negligible error. The corresponding plots are in Fig. 6.11 and Fig. 6.12. Again, the reconstruction error is very small, the rms value of the error equals 0.01411.
Fig. 6.11: Magnitude Bode plots in dB versus $\omega/2\pi$: Example 6.3:

$|F_0|$ (solid) and $|F_1|$ (dot)

Fig. 6.12: Example 6.3: $x(k)$ (solid) and $e(k)$ (dash) versus $k$
The examples illustrate that synthesis filters can be designed to get arbitrarily close to perfect reconstruction. The tradeoff in accuracy is the order of the synthesis filters. Also, the quality of the analysis filters (flatness on the passband and stopband, sharpness of cutoff) affects the order of the synthesis filters.

6.2 FILTER BANK DESIGN CONVERSION TO $H_\infty$ OPTIMIZATION

Conversion of synthesis filter bank design problem to one of $H_\infty$ optimization is considered in this section. The Hardy space $H_\infty$ consists of all complex-valued functions of $z$ that are analytic and bounded outside the unit disc, that is for $|z| > 1$. Thus $H_\infty$ is the space of transfer functions of causal LTI systems that are stable (bounded-input, bounded-output stable on $\ell_2$). The norm of a function $G(z)$ in $H_\infty$ is its suprema magnitude on the unit circle: $\|G\|_\infty = \sup_{0 \leq \omega \leq 2\pi} |G(e^{j\omega})|$, thus $\|G\|_\infty$ equals the peak magnitude on the Bode plot. If $G(z)$ is the transfer function of a stable, causal LTI system with input $x(k)$ and output $y(k)$, then the $\ell_2$-induced norm equals the $H_\infty$-norm of $G(z)$, that is,

$$\sup_{\|x\|_2} \|y\|_2 = \|G\|_\infty$$

(6.2)

This definition extends to matrix-valued functions, in which case the magnitude is replaced by the maximum singular value:

$$\|G\|_\infty = \sup_{\omega} \sigma_{\max}[G(e^{j\omega})]$$
To extend (6.2), let $\ell_2^n$ denote the direct sum of $n$ copies of $\ell_2$, thus elements of $\ell_2^n$ are square-summable vector-valued signals. The norm of a signal $x(k)$ in $\ell_2^n$ is defined to be

$$\|x\|_2 = \left(\sum_k x(k)^* x(k)\right)^{1/2}$$

where $^*$ denotes complex-conjugate transpose.

If $G(z)$ is the transfer function of a stable, causal LTI system with input $x(k)$ of dimension $m$ and output $y(k)$ of dimension $p$, so that $G(z)$ is $p \times m$, then the induced norm from $\ell_2^n$ to $\ell_2^p$ equals the $H_\infty$-norm of $G(z)$, that is, (6.2) holds. As with related multirate problems [5], Fig. 6.2 is simplified by means of polyphase representations. First, bring in the $2 \times 2$ matrix $E(z)$ whose elements are the type-1 polyphase components of the analysis filters:

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = E(z) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

Next, the $2 \times 2$ matrix $R(z)$ whose elements are the type-2 polyphase components of the synthesis filters:

$$\begin{bmatrix} F_0(z) & F_1(z) \end{bmatrix} = \begin{bmatrix} z^{-1} & 1 \end{bmatrix} R(z^2)$$

(6.3)

Use of these in Fig. 6.2 together with the standard identities leads to the input-output equivalent system in Fig. 6.13. Now block the input $x$ and output $e$ in Fig. 6.13 as shown in Fig. 6.14. This latter system has 2-dimensional input and output:

$$y = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}, w = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$$
Fig. 6.13: Equivalent system in terms of poly phase components

Fig. 6.14: Input and output blocking
Blocking preserves \( \ell_2 \) norm i.e., the norm of \( x \) in \( \ell_2 \) equals the norm of \( v \) in \( \ell_2^2 \) and the norm of \( e \) in \( \ell_2 \) equals the norm of \( w \) in \( \ell_2^2 \). Thus the \( \ell_2 \)-induced norm of the system in Fig. 6.13 equals the \( \ell_2^2 \)-induced norm of the system in Fig. 6.14.

The reason of using blocking is that the system in Fig. 6.14 from \( v \) to \( w \) is no longer time-varying- it is LTI. Indeed, Fig. 6.14 is equivalent to the final system, shown in Fig. 6.15, where each of the three matrices \( E(z) \), \( R(z) \), and \( W(z) \) is \( 2 \times 2 \).

The matrix \( W(z) \) is the transfer matrix from input to output in Fig. 6.16. It has a simple form as a consequence of the fact that \( T_d(z) = z^m \). Actually, \( W(z) \) has two possible forms, depending on whether \( m \) is even or odd:

\[
W(z) = \begin{cases} 
  z^{-d} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \text{if } m = 2d + 1, \\
  z^{-d} \begin{bmatrix} 0 & z \\ 1 & 0 \end{bmatrix}, & \text{if } m = 2d
\end{cases}
\]

From the design problem it can be concluded that minimizing the \( \ell_2 \)-induced norm from \( x \) to \( e \) in Fig. 6.2, is equivalent to minimizing the \( \ell_2^2 \)-induced norm from \( v \) to \( w \) in Fig. 6.15. But the latter induced norm equals \( \|W-RE\|_\infty \), the \( H_\infty \)-norm of the \( 2 \times 2 \) transfer matrix \( W(z) - R(z)E(z) \).
The preceding can be summarized as follows:

**Theorem 6.1**: Assume $T_d(z) = z^m$. Let $E(z)$ and $R(z)$ be the $2 \times 2$ transfer matrices corresponding to polyphase representations of, respectively, the analysis and synthesis filters. Define
Then

\[
W(z) = \begin{cases} 
  z^{-d} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \text{if } m = 2d+1, \\
  z^{-d} \begin{bmatrix} 0 & z \\ 1 & 0 \end{bmatrix}, & \text{if } m = 2d 
\end{cases}
\]

Then

\[
J_{\text{opt}} = \inf_{R(z)} \sup_{\|e\|_2} \|W - RE\|_{\infty}
\]

The latter optimization is over all matrices \( R(z) \) that are analytic and bounded outside the unit disc.

6.3 STATE SPACE REPRESENTATION

State-space representation of transfer functions is useful for processing programs of \( H_\infty \) optimization using Matlab. In this section state-space formulas relevant for the design program are presented, which is a procedure with Input: an integer \( m > 0 \) and state models of \( H_0(z), H_1(z) \) and Output: \( J_{\text{opt}}(m) \) and state models of \( F_0(z), F_1(z) \). To simplify the presentation packed notation is used. Thus

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

stands for the transfer matrix \( D + C(zI - A)^{-1}B \). State representations for the analysis filters is

\[
H_0(z) = \begin{bmatrix} A_{H_0} & B_{H_0} \\ C_{H_0} & D_{H_0} \end{bmatrix}, H_1(z) = \begin{bmatrix} A_{H_1} & B_{H_1} \\ C_{H_1} & D_{H_1} \end{bmatrix},
\]

and for \( W(z) \),
\[ W(z) = \begin{bmatrix} A_w & B_w \\ C_w & D_w \end{bmatrix} \]

The transfer matrix associated with their polyphase representation is then

\[ E(z) = \begin{bmatrix} A_h & 0 & A_h B_h & A_h^2 B_h \\ 0 & A_h^2 & A_h B_h & A_h^2 B_h \\ C_h & 0 & D_h & C_h B_h \\ 0 & C_h & D_h & C_h B_h \end{bmatrix} \]

Instead of Fig. 6.15, Matlab requires a system of the form in Fig. 6.17.

![System diagram](image)

**Fig. 6.17: System for Matlab**

To make Fig. 6.15 and Fig. 6.17 equivalent, take

\[ G(z) = \begin{bmatrix} W(z) & -I \\ E(z) & 0 \end{bmatrix} \]

This has the realization

\[ G(z) = \begin{bmatrix} A_G & B_G \\ C_G & D_G \end{bmatrix} = \begin{bmatrix} A_w & 0 & B_w & 0 \\ 0 & A_e & B_e & 0 \\ C_w & 0 & D_w & -I \\ 0 & C_e & D_e & 0 \end{bmatrix} \]
The Matlab function hinfsyn accepts as input a realization for \( G(z) \) and outputs a realization of \( R(z) \) with the same order as \( G(z) \) that minimizes the \( H_\infty \) norm from \( v \) to \( w \) in Fig. 6.17. Let the realization of \( R(z) \) be

\[
R(z) = \begin{bmatrix} A_R & B_R \\ C_R & D_R \end{bmatrix}
\]

Then

\[
R(z^2) = \begin{bmatrix} A_{R2} & B_{R2} \\ C_{R2} & D_{R2} \end{bmatrix} = \begin{bmatrix} 0 & I \\ A_R & 0 \\ C_R & 0 \\ 0 & D_R \end{bmatrix}
\]

Finally, from (6.3)

\[
\begin{bmatrix} F_0(z) & F_1(z) \end{bmatrix} = \begin{bmatrix} A_{R2} & 0 \\ 1 & 0 \\ C_{R2} & 1 \\ 0 & D_{R2} \end{bmatrix}
\]

For a dimension count, let \( n_H \) denote the size of the \( A \)-matrix in a realization of a transfer matrix \( H(z) \).

Adding up the dimensions in the above formulas, it can be stated that

\[
n_{F_0} = n_{F_1} = 2(n_w + n_{H_0} + n_{H_1}) + 1
\]

since \( n_w = m - 1 \), this yields

\[
n_{F_0} = n_{F_1} = 2(m + n_{H_0} + n_{H_1}) - 1
\]

It may be possible to get lower order synthesis filters if minimal realizations are used at each appropriate stage that is discard uncontrollable or unobservable states.
6.4 PERFORMANCE & LIMITATIONS

In this section it is proved that, under a mild condition on the analysis filters, arbitrarily good reconstruction is possible if a sufficiently large time delay can be tolerated. The mild condition is on the $2 \times 2$ transfer matrix $E(z)$.

**Lemma 6.1:** A rational transfer matrix $G(z)$ is said to be stable if it is analytic in $|z| \geq 1$.

Let $G(z)$ be a rational matrix with no poles on the unit circle. Let $Y(z) = z^d G(z)$ and $K(d) = \inf_{X(z)} \|Y - X\|_\infty$, where the infimum is over all $X(z)$s that are analytic and bounded outside the unit disc that is $K(d)$ equals the distance from $Y(z)$ to $H_\infty$. Then $\lim_{d \to \infty} K(d) = 0$.

**Proof:** By Nehari's theorem [91] [100] $K(d)$ equals the norm of the Hankel operator, $\Gamma_Y$, associated with $Y(z)$. The series expansion of $G(z)$ is used to write down $\Gamma_Y$ explicitly

$$G(z) = ... + G_2 z^2 + G_1 z + G_0 + G_1 z^{-1} + G_2 z^{-2} + .......$$

Then the Hankel operator associated with $G(z)$ is

$$\Gamma_G = \begin{bmatrix}
G_{-1} & G_{-2} & G_{-3} & ... \\
G_{-2} & G_{-3} & G_{-4} & ... \\
G_{-3} & G_{-4} & G_{-5} & ... \\
... & ... & ... & ...
\end{bmatrix}$$

and likewise

$$\Gamma_Y = \begin{bmatrix}
G_{-(d+1)} & G_{-(d+2)} & G_{-(d+3)} & ... \\
G_{-(d+2)} & G_{-(d+3)} & G_{-(d+4)} & ... \\
G_{-(d+3)} & G_{-(d+4)} & G_{-(d+5)} & ... \\
... & ... & ... & ...
\end{bmatrix}$$
Being rational, the series \( \ldots + G_2 z^2 + G_1 z \) (i.e., the anti-causal part of \( G(z) \)) has a state realization: \( \ldots + G_2 z^2 + G_1 z = z \left(C(I-zA)^{-1}B, \right. \text{that is,} \left. G_{-(k+1)} = CA^k B \right. \). All the eigenvalues of \( A \) lie strictly inside the unit disc, and so \( A^k \to 0 \) as \( k \to \infty \). Hence

\[
\Gamma_r = \begin{bmatrix}
CA^d B & CA^{d+1} B & CA^{d+2} B & \ldots \\
CA^{d+1} B & CA^{d+2} B & CA^{d+3} B & \ldots \\
CA^{d+2} B & CA^{d+3} B & CA^{d+4} B & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

\[= \begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots
\end{bmatrix} A^d \begin{bmatrix}
B \\
AB \\
A^2 B \\
\vdots
\end{bmatrix} \quad (6.4)
\]

The (observability and controllability) operators

\[
\Xi_0 = \begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots
\end{bmatrix}, \quad \Xi_c = \begin{bmatrix}
B & AB & A^2 B & \ldots
\end{bmatrix}
\]

are both bounded. So from (6.4) \( \|\Gamma_r\| \leq \|\Xi_0\| \cdot \|A^d\| \cdot \|\Xi_c\| \). The right-hand side converges to 0 as \( d \to \infty \).

**Theorem 6.2**: Assume \( E(e^{i\omega}) \) is nonsingular for all \( 0 \leq \omega \leq 2\pi \). Then

\[
\lim_{m \to \infty} J_{\text{opt}}(m) = 0.
\]

**Proof**: Every stable transfer function can be factored into the product of an all pass function (unit magnitude at all frequencies, also called "lossless" function) and a minimum phase function (no zeros in \(|z| > 1\)). This fact is also true for matrix-valued transfer functions.
The first step in the proof of the theorem is to factor \( E(z) \) into an all pass part, \( E_{ap}(z) \), and a minimum phase part, \( E_{mp}(z) \). The procedure is as follows: The minimum phase factor arises from spectral factorization: \( E(z)E(z^{-1})^T = E_{mp}(z)E_{mp}(z^{-1})^T \). By the assumption on \( E(z) \) and by construction, \( E_{mp}(z) \) and \( E_{mp}(z)^{-1} \) are stable. If \( E_{ap}(z) = E_{mp}(z)^{-1}E(z) \) then \( E_{ap}(z)E_{ap}(z^{-1})^T = 1 \), so \( E_{ap}(z) \) is unitary on the unit circle.

Hence

\[
J_{ap}(m) = \inf_{R(z)} \|W - RE\|_\infty
= \inf_{R(z)} \|W - RE_{mp}E_{ap}\|_\infty
= \inf_{R(z)} \|(WE_{ap}^{-1} - RE_{mp})E_{ap}\|_\infty
\]

Since \( E_{ap}(e^{j\omega}) \) is unitary, \( \|(WE_{ap}^{-1} - RE_{mp})E_{ap}\|_\infty = \|WE_{ap}^{-1} - RE_{mp}\|_\infty \), and so

\[
J_{ops}(m) = \inf_{R(z)} \|WE_{ap}^{-1} - RE_{mp}\|_\infty \cdot E_{mp}(z) \text{ and } E_{mp}(z)^{-1} \text{ are stable. Define } X(z) = R(z)E_{mp}(z) \text{ and equally minimize over } X(z) \text{ in } H_\infty \text{ then } J_{ops}(m) = \inf_{X(z)} \|WE_{ap}^{-1} - X\|_\infty \text{. The result now follows from Lemma 6.1.}
\]

Lemma 6.2: For a fixed \( \omega \) the matrix \( E(e^{j\omega}) \) is non-singular if and only if

\[ H_0(e^{j\omega/2})H_1(-e^{j\omega/2}) \neq H_0(-e^{j\omega/2})H_1(e^{j\omega/2}) \]

Proof: By definition of \( E(z) \),

\[
\begin{bmatrix}
H_0(z) \\
H_1(z)
\end{bmatrix} = E(z^3) \begin{bmatrix}
1 \\
z^{-1}
\end{bmatrix}
\]

Replacing \( z \) by \( -z \),

\[
\begin{bmatrix}
H_0(-z) \\
H_1(-z)
\end{bmatrix} = E(z^3) \begin{bmatrix}
1 \\
-z^{-1}
\end{bmatrix}
\]

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Combine the preceding two equations

\[
\begin{bmatrix}
H_0(z) & H_0(-z) \\
H_1(z) & H_1(-z)
\end{bmatrix} = E(z^2)\begin{bmatrix}
1 & z^{-1} \\
1 & -z^{-1}
\end{bmatrix}
\]

Since the right matrix on the right is nonsingular, \(E(z^2)\) is nonsingular if and only if the matrix on the left is nonsingular, i.e., \(\det E(z^2) \neq 0 - H_0(z)H_1(-z) \neq H_0(-z)H_1(z)\), from which the result follows. The condition \(H_0(e^{i\omega/2})H_1(-e^{i\omega/2}) \neq H_0(-e^{i\omega/2})H_1(e^{i\omega/2})\) for \(\omega \leq \pi\), or equivalently

\[
H_0(e^{i\omega})H_1(-e^{i\omega}) \neq H_0(-e^{i\omega})H_1(e^{i\omega}) \quad (6.5)
\]

for \(\omega \leq \pi/2\), is very mild. For example, if \(H_0(z)\) is low-pass, \(H_1(z)\) is high-pass, and both have cutoff frequency \(\pi/2\), then \(|H_0(e^{i\omega})| > |H_0(-e^{i\omega})|\) and \(|H_1(-e^{i\omega})| > |H_1(e^{i\omega})|\) for \(\omega < \pi/2\) and hence (6.5) holds for \(\omega < \pi/2\). In the special case that \(H_1(z) = H_0(-z)\), as in the three examples, the condition reduces to \(H_0(e^{i\omega/2}) \neq \pm H_1(e^{i\omega/2})\) for \(\omega \leq \pi\), or equivalently \(H_0(e^{i\omega}) \neq \pm H_1(e^{i\omega})\) for \(\omega \leq \pi/2\). If \(H_0(z)\) is low-pass with cutoff frequency \(\pi/2\), then \(|H_0(e^{i\omega})| > |H_1(e^{i\omega})|\) for \(\omega = \pi/2\), \(H_0(e^{i\omega}) \neq \pm H_1(e^{i\omega})\) because of phase lag.

The limitation of the design method is that quite high order synthesis filters may result. In this case order of the synthesis filters can be reduced or the analysis filters can be redesigned so as not to be so demanding.

There are many ways to reduce the order of a filter, but only a few give a guaranteed error bound on the frequency response, and hence on the \(\ell_2\)-induced norm.
One such way to do order reduction is to use the Matlab function sysbal.

To state the error bound, suppose $G(z)$ is a stable, causal filter of McMillan degree $n$. The Hankel operator associated with $G(z)$ has $n$ nonzero singular values, denoted in decreasing order by $\sigma_1, \ldots, \sigma_n$. Suppose $G_r(z)$ is the approximant of McMillan degree $m < n$. Then $\|G - G_r\|_\infty \leq 2(\sigma_{m+1} + \ldots + \sigma_n)$. The left-hand side is the maximum frequency-response error, so one should choose $m$ so that the right-hand sum is sufficiently small. The filters $F_0(z)$ and $F_1(z)$ in the three examples in the introduction were reduced in this way. For example, Fig. 6.18 shows plots of $\sigma_k$ versus $k$ for the filter $F_0(z)$. Note that for Example 6.1, $\sigma_{12}$ is much smaller than preceding ones, leading to a reduced filter of order 11, similarly for the other examples.

![Fig. 6.18: Hankel singular values $\sigma_k$ versus $k$ for $F_0(z)$: Example 6.1 (solid), Example 6.2 (dash), Example 6.3 (dot).](image)
6.5 CONCLUSIONS

In this chapter a procedure is proposed for design of synthesis filters in a multirate filter bank. Given causal, stable analysis filter and given a tolerable time delay $m$ a causal stable synthesis filter is designed which minimizes error using $H_m$ optimization technique. Two channel multirate filter bank is considered but it can be extended easily to M-Channel case.

The advantage of proposed method is that both the analysis and synthesis filters need not be designed i.e., for an existing analysis filter (which is designed depending on coding of input signal) a synthesis filter can be designed for good reconstruction. In this chapter it is proved that under mild conditions on the analysis filters, error converges to 0 as $m$ tends to $\infty$ i.e., if a large time delay (between input and output) is tolerable then synthesis filters can be designed to give reconstruction that is close to perfect.