Chapter 2

Structure Graceful Index of a Graph

2.1 Introduction

S.W. Golomb\(^8\) and G.J. Simmons\(^9\) proved that the complete graph \(K_n\) is graceful if and only if \(n \leq 4\). Beutner and Harborth\(^{16}\) showed that \(K_n - e\) (\(K_n\) with an edge deleted) is graceful only if \(n \leq 5\), and \(K_n - 2e\) (\(K_n\) with two edges deleted) is graceful only if \(n \leq 6\) and any \(K_n - 3e\) is graceful only if \(n \leq 6\). Both Rosa\(^{10}\) and Golomb\(^8\) proved that the complete bipartite graphs \(K_{m,n}\) are graceful. It has been conjectured that \(K_{m,n}\) is the largest graceful subgraph of \(K_n\).

This motivated a study on some sort of gracefulness of \(K_n\). The concept of graph structure\(^3\) and structure graceful index\(^{17}\) seem to suit this situation. Before entering into this discussion a short analysis has been done on weaker versions of gracefulness. It is proved that the SGI of the n-cone \((C_m + \overline{K_n})\) is at most 2. A new family of graphs \(G_n\) has been identified.
2.2 Weaker Versions of Gracefulness

Definition 2.2.1 A graph $G$ is said to be nearly graceful if there exists an injective function $f : V(G) \rightarrow \{0, 1, 2, ..., q+1\}$, while the edge labels induced by the absolute value $|f(u) - f(v)|$ are $\{1, 2, ..., q - 1, q\}$ or $\{1, 2, ..., q - 1, q + 1\}$.

Theorem 2.2.2 Every nearly graceful graph $G$ gives rise to a graceful graph $H$ which is got by either adding an edge or by a vertex together with an edge to $G$.

Proof: Let $G$ be a nearly graceful graph with $p$ vertices and $q$ edges. Let $H$ be a graceful graph which is got by either adding an edge or by a vertex together with an edge to $G$.

Claim: $H$ is graceful.

Let $V(G) = \{v_1, v_2, ..., v_p\}$ and $E(G) = \{e_1, e_2, ..., e_q\}$.

Let $f : V(G) \rightarrow \{0, 1, ..., q + 1\}$ be a nearly graceful labeling of $G$.

Case (i) : $f(E(G)) = \{1, 2, ..., q - 1, q\}$.

$q \in f(E(G))$ can be realized as an edge label as $|q - 0|$ or $|(q + 1) - 1|$.

Subcase (i) : Suppose $f(e) = q$, $e = v_1v_2$ with $f(v_1) = q$, $f(v_2) = 0$

Note that $f(v_i) = q + 1 \Rightarrow v_iv_2 \notin E(G)$.

(since $f(v_i) = q + 1$ and $v_iv_2 \in E(G) \Rightarrow f(v_iv_2) = q + 1 \in f(E(G))$).

If $q + 1 \in f(V(G))$, then let $H$ be the graph with $V(H) = V(G)$ and $E(H) = E(G) \cup \{v_iv_2\}$. Then $f$ is a graceful labeling for $H$.

If $q + 1 \notin f(V(G))$, then let $V(H) = V(G) \cup \{v_{p+1}\}$ and $E(H) = E(G) \cup \{v_{p+1}v_2\}$.

Define $f(v_{p+1}) = q + 1$. Then $f$ is a graceful labeling for $H$. 

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Subcase (ii) : Suppose \( f(e) = q, e = v_1v_2 \) with \( f(v_1) = q + 1, f(v_2) = 1 \)

Note that \( f(v_i) = 0 \Rightarrow v_i v_1 \notin E(G) \).

(since \( f(v_i) = 0 \) and \( v_i v_1 \in E(G) \Rightarrow f(v_i v_1) = q + 1 \in f(E(G)) \).

If \( 0 \in f(V(G)) \), then let \( H \) be the graph with \( V(H) = V(G) \) and \( E(H) = E(G) \cup \{v_i v_1\} \). Then \( f \) is a graceful labeling for \( H \).

If \( 0 \notin f(V(G)) \), then let \( V(H) = V(G) \cup \{v_{p+1}\} \) and \( E(H) = E(G) \cup \{v_{p+1} v_1\} \).

Define \( f(v_{p+1}) = 0 \). Then \( f \) is a graceful labeling for \( H \).

Case (ii) : \( f(E(G)) = \{1, 2, \ldots, q - 1, q + 1\} \)

\( q + 1 \in f(E(G)) \Rightarrow q + 1, 0 \in f(V(G)) \).

Let \( f(v_1) = q + 1, f(v_2) = 0 \).

Note that \( f(v_i) = q \Rightarrow v_i v_2 \notin E(G) \).

(since \( f(v_i) = q \) and \( v_i v_2 \in E(G) \Rightarrow f(v_i v_2) = q \in f(E(G)) \).

If \( q \in f(V(G)) \), then let \( H \) be the graph with \( V(H) = V(G) \) and \( E(H) = E(G) \cup \{v_i v_2\} \). Then \( f \) is a graceful labeling for \( H \).

If \( q \notin f(V(G)) \), then let \( V(H) = V(G) \cup \{v_{p+1}\} \) and \( E(H) = E(G) \cup \{v_{p+1} v_2\} \).

Define \( f(v_{p+1}) = q \). Then \( f \) is a graceful labeling for \( H \).

**Definition 2.2.3** A graph \( G \) is said to be almost graceful if there exists an injective function \( f : V(G) \rightarrow \{0, 1, 2, \ldots, q - 1, q + 1\} \), while the edge labels induced by the absolute value \( |f(u) - f(v)| \) are \( \{1, 2, \ldots, q - 1, q + 1\} \).

**Theorem 2.2.4** Every almost graceful graph \( G \) gives rise to a graceful graph \( H \) which is got by either adding an edge or by a vertex together with an edge to \( G \).
Proof: Let $G$ be a almost graceful graph with $p$ vertices and $q$ edges. Let $H$ be a graceful graph which is got by either adding an edge or by a vertex together with an edge to $G$.

Let $V(G) = \{v_1, v_2, ..., v_p\}$ and $E(G) = \{e_1, e_2, ..., e_q\}$.

Let $f : V(G) \to \{0, 1, ..., q - 1, q + 1\}$ be an almost graceful labeling of $G$ and $f(E(G)) = \{1, 2, ..., q - 1, q + 1\}$.

Claim: $H$ is graceful

$q + 1 \in f(E(G)) \Rightarrow q + 1, 0 \in f(V(G))$.

Let $f(v_1) = 0, f(v_2) = q + 1$.

Note that $f(v_i) = 1 \Rightarrow v_i v_2 \notin E(G)$.

(since $f(v_i) = 1$ and $v_i v_2 \in E(G) \Rightarrow f(v_i v_2) = q \in f(E(G))$).

If $1 \in f(V(G))$, then let $H$ be the graph with $V(H) = V(G)$ and $E(H) = E(G) \cup \{v_i v_2\}$. Then $f$ is a graceful labeling for $H$.

If $1 \notin f(V(G))$, then let $V(H) = V(G) \cup \{v_{p+1}\}$ and $E(H) = E(G) \cup \{v_{p+1} v_2\}$.

Define $f(v_{p+1}) = 1$. Then $f$ is a graceful labeling for $H$.

Definition 2.2.5 A graph $G$ is said to be Pseudo graceful if there exists an injective function $f : V(G) \to \{0, 1, 2, ..., q - 1, q + 1\}$, while the edge labels induced by the absolute value $|f(u) - f(v)|$ are $\{1, 2, ..., q - 1, q\}$.

Theorem 2.2.6 Every pseudo graceful graph $G$ gives rise to a graceful graph $H$ which is got by adding an edge or by a vertex together with an edge to $G$.

Proof: Let $G$ be a pseudo graceful graph with $p$ vertices and $q$ edges. Let $H$ be a graceful graph which is got by either adding an edge or by a vertex together with an edge to $G$. 
Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $E(G) = \{e_1, e_2, \ldots, e_q\}$.

Let $f : V(G) \to \{0, 1, \ldots, q - 1, q + 1\}$ be a pseudo graceful labeling of $G$ and $f(E(G)) = \{1, 2, \ldots, q\}$.

Claim: $H$ is graceful

$q \in f(E(G)) \Rightarrow q + 1, 1 \in f(V(G))$.

Let $f(v_1) = 1, f(v_2) = q + 1$.

Note that $f(v_i) = 0 \Rightarrow v_i v_2 \notin E(G)$.

(since $f(v_i) = 0$ and $v_i v_2 \in E(G) \Rightarrow f(v_i v_2) = q + 1 \in f(E(G))$).

If $0 \in f(V(G))$, then let $H$ be the graph with $V(H) = V(G)$ and $E(H) = E(G) \cup \{v_i v_2\}$. Then $f$ is a graceful labeling for $H$.

If $0 \notin f(V(G))$, then let $V(H) = V(G) \cup \{v_{p+1}\}$ and $E(H) = E(G) \cup \{v_{p+1} v_2\}$.

Define $f(v_{p+1}) = 0$. Then $f$ is a graceful labeling for $H$.

Note 2.2.7 $K_n$ is nearly graceful for $n \leq 5$.

The nearly graceful labeling of $K_5$ is as follows:

![Figure 2.1: Nearly gracefulness of $k_5$](image-url)
Theorem 2.2.8  $K_n$ is not nearly graceful for $n \geq 6$.

Proof: For $n \geq 6$, the graph $K_n$ has $q \geq 15$ edges.

If $K_n$ were nearly graceful, we could assign a subset of the numbers $\{0, 1, 2, ..., q + 1\}$ to the nodes in such a way that the edges receive each of the numbers $\{1, 2, ..., q\}$ or $\{1, 2, ..., q - 1, q + 1\}$.

Case(i):

To get the edge label $\{1, 2, ..., q\}$, the admissible values for the nodes are analysed in the following table. In order to have an edge number $q$ for $K_n$ both 0 and $q$ must be node numbers or 1 and $q + 1$ must be node numbers.

Subcase(i): Suppose 0 and $q$ are node numbers. Then we have the following possibilities:

<table>
<thead>
<tr>
<th>To get the edge label</th>
<th>Admissible value for nodes</th>
<th>Edge labels</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$</td>
<td>0, $q$</td>
<td>$q$</td>
</tr>
<tr>
<td>$q-1$</td>
<td>0, $q$, $q-1$</td>
<td>$q$, $q-1$, 1</td>
</tr>
<tr>
<td></td>
<td>0, $q$, 1</td>
<td>$q$, $q-1$, 1</td>
</tr>
<tr>
<td>$q-2$</td>
<td>0, $q$, $q-1$, $q-2$</td>
<td>$q$, $q-1$, 2, $q-2$, 1</td>
</tr>
<tr>
<td></td>
<td>0, $q$, $q-1$, 2</td>
<td>$q$, $q-1$, 2, $q-3$, 2, 1</td>
</tr>
<tr>
<td></td>
<td>0, $q$, $q-1$, 1</td>
<td>$q$, $q-1$, $q-2$, 1, 1</td>
</tr>
<tr>
<td>$q-4$</td>
<td>0, $q$, $q-1$, 2, $q-4$</td>
<td>$q$, $q-1$, $q-2$, $q-3$, 2, 1</td>
</tr>
</tbody>
</table>

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With nodes numbered 0, q, q−1, 2, q−4 we have edge numbers 1, 2, 3, 4, q−6, q−4, q−3, q−2, q−1, q. Now there is no way to get the edge number q−5(q ≥ 10), because each of the ways to obtain q−5 as a difference of two numbers contain at least one impossible node number. So $K_n$ is not nearly graceful for $n ≥ 6$.

Subcase(ii): Suppose 1 and q + 1 are node numbers. Then we have the following possibilities:

<table>
<thead>
<tr>
<th>To get the edge label</th>
<th>Admissible value for nodes</th>
<th>Edge labels</th>
</tr>
</thead>
<tbody>
<tr>
<td>q</td>
<td>1, q+1</td>
<td>q</td>
</tr>
<tr>
<td>q−1</td>
<td>1, q+1, q</td>
<td>q, q−1, 1</td>
</tr>
<tr>
<td></td>
<td>1, q+1, 2</td>
<td>q, q−1, 1</td>
</tr>
<tr>
<td>q−2</td>
<td>1, q+1, q, q−1</td>
<td>q, q−1, q−2, 2, 1, 1</td>
</tr>
<tr>
<td></td>
<td>impossible</td>
<td>q, q−1, q−2, q−3, 2, 1</td>
</tr>
<tr>
<td></td>
<td>1, q+1, q, 3</td>
<td>q, q−1, q−2, 1, 1</td>
</tr>
<tr>
<td></td>
<td>impossible</td>
<td>q, q−1, q−2, 2, 1, 1</td>
</tr>
<tr>
<td></td>
<td>1, q+1, 2, q</td>
<td>q, q−1, q−2, 2, 1, 1</td>
</tr>
<tr>
<td></td>
<td>impossible</td>
<td>q, q−1, q−2, 2, 1, 1</td>
</tr>
<tr>
<td></td>
<td>1, q+1, 2, q−1</td>
<td>q, q−1, q−2, q−3, 2, 1</td>
</tr>
<tr>
<td></td>
<td>impossible</td>
<td>q, q−1, q−2, q−3, 2, 1</td>
</tr>
</tbody>
</table>
To get the edge label | Admissible value for nodes | Edge labels |
---|---|---|
$1, q+1, q, 3, q-3$ | $q,q-1,q-2,q-3,q-4,q-6,$ | $4,3,2,1$ |
$1,q+1,q,3,5$ | $q,q-1,q-2,q-3,q-4,q-5,$ | $4,2,2,1$ |
impossible | | |
$1,q+1,q,3,4$ | $q,q-1,q-2,q-3,q-3,q-4,$ | $3,2,1,1$ |
impossible | | |
$q-4$ $1,q+1,q,3,q-1$ | $q,q-1,q-2,q-3,q-4,q-4,$ | $2,2,1,1$ |
impossible | | |
$1,q+1,q-1,2,q-3$ | $q,q-1,q-2,q-3,q-4,q-5,$ | $4,2,2,1$ |
impossible | | |
$1,q+1,q-1,2,5$ | $q$ to $q-4,q-6,q-4,3,2,1$ | | |
$1,q+1,q-1,2,3$ | $q,q-1,q-2,q-3,q-4,q-4,$ | $2,2,1,1$ |
impossible | | |
$1,q+1,q-1,2,q-2$ | $q,q-1,q-2,q-3,q-3,q-4,$ | $3,2,1,1$ |
impossible | | |

With nodes numbered $1, q+1, q, 3, q-3$ or $1, q+1, q-1, 2, 5$ we have edge numbers $1, 2, 3, 4, q-6, q-4, q-3, q-2, q-1, q$. Now there is no way to get the edge number $q-5 (q \geq 10)$, because each of the ways to obtain $q - 5$ as a difference
of two numbers contain at least one impossible node number. So $K_n$ is not nearly graceful for $n \geq 6$ in this case also.

Case(ii):

To get edge label $\{1, 2, 3, \ldots, q - 1, q + 1\}$, the admissible values for the nodes are analysed in the following table.

<table>
<thead>
<tr>
<th>To get the edge label</th>
<th>Admissible value for nodes</th>
<th>Edge labels</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q+1$</td>
<td>$0, q+1$</td>
<td>$q+1$</td>
</tr>
<tr>
<td>$q-1$</td>
<td>$0, q+1, q-1$</td>
<td>$q+1, q-1, 2$</td>
</tr>
<tr>
<td></td>
<td>$0, q+1, 2$</td>
<td>$q+1, q-1, 2$</td>
</tr>
<tr>
<td>$q-2$</td>
<td>$0, q+1, q-1, q-2$</td>
<td>$q+1, q-1, q-2, 3, 2, 1$</td>
</tr>
<tr>
<td></td>
<td>$0, q+1, q-1, 3$</td>
<td>$q+1, q-1, q-2, q-4, 3, 2$</td>
</tr>
<tr>
<td></td>
<td>$0, q+1, q-1, 1$</td>
<td>$q+1, q, q-1, q-2, 2, 1$</td>
</tr>
<tr>
<td></td>
<td>impossible</td>
<td>$q+1, q-1, q-2, q-4, 3, 2$</td>
</tr>
<tr>
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<td>$0, q+1, 2, q$</td>
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</tr>
<tr>
<td></td>
<td>$0, q+1, 2, 3$</td>
<td>$q+1, q-1, q-2, 3, 2, 1$</td>
</tr>
<tr>
<td></td>
<td>$0, q+1, 2, q$</td>
<td>$q+1, q, q-1, q-2, 2, 1$</td>
</tr>
<tr>
<td></td>
<td>impossible</td>
<td>$q+1, q, q-1, q-2, 2, 1$</td>
</tr>
<tr>
<td>To get the edge label</td>
<td>Admissible value for nodes</td>
<td>Edge labels</td>
</tr>
<tr>
<td>-----------------------</td>
<td>---------------------------</td>
<td>-------------</td>
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<tr>
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<td>0, q+1, q-1, 3</td>
<td>q+1, q-1, q-2, q-4, 3, 2</td>
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<td>0, q+1, q-1</td>
<td>q+1, q-1, q-2, 2, 1</td>
</tr>
<tr>
<td></td>
<td>impossible</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0, q+1, q-2</td>
<td>q+1, q-1, q-2, q-4, 32</td>
</tr>
<tr>
<td></td>
<td>0, q+1, q-3</td>
<td>q+1, q-1, q-2, 3, 2, 1</td>
</tr>
<tr>
<td></td>
<td>impossible</td>
<td></td>
</tr>
<tr>
<td>q-3</td>
<td>0, q+1, q-1, q-2, q-3</td>
<td>q+1, q-1, q-2, q-3, 4, 3, 1</td>
</tr>
<tr>
<td></td>
<td>impossible</td>
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</tr>
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<tr>
<td></td>
<td>impossible</td>
<td>4, 2, 2, 1</td>
</tr>
<tr>
<td></td>
<td>1, q+1, q, 3, 4</td>
<td>q+1, q-1, q-2, q-3, q-3, q-4, 1</td>
</tr>
<tr>
<td></td>
<td>impossible</td>
<td>3, 2, 1, 1</td>
</tr>
<tr>
<td></td>
<td>1, q+1, q, 3, q-1</td>
<td>q+1, q-1, q-2, q-3, q-4, q-1, 1</td>
</tr>
<tr>
<td></td>
<td>impossible</td>
<td>2, 2, 1, 1</td>
</tr>
</tbody>
</table>
To get the edge label | Admissible value for nodes | Edge labels
---|---|---
q-4 | 1,q+1,q-1,2,q-3 | q,q-1,q-2,q-3,q-4,q-5,
 | impossible | 4,2,2,1
 | 1,q+1,q-1,2,5 | q to q-4,q-6,4,3,2,1
 | 1,q+1,q-1,2,3 | q,q-1,q-2,q-3,q-4,q-4,
 | impossible | 2,2,1,1
 | 1,q+1,q-1,2,q-2 | q,q-1,q-2,q-3,q-3,q-4,
 | impossible | 3,2,1,1

With nodes numbered 0, q+1, q−1, 3, 4, q−6 we have edge numbers 1, 2, 3, 4, 5, 7, q−10, q−9, q−6, q−5, ..., q−1, q+1. With nodes numbered 0, q+1, 2, q−2, q−3, 7 we have edge numbers 1, 2, 3, 4, 5, 7, q−10, q−7, ..., q−1, q+1. Now there is no way to get the edge number q−7 and q−8 respectively, (q ≥ 15), because each of the ways to obtain q−7 and q−8 respectively as a difference of two numbers contain at least one impossible node number. So $K_n$ is not nearly graceful for $n ≥ 6$ in this case also. Hence $K_n$ is not nearly graceful for $n ≥ 6$.

**Theorem 2.2.9** $K_n$ is not almost graceful for $n ≥ 7$.

**Proof:** For $n ≥ 7$, the graph $K_n$ has $q ≥ 21$ edges.

**Claim:** $f(E(K_n)) \neq \{1, 2, ..., q−1, q+1\}$.

To get $f(E(K_n)) = \{1, 2, ..., q−1, q+1\}$, the admissible values for $f(V(K_n))$ are
analysed in the following table.

<table>
<thead>
<tr>
<th>To get the edge label</th>
<th>Admissible value for nodes</th>
<th>Edge labels</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q+1$</td>
<td>$q+1, 0$</td>
<td>$q+1$</td>
</tr>
<tr>
<td>$q-1$</td>
<td>$q+1, 0, 2$</td>
<td>$q+1, q-1, 2$</td>
</tr>
<tr>
<td></td>
<td>$q+1, 0, q-1$</td>
<td>$q+1, q-1, 2$</td>
</tr>
<tr>
<td>$q-2$</td>
<td>$q+1, 0, 2, 3$</td>
<td>$q+1, q-1, q-2, 3, 2, 1$</td>
</tr>
<tr>
<td></td>
<td>$q+1, 0, 2, q-2$</td>
<td>$q+1, q-1, q-2, q-4, 3, 2$</td>
</tr>
<tr>
<td></td>
<td>$q+1, 0, q-1, 3$</td>
<td>$q+1, q-1, q-2, q-4, 3, 2$</td>
</tr>
<tr>
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<td>$q+1, 0, q-1, q-2$</td>
<td>$q+1, q-1, q-2, 3, 2, 1$</td>
</tr>
<tr>
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<td>$q+1, 0, q-1, 1$</td>
<td>$q+1, q, q-1, q-2, 2, 1$</td>
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<tr>
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<td>$q+1, q, q-1, q-2, 2, 1$</td>
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<tr>
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<td>$q+1, 0, 2, 3, q-1,$</td>
<td>$q+1, q-1, q-2, q-3, q-4, 3, 2, 1$</td>
</tr>
<tr>
<td></td>
<td>impossible</td>
<td>$q+1, q-1, q-2, q-3, q-4, 3, 2, 1$</td>
</tr>
<tr>
<td>To get the edge label</td>
<td>Admissible value for nodes</td>
<td>Edge labels</td>
</tr>
<tr>
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<td>Edge labels</td>
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</tr>
<tr>
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<td>(q+1, q-1) to (q-7, q-3, 4, 3, 2, 1)</td>
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<td>To get the edge label</td>
<td>Admissible value for nodes</td>
<td>Edge labels</td>
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<td>$q-6$</td>
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<td>$q+1, q-1$ to $q-6, q-8$, 7, 4, 3, 2, 1</td>
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<td>$q+1, q-1$ to $q-6, q-9, q-10$, 7, 5, 4, 3, 2, 1</td>
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<td>$q+1, q-1$ to $q-7, q-3, 4, 4$, 3, 2, 1</td>
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<td>$q+1, q-1$ to $q-6, q-2, q-5$, 4, 3, 2, 1</td>
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<td>$q+1, q-1$ to $q-7, q-9, q-10$, q-14, 8, 7, 5, 4, 3, 2, 1</td>
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<td>$q+1, q-1$ to $q-7, q-5, q-9, q-10$, q-12, 7, 6, 5, 4, 3, 2, 2, 1</td>
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<td>$q+1, q-1$ to $q-10, q-4$, 7, 5, 5, 4, 3, 2, 2, 1</td>
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<td>To get the edge label</td>
<td>Admissible value for nodes</td>
<td>Edge labels</td>
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<td>$q-7$</td>
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<td>$q+1, q-1$ to $q-7, q-3, 7, 4$</td>
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<td>$q-9, q-10, 7, 5, 4, 4, 3, 3, 2, 2, 1$</td>
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<td>$q+1, q-1$ to $q-7, q-9, q-9, q-10, q-14, 8, 7, 5, 4, 4, 3, 2, 1$</td>
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<tr>
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<td>impossible</td>
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<td>$q+1, q-1$ to $q-7, q-5, q-9, q-10, q-12, 7, 6, 5, 4, 3, 3, 2, 2, 1$</td>
</tr>
<tr>
<td></td>
<td>impossible</td>
<td>$q+1, q-1$ to $q-8, q-4, q-9, q-10, q-10, 7, 5, 4, 3, 3, 2, 2, 1$</td>
</tr>
<tr>
<td></td>
<td>$q+1, 0, q-1, 3, 4, q-6, 4$</td>
<td>$q+1, q-1$ to $q-7, q-3, q-6, q-9, q-10, 7, 5, 4, 4, 3, 3, 2, 2, 1$</td>
</tr>
<tr>
<td></td>
<td>impossible</td>
<td>$q+1$ to $q-7, q-2, q-9, q-10, 7, 5, 4, 3, 3, 2, 2, 1, 1$</td>
</tr>
</tbody>
</table>

From the table, \( f(V(K_n)) = \{q + 1, 0, 2, q - 2, q - 3, 7\} \) or \( \{q + 1, 0, q - 1, 3, 4, q - 6\} \) \( \Rightarrow \)

\( f(E(K_n)) = \{1, 2, 3, 4, 5, 7, q + 1, q - 1, q - 2, q - 3, q - 4, q - 5, q - 6, q - 9, q - 10\} \).

Then \( q - 7 \in f(E(K_n)) \) \( \Rightarrow \) there is at least one impossible number in \( f(V(K_n)) \).
So, $K_n$ is not almost graceful for $n \geq 7$.

**Theorem 2.2.10** $K_n$ is not pseudo graceful for $n \geq 6$.

**Proof:** For $n \geq 5$, the graph $K_n$ has $q \geq 15$ edges.

Claim: $f(E(K_n)) \neq \{1, 2, \ldots, q\}$

To get $f(E(K_n)) = \{1, 2, \ldots, q\}$, the admissible values for $f(V(K_n))$ are analysed in the following table.

<table>
<thead>
<tr>
<th>To get the edge label</th>
<th>Admissible value for nodes</th>
<th>Edge labels</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$</td>
<td>$q+1,1$</td>
<td>$q$</td>
</tr>
<tr>
<td>$q-1$</td>
<td>$q+1,1,2$</td>
<td>$q$, $q-1,1$</td>
</tr>
<tr>
<td>$q-2$</td>
<td>$q+1,1,2,q-1$</td>
<td>$q$, $q-1$, $q-2,1,1$</td>
</tr>
<tr>
<td>$q-4$</td>
<td>$q+1,1,2,q-1,5$</td>
<td>$q$ to $q-4$, $q-6$, $4$, $3$, $2$, $1$</td>
</tr>
<tr>
<td></td>
<td>$q+1,1,2,q-1,5$</td>
<td>$q$ to $q-5$, $4$, $2$, $2$, $1$</td>
</tr>
</tbody>
</table>
To get the edge label | Admissible value for nodes | Edge labels |
---|---|---|
q-4 | $q+1,1,2,q-1, q-2$ | q to q-4, q-3, 3, 2, 1, 1 |
| impossible | | 2,2,1,1 |
| $q+1,1,2,q-1,3$ | q to q-4, q-2, | |
| impossible | | |
q-5 | $q+1,1,2,q-1,5,6$ | q to q-7,5,4, 4,3,2,1,1 |
| impossible | | 4,3,2,1,1 |
| $q+1,1,2,q-1,5,q-4$ | q to q-6,q-6,q-9, | |
| impossible | | 5,4,3,3,2,1 |
| $q+1,1,2,q-1,5,q-3$ | q to q-6,q-4,q-8, | |
| impossible | | 4,4,3,2,2,1 |
| $q+1,1,2,q-1,5,4$ | q to q-5,q-3,q-6, | |
| impossible | | 4,3,3,2,2,1,1 |

From the table,

\[ f(V(K_n)) = \{q + 1, 1, 2, q - 1, 5\} \Rightarrow f(E(K_n)) = \{1, 2, 3, 4, q - 6, q - 4, q - 3, q - 2, q - 1, q\}. \]

Then \(q - 5 \in f(E(K_n)) \Rightarrow \) there is at least one impossible number in \(f(V(K_n))\).

So, \(K_n\) is not pseudo graceful for \(n \geq 6\).
2.3 Graceful Graph Structure

Definition 2.3.1 A graph structure \( G = (V; R_1, R_2, \ldots, R_k) \) is said to be graceful if there exists an injective function \( f : V(G) \to Z_{q+1}^k \) such that \( f_i : V(G[E_i]) \to Z_{q+1} \) defined by \( f_i(v) = (f(v))_i, 1 \leq i \leq k \) is a graceful numbering of \( G[E_i] \).

Note 2.3.2 Any graph can be viewed as a graceful graph structure for some \( k \).

Proof: Let \( G = (V, E) \) be a \((p, q)\) graph. Let \( G' = (V; R_1, R_2, \ldots, R_k) \) be a graph structure, where each \( R_i \) contains exactly one edge from \( E(G) \). Then clearly, \( G' \) is graceful for \( k = q \).

Definition 2.3.3 A graph \( G = (V, E) \) is said to be \( k \)-structure graceful if \( E \) can be partitioned into \( k \) disjoint subsets \( R_1, R_2, \ldots, R_k \) such that the graph structure \((V(G); R_1, R_2, \ldots, R_k) \) is graceful.

Note 2.3.4

1. For \( k = 1 \), it is the usual graceful numbering.

2. If \( G \) is \( k \)-structure graceful, then \( G \) is \( m \)-structure graceful for any \( m \geq k \).

Example 2.3.5 Consider \( C_5 \).

Figure 2.2: Cycle \( C_5 \)
$E(C_5) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$.

Let $R_1 = \{v_1v_2, v_2v_3\}; R_2 = \{v_3v_4, v_4v_5\}; R_3 = \{v_5v_1\}$.

Consider graph structure $(V; R_1, R_2, R_3)$.

![Figure 2.3: C₅ as a graceful graph structure](image)

Note that the induced subgraphs $G[R_1]$, $G[R_2]$ and $G[R_3]$ are graceful under the labelings induced by the above labeling.

![Figure 2.4: Induced graceful numberings of G[Rᵢ]](image)

Hence $C_5$ is 3-structure graceful.

$E(C_5)$ can also be written as $E = F_1 \cup F_2$, where $F_1 = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5\}$ and $F_2 = \{v_5v_1\}$.
(V(G); F_1, F_2) is also a graceful graph structure. Hence C_5 is also 2-structure graceful.

Definition 2.3.6 The structure graceful index of a graph G is defined as the minimum k, for which G is k-structure graceful. Let us denote it by SGI(G).

The structure graceful index of a cycle and C_p ∪ C_q have been already determined by R. B. Gnanajothi [17]. She also determined the SGI of C_p ∪ C_q when p + q ≡ 1, 2 (mod 4).

Definition 2.3.7 The n-cone (also called the n-point suspension of C_m) is (C_m + K_n), where C_m is a cycle of length m and K_n is a complete graph on n vertices.

Theorem 2.3.8 The SGI (C_m + K_n) ≤ 2.

Proof: Consider (C_m + K_n).
Let V(C_m) = \{v_1, v_2, ..., v_m\} and E(C_m) = \{v_1v_2, v_2v_3, ..., v_{m-1}v_m, v_mv_1\}
V(K_n) = \{u_1, u_2, ..., u_n\} and E(K_n) = \{\}
Then V(C_m + K_n) = \{v_1, v_2, ..., v_m, u_1, u_2, ..., u_n\} and
E(C_m + K_n) = \{v_1v_2, v_2v_3, ..., v_{m-1}v_m, v_mv_1\}.
The \( n \)-cone \( (C_m + K_n) \) is graceful when \( m \equiv 0 \) or \( 3 \pmod{12} \)\(^{[1]}\).

So, when \( m \not\equiv 0 \) or \( 3 \pmod{12} \), partition the edges of \( (C_m + K_n) \) as \( E(C_m + K_n) = F_1 \cup F_2 \) where \( F_1 = E(P_{m-1} + K_n) \) and \( F_2 = \{v_1v_m\} \) where \( P_{m-1} \) which is a tree, is a path of length \( m - 1 \).

If \( T \) is any graceful tree, then \( T + K_n \) is graceful \(^{[18]}\). Therefore \( P_{m-1} + K_n \) is graceful. That is \((C_m + K_n)[F_1]\)is graceful. \((C_m + K_n)[F_2]\) is also graceful.

Let \( f = (f_1, f_2) \) denote the graceful labelings of \((C_m + K_n)[F_1]\) and \((C_m + K_n)[F_2]\) respectively. \( f_1, f_2 \) together will give the structure graceful labeling for \((C_m + K_n)\).

So \( SGI(C_m + K_n) = 2 \). Hence \( SGI(C_m + K_n) \leq 2 \), in general.

**Definition 2.3.9** A graph consisting of \( n \) triangles \( t_1, t_2, \ldots, t_n \) such that triangle \( t_i \) circumscribes \( t_{i-1} \) for \( i = 2, 3, \ldots, n \) is called a triangulated graph.

**Theorem 2.3.10** Let \( G \) be a triangulated graph. Then \( SGI(G) = 2 \), when \( n \) is even.

**Proof:** Let \( G \) be a triangulated graph in which \( n \) is even.

A triangulated graph is graceful iff \( n \) is odd \(^{[1]}\). Therefore \( G \) is not graceful.

Partition the edge set of \( G \) as \( E(G) = F_1 \cup F_2 \), where \( F_1 = E(G \backslash t_n) \) and \( F_2 = E(t_n) \), \( t_n \) is the \( n \)th triangle in \( G \). Then \( G[F_1] \) and \( G[F_2] \) are graceful, as each \( G[F_i], i = 1, 2 \) has odd number of triangles. Hence \( SGI(G) = 2 \), when \( n \) is even.

**2.4 \( G_n \) graph and \( G_{m,n} \) graph**

**Definition 2.4.1** The graph with vertex set \( \{v_1, v_2, \ldots, v_n\} \) and edge set \( \{v_1v_i/i > 1\} \cup \{v_2v_i/i > 2\} \cup \{v_3v_i/i > 3\} \cup \{v_jv_n/5 \leq j \leq n - 1\} \), for \( n > 4 \) is called \( G_n \)-graph.
Theorem 2.4.2  $G_n$-graph is graceful for $n > 4$.

Proof: The $G_n$-graph has $n$ vertices and $4n - 11$ edges.

Let $V(G_n) = \{v_1, v_2, \cdots, v_n\}$ and

$E(G_n) = \{v_1v_{i}/i > 1\} \cup \{v_2v_{i}/i > 2\} \cup \{v_3v_{i}/i > 3\} \cup \{v_jv_{n}/5 \leq j \leq n - 1\}$, for $n > 4$.

Define $f : V(G_n) \to \{0, 1, 2, \ldots, 4n - 11\}$ by

$$f(v_i) = \begin{cases} 0, & \text{if } i = 1 \\ 4n - 11, & \text{if } i = 2 \\ 4n - 12, & \text{if } i = 3 \\ 2(i - 3), & \text{if } 4 \leq i \leq n - 1 \\ 2n - 5, & \text{if } i = n \end{cases}$$

Claim: $f$ is injective

$f(v_1) \neq f(v_2)$, since $f(v_1)$ is an even number and $f(v_2)$ is an odd number.

$$f(v_1) = f(v_3) \Rightarrow 0 = 4n - 12$$
$$\Rightarrow n = 3, \text{ which is impossible, because } n > 4.$$ 

Therefore $f(v_1) \neq f(v_3)$.

Note that $f(v_i)$ is odd for $i = 2, n$ and it is even for all other vertices.

$$f(v_2) = f(v_n) \iff 4n - 11 = 2n - 5$$
$$\iff 2n = 6$$
$$\iff n = 3$$

Therefore $n > 4 \Rightarrow f(v_2) \neq f(v_n)$.

For $4 \leq i < j \leq n - 1$,
we have $2(i - 3) < 2(j - 3)$ and hence $f(v_i) < f(v_j)$.

Therefore $f(v_i) \neq f(v_j)$ for $4 \leq i \neq j \leq n - 1$.

For $4 \leq i \leq n - 1$, $2 \leq f(v_i) \leq 2n - 8$ and hence $f(v_1) = 0 < f(v_i) < 4n - 12 = f(v_3)$.

Therefore $f(v_1) \neq f(v_3)$, $f(v_i) \neq f(v_j)$ and $f(v_i) \neq f(v_3)$ for $4 \leq i \leq n - 1$.

Hence the claim.

The edges receiving even labels are $\{v_1v_i : 4 \leq i \leq n - 1\}, v_2v_n, \{v_3v_i : 4 \leq i \leq n - 1\}$ and $v_1v_3$.

\[
\{l(v_1v_i) : 4 \leq i \leq n - 1\} = \{|l(v_1) - l(v_i)| : 4 \leq i \leq n - 1\} \\
= \{(2i - 6) : 4 \leq i \leq n - 1\} \\
= \{2, 4, 6, ..., 2n - 8\}
\]

\[
l(v_2v_n) = |(4n - 11) - (2n - 5)| \\
= 2n - 6
\]

\[
\{l(v_3v_i) : 4 \leq i \leq n - 1\} = \{|l(v_3) - l(v_i)| : 4 \leq i \leq n - 1\} \\
= \{4n - 12) - (2i - 6) : 4 \leq i \leq n - 1\} \\
= \{4n - 2i - 6 : 4 \leq i \leq n - 1\} \\
= \{4n - 14, 4n - 16, ..., 2n - 4\}
\]

\[
l(v_1v_3) = 4n - 12.
\]

Hence all the even numbers $2, 4, 6, ..., (4n - 12)$ appear as edge labels.

The edges receiving odd labels are $v_2v_3, \{v_nv_i : 5 \leq i \leq n - 1\}, v_3v_n, v_1v_n, \{v_2v_i : 4 \leq i \leq n - 1\}$ and $v_1v_2$. 

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\begin{align*}
l(v_2v_3) &= |(4n - 11) - (4n - 12)| = 1 \\
\{l(v_nv_i) : 5 \leq i \leq n - 1\} &= \{|l(v_n) - l(v_i)| : 5 \leq i \leq n - 1\} \\
&= \{|(2n - 5) - (2i_0)| : 5 \leq i \leq n - 1\} \\
&= \{|2n - 2i + 1| : 5 \leq i \leq n - 1\} \\
&= \{2n - 9, 2n - 11, ..., 3\} \\
l(v_2v_n) &= |(4n - 11) - (2n - 5)| = 2n - 6 \\
\{l(v_3v_i) : 4 \leq i \leq n - 1\} &= \{|l(v_3) - l(v_i)| : 4 \leq i \leq n - 1\} \\
l(v_3v_n) &= 2n - 7 \\
l(v_1v_n) &= 2n - 5 \\
\{l(v_2v_i) : 4 \leq i \leq n - 1\} &= \{|(4n - 11) - (2i - 6)| : 4 \leq i \leq n - 1\} \\
&= \{|4n - 2i - 5| : 4 \leq i \leq n - 1\} \\
&= \{4n - 13, 4n - 15, ..., 2n - 3\} \\
l(v_1v_2) &= 4n - 11.
\end{align*}

Hence all the odd numbers 1, 3, 5, ..., \(4n - 11\) appear as edge labels.

Thus \(f : V(G_n) \to \{0, 1, 2, ..., 4n - 11\}\) is an injective function and the edges receive the labels from 1 to \(4n - 11\). Therefore \(f\) is a graceful labeling of \(G_n\). Hence \(G_n\)-graph is graceful.
Illustration 2.4.3

Figure 2.6: Gracefulness of $G_9$

Definition 2.4.4 The graph with vertex set \{v_1, v_2...v_{m+n}\} and edge set \{v_1v_i/ 2 \leq i \leq m+n\} \cup \{v_jv_k/ 2 \leq j < k \leq m\} is called $G_{m,n}$-graph.

Lemma 2.4.5 $G_{m,n}$-graph is graceful for $2 < m < 7$.

Proof: Let $V(G) = \{v_1, v_2...v_{m+n}\}$ and $E(G) = \{v_1v_i/ 2 \leq i \leq m+n\} \cup \{v_jv_k/ 2 \leq j < k \leq m\}$.

Case(i) : When $m = 3$

Define $f(v_i) = \begin{cases} 0, & i = 1 \\ i, & 2 \leq i \leq 3+n \end{cases}$
Clearly \( f \) is injective.

Let \( A_1 = \{\ell(v_1)/2 \leq i \leq 3 + n\} \)

\[
= \{\|\ell(v_1) - \ell(v_i)\|/2 \leq i \leq 3 + n\}
= \{|\ell(v_1) - \ell(v_i)|\} \cup \{|\ell(v_1) - \ell(v_2)|\} \cup \ldots \cup \{|\ell(v_1) - \ell(v_{3+n})|\}
= \{2, 3, \ldots 3 + n\} \quad (2.1)
\]

\( A_2 = \{\ell(v_2v_3)\} = \{|\ell(v_2) - \ell(v_3)|\} = \{|2 - 3|\} = \{1\}. \quad (2.2) \]

From (2.1) and (2.2), \( A_1 \cup A_2 = \{1, 2, \ldots, 3 + n\} \).

Thus \( f : V(G) \to \mathbb{Z}_{3+n} \) is an injective function and the edges receive the labels from \( \{1, 2, \ldots 3 + n\} \). Hence \( G_{m,n} \)-graph is graceful.

Case (ii) : When \( m = 4 \)

Define \( f(v_i) = \begin{cases} 
0, & i = 1 \\
6, & i = 2 \\
5, & i = 3 \\
2, & i = 4 \\
i + 2, & 5 \leq i \leq 4 + n
\end{cases} \)

Claim: \( f \) is injective

\( f(v_1) \neq f(v_i), \text{ for } i > 1, \text{ since } f(v_1) \text{ is } 0 \text{ and } f(v_i) \text{ is not } 0 \text{ for } i > 1. \)

For \( i \geq 5, \quad f(v_i) \geq 7. \text{ Therefore } f(v_i) \neq f(v_j) \text{ when } i \geq 5 \text{ and } 1 \leq j \leq 4. \)

Clearly, \( f(v_i) \neq f(v_j) \) for \( 5 \leq i < j \leq 4 + n. \) Therefore \( f \) is an injective function.
Let \( A_3 = \{ \ell(v_1 v_i)/ 2 \leq i \leq 4 + n \} \)
\[
= \{ |\ell(v_1) - \ell(v_i)|/2 \leq i \leq 4 + n \}
= \{ |\ell(v_1) - \ell(v_2)| \} \cup \{ |\ell(v_1) - \ell(v_3)| \} \cup \ldots \cup \{ |\ell(v_1) - \ell(v_{4+n})| \}
= |0 - 6| \cup |0 - 5| \cup |0 - 2| \cup |0 - 7| \cup |0 - 8| \cup \ldots \cup |0 - (n + 6)|
= \{6, 5, 2, 7, 8, \ldots, n + 6\}
\] (2.3)
\[
A_4 = \{ \ell(v_2 v_i)/ 3 \leq i \leq m \}
= \{ |\ell(v_2) - \ell(v_i)|/3 \leq i \leq m \}
= \{ |\ell(v_2) - \ell(v_3)| \} \cup \{ |\ell(v_2) - \ell(v_4)| \}
= \{ |6 - 5| \cup |6 - 2| \} = \{1, 4\}
\] (2.4)
\[
A_5 = \{ \ell(v_3 v_4) \} = \{ |\ell(v_3) - \ell(v_4)| \} = \{ |5 - 2| \} = \{3\}
\] (2.5)

From (2.3), (2.4), and (2.5), \( A_3 \cup A_4 \cup A_5 = \{1, 2, \ldots, n + 6\} \).

Thus \( f : V(G) \rightarrow \mathbb{Z}_{6+n} \) is an injective function and the edges receive the labels from \( \{1, 2, \ldots, 6 + n\} \). Hence \( G_{m,n} \)-graph is graceful.

Case(iii) : When \( m=5 \)

Define \( f(v_i) = \)
\[
\begin{align*}
0, & \quad i = 1 \\
11, & \quad i = 2 \\
10, & \quad i = 3 \\
2, & \quad i = 4 \\
7, & \quad i = 5 \\
6, & \quad i = 6 \\
i + 5, & \quad 7 \leq i \leq 5 + n
\end{align*}
\]
Claim: \( f \) is injective

\[ f(v_1) \neq f(v_i), \text{ since } f(v_1) \text{ is 0 and } f(v_i) \text{ is a positive integer for } i > 1. \]

Also \( f(v_i) \neq f(v_j), \text{ for } 2 \leq i < j \leq 5 + n. \)

Therefore \( f \) is an injective function.

Let \( A_6 = \{ \ell(v_i v_i)/ 2 \leq i \leq 5 + n \} = \{ |\ell(v_1) - \ell(v_i)|/2 \leq i \leq 5 + n \} \)

\[ = \{ |\ell(v_1) - \ell(v_2)| \} \cup \{ |\ell(v_1) - \ell(v_3)| \} \cup ... \cup \{ |\ell(v_1) - \ell(v_{5+n})| \} \]

\[ = \{ -11 \} \cup \{ -10 \} \cup \{ -2 \} \cup \{ -7 \} \cup \{ -6 \} \cup \{ -12 \} \cup \{ -13 \} \]

\[ \cup ... \cup \{ -(n + 10) \} = \{ 11, 10, 2, 7, 6, 12, 13, ..., n + 10 \} \]  (2.6)

\[ A_7 = \{ \ell(v_i v_j)/ 2 \leq i \leq 5 \} = \{ |\ell(v_i) - \ell(v_j)|/2 \leq i \leq 5 \} \]

\[ = \{ |\ell(v_2) - \ell(v_3)| \} \cup \{ |\ell(v_2) - \ell(v_4)| \} \cup \{ |\ell(v_2) - \ell(v_5)| \} \cup \]

\[ \{ |\ell(v_3) - \ell(v_4)| \} \cup \{ |\ell(v_3) - \ell(v_5)| \} \cup \{ |\ell(v_4) - \ell(v_5)| \} \]

\[ = \{ -11 \} \cup \{ -10 \} \cup \{ -2 \} \cup \{ -7 \} \cup \{ -10 \} \cup \{ -7 \} \cup \{ -12 \} \]

\[ = \{ 1, 9, 4, 8, 3, 5 \} \]  (2.7)

From (2.6) and (2.7), \( A_6 \cup A_7 = \{ 1, 2, ..., 10 + n \} \)

Thus \( f : V(G) \rightarrow Z_{10+n} \) is an injective function and the edges receive the labels
from \( \{ 1, 2, ..., 10 + n \} \). Hence \( G_{m,n}-\text{graph} \) is graceful.

Case(iv) : When \( m=6 \)

Define \( f(v_i) = \)

\[
\begin{align*}
0, & \quad i = 1 \\
17, & \quad i = 2 \\
16, & \quad i = 3 \\
2, & \quad i = 4 \\
13, & \quad i = 5 \\
7, & \quad i = 6 \\
8, & \quad i = 7 \\
12, & \quad i = 8 \\
i + 9, & \quad 9 \leq i \leq 6 + n
\end{align*}
\]
Claim: $f$ is injective

$f(v_1) \neq f(v_i)$, since $f(v_1)$ is 0 and $f(v_i)$ is positive integer for $i > 1$.

Also $f(v_i) \neq f(v_j)$, for $2 \leq i < j \leq 6 + n$.

Therefore $f$ is an injective function.

Let $A_8 = \{\ell(v_i)/2 \leq i \leq 6 + n\} = \{|\ell(v_1) - \ell(v_i)|/2 \leq i \leq 6 + n\}$

$A_8 = \{|\ell(v_1) - \ell(v_2)|\} \cup \{|\ell(v_1) - \ell(v_3)|\} \cup \cdots \cup \{|\ell(v_1) - \ell(v_{6+n})|\}$

$A_8 = \{|0 - 17| \cup |0 - 16| \cup |0 - 2| \cup |0 - 13| \cup |0 - 7| \cup |0 - 8| \cup$

$|0 - 12| \cup |0 - 18| \cup |0 - 19| \cup \cdots \cup |0 - (n + 15)|\}$

$A_8 = \{17, 16, 13, 7, 8, 12, 18, 19, \ldots, n + 15\}$ \hspace{1cm} (2.8)

$A_9 = \{\ell(v_i)/2 \leq i \leq 6\} = \{|\ell(v_i) - \ell(v_j)|/2 \leq i \leq 6\}$

$A_9 = \{|\ell(v_2) - \ell(v_3)|\} \cup \{|\ell(v_2) - \ell(v_4)|\} \cup \cdots \cup \{|\ell(v_2) - \ell(v_6)|\}$

$A_9 = \{|17 - 16| \cup |17 - 2| \cup |17 - 13| \cup |17 - 7| \cup |16 - 2| \cup$

$|16 - 13| \cup |16 - 7|\} \cup \{|2 - 13| \cup |2 - 7|\} \cup \{|13 - 7|\}$

$A_9 = \{1, 15, 4, 10\} \cup \{14, 3, 9\} \cup \{11, 5\} \cup \{6\}$

$A_9 = \{1, 3, 4, 5, 6, 9, 10, 11, 14, 15\}$ \hspace{1cm} (2.9)

From (2.8) and (2.9), $A_8 \cup A_9 = \{1, 2, \ldots, n + 15\}$.

Thus $f : V(G) \rightarrow Z_{15+n}$ is an injective function and the edges receive labels from

$\{1, 2, \ldots, 15 + n\}$. Hence $G_{m,n}$-graph is graceful.
Theorem 2.4.6 One-point union of $K_2$ and $K_6$ is not graceful.

Proof: Let $G$ denote one point union of $k_2$ and $k_6$. Let $V(G) = \{u_1,v_1,v_2,\ldots,v_6\}$ and $E(G) = \{u_1v_1\} \cup \{v_iv_j/1 \leq i, j \leq 6, i \neq j\}$. Then $G$ contains 7 vertices and 16 edges. In order to get a graceful labeling of $G$, we have to assign $\{0, 1, 2, \ldots, 16\}$ to the nodes of $G$ in such a way that the edges receive the labels $\{1, 2, \ldots, 16\}$. The edge label 16 can be got only when 0 and 16 are assigned to two vertices of $G$. Then 0 and 16 can be assigned to the vertices of $G$ in the following ways.

Cases:

(i) 0 to the node $v_1$ and 16 to the node $u_1$

(ii) 0 to the node $u_1$ and 16 to the node $v_1$

(iii) Both 0 and 16 to the nodes in $K_6$

In case(i), edge $u_1v_1$ receives the label 16 and we are left with $K_6$. But we know that $K_6$ is not graceful. So this case does not yield a graceful labeling of $G$.

In case(ii) also, the edge $u_1v_1$ receives the label 16 and $l(u_1) = 0$, $l(v_1) = 16$. Then we have to assign the labels $\{1, 2, \ldots, 15\}$ to $v_2,v_3,v_4,v_5$ and $v_6$ to get the edge labels as $\{1, 2, \ldots, 15\}$.

Assume that it is possible. Let $l$ be such a labeling.

Then $\{l(v_i) : i = 1, 2, 3, 4, 5, 6\} \subseteq \{1, 2, \ldots, 16\}$ and

$\{l(v_i) - l(v_j) : 1 \leq i \neq j \leq 6\} = \{1, 2, \ldots, 15\}$.

Let $l'(v_i) = l(v_i) - 1$, $i = 1, 2, \ldots, 6$.

Clearly $\{l'(v_i) : 1 \leq i \leq 6\} \subseteq \{0, 1, 2, \ldots, 15\}$ and

$\{|l'(v_i) - l'(v_j)| : 1 \leq i \neq j \leq 6\} = \{|l(v_i) - l(v_j)| : 1 \leq i \neq j \leq 6\} = \{1, 2, \ldots, 15\}$.
Then $l'$ is a graceful labeling of $K_6$.

$\Rightarrow \Leftarrow$ Since $K_6$ is not graceful. Hence we are left with case(iii) alone.

Case(iii):

Without loss of generality, let $l(v_1) = 0$ and $l(v_2) = 16$. This gives the edge label 16 to the edge $v_1v_2$. There are 14 edges in $K_6$ and the edge $u_1v_1$ which are to be labeled with $\{1,2,...,15\}$. Then the following subcases arise:

Subcase:

(i) $l(u_1v_1)$ is even
(ii) $l(u_1v_1)$ is odd

\{l(v_i v_j) : 1 \leq i \neq j \leq 6\} = A(say).

Let $A_1 = \{\text{even numbers in } A\}$ and \\
$A_2 = \{\text{odd numbers in } A\}$.

In subcase(i), $|A_1|=7$ and $|A_2|=8$.
In subcase(ii), $|A_1|=8$ and $|A_2|=7$.

In both subcases, the labeling of $v_3,v_4,v_5,v_6$ should fall under one of the following possibilities:

(a) $|A_1| = 15$ and $|A_2| = 0$
(b) $|A_1| = 10$ and $|A_2| = 5$
(c) $|A_1| = 7$ and $|A_2| = 8$
(d) $|A_1| = 6$ and $|A_2| = 9$
(e) $|A_1| = 7$ and $|A_2| = 8$

None of the possibilities (a) to (e) results in the subcase(ii) and the possibilities (c) and (e) result in subcase(i).
The various possibilities of $l(v_3), l(v_4), l(v_5), l(v_6)$ and $l(u_1)$ to obtain a graceful labeling of $G$ are analysed in the following table.

<table>
<thead>
<tr>
<th>To get the edge label</th>
<th>Admissible value for nodes</th>
<th>Edge label</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0 16 - - - - -</td>
<td>16</td>
</tr>
<tr>
<td>15</td>
<td>0 16 15 - - - - -</td>
<td>16,15,1</td>
</tr>
<tr>
<td></td>
<td>0 16 1 - - - - -</td>
<td>16,15,1</td>
</tr>
<tr>
<td>14</td>
<td>0 16 15 2 - - - -</td>
<td>16,15,14,13,2,1</td>
</tr>
<tr>
<td></td>
<td>0 16 114 - - - - -</td>
<td>16,15,14,13,2,1</td>
</tr>
<tr>
<td></td>
<td>0 16 15 - - 14</td>
<td>16,15,14,11</td>
</tr>
<tr>
<td></td>
<td>0 16 1 - - 14</td>
<td>16,15,14,11</td>
</tr>
<tr>
<td>13</td>
<td>0 16 15 2 12 - - - -</td>
<td>16,15,14,13,12,10,4,3,2,1</td>
</tr>
<tr>
<td></td>
<td>0 16 114 4 - - - - -</td>
<td>16,15,14,13,12,10,4,3,2,1</td>
</tr>
<tr>
<td></td>
<td>0 16 15 13 - - - - -13</td>
<td>16-13,3,2,1</td>
</tr>
<tr>
<td></td>
<td>0 16 15 3 - - 14</td>
<td>16-12,3,1</td>
</tr>
<tr>
<td></td>
<td>0 16 113 - - 14</td>
<td>16-12,3,1</td>
</tr>
<tr>
<td></td>
<td>0 16 1 3 - - 14</td>
<td>16-13,3,2,1</td>
</tr>
<tr>
<td>12</td>
<td>0 16 15 2 12 - - - -</td>
<td>16,15,14,13,12,10,4,3,2,1</td>
</tr>
</tbody>
</table>

47
<table>
<thead>
<tr>
<th>To get the edge label</th>
<th>Admissible value for nodes</th>
<th>Edge label</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0 16 1 14 4 - -</td>
<td>16,15,14,13,12,10,4,3,2,1</td>
</tr>
<tr>
<td></td>
<td>0 16 15 3 - - 14</td>
<td>16-12,3,1</td>
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<tr>
<td></td>
<td>0 16 1 13 - - 14</td>
<td>16-12,3,1</td>
</tr>
<tr>
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<td>16-11,9,4,3,2,1</td>
</tr>
<tr>
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<td>0 16 1 3 12 - 14</td>
<td>16-11,9,4,3,2,1</td>
</tr>
<tr>
<td>11</td>
<td>0 16 15 13 4 - 14</td>
<td>16-11,9,4,3,2,1</td>
</tr>
<tr>
<td></td>
<td>0 16 15 3 11 - 14</td>
<td>16-11,8,5,4,3,1</td>
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<td>0 16 15 3 5 - 14</td>
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</tr>
<tr>
<td>10</td>
<td>0 16 15 13 4 - 14</td>
<td>impossible</td>
</tr>
<tr>
<td></td>
<td>0 16 15 3 11 - 14</td>
<td>impossible</td>
</tr>
<tr>
<td></td>
<td>0 16 1 13 5 - 14</td>
<td>impossible</td>
</tr>
<tr>
<td></td>
<td>0 16 1 3 12 - 14</td>
<td>impossible</td>
</tr>
<tr>
<td></td>
<td>0 16 15 3 5 - 14</td>
<td>16-10,5,3,2,1</td>
</tr>
<tr>
<td></td>
<td>0 16 1 13 11 - 14</td>
<td>16-10,5,3,2,1</td>
</tr>
</tbody>
</table>
None of the admissible values of nodes yield a graceful labeling. Hence one-point union of $K_2$ and $K_6$ is not graceful.

**Theorem 2.4.7** $SGI(K_n) = 2, 4 < n < 10.$

**Proof:** We know that $K_n$ is not graceful for $n > 4$.

Partition the edges of $K_n$ as follows:

Let $R_1 = E(G_n), \quad R_2 = E(K_n) \setminus E(G_n)$.

Then $K_n[R_1]$, the subgraph of $K_n$ induced by $R_1$ is $G_n$ and

$K_n[R_2]$, the subgraph of $K_n$ induced by $R_2$ is one-point union of $K_2$ and $K_{n-4}$.

By theorem 2.4.2, $G_n$ is graceful.

The gracefulness of $K_n[R_2]$ for $n = 5, 6, 7, 8, 9$ is as follows:
Figure 2.7: Gracefulness of $K_n[R_2]$ for $n=5,6,7,8,9$
Hence $SGI(K_n) = 2$ for $4 < n < 10$.

**Illustration 2.4.8**

![Graph of 2-Structure graceful labeling of $K_9$](image)

**Figure 2.8**: 2-Structure graceful labeling of $K_9$

**Note 2.4.9** The partition used in theorem 2.4.7 will not work for $K_{10}$.

**Proof**: As in theorem 2.4.7, we have the partition of $K_{10}$ as $E(G_{10})$ and $E(K_{10} \setminus G_{10})$.

Let $A = E(G_{10})$ and $B = E(K_{10} \setminus G_{10})$ and let $G_{10}^{(1)} = A$ and $G_{10}^{(2)} = B$.

By theorem 2.4.2, $G_{10}^{(1)}$ is graceful. But $G_{10}^{(2)}$ is a one-point union of $K_2$ and $K_6$.

By theorem 2.4.6, $G_{10}^{(2)}$ is not graceful.
Theorem 2.4.10 \( SGI(K_{10}) = 2 \).

Proof: The structure graceful labeling of \( K_{10} \) is as follows:

In this partition \( K_{10} \) is 2-structure graceful. So \( SGI(K_{10}) = 2 \).
Theorem 2.4.11 \( SGI(K_n) \leq \begin{cases} \left\lceil \frac{n-5}{4} \right\rceil + 1, & \text{when } n \equiv 1(\mod 4). \\ \left\lceil \frac{n-6}{4} \right\rceil + 1, & \text{when } n \equiv 2(\mod 4). \\ \left\lceil \frac{n-7}{4} \right\rceil + 2, & \text{when } n \equiv 3(\mod 4) \\ \left\lceil \frac{n-8}{4} \right\rceil + 2, & \text{when } n \equiv 0(\mod 4) \end{cases} \)

where \( n > 10 \).

Proof: Partition the edges of \( K_n \) (ie) \( E(K_n) \) into two sets namely, \( E(G) \) and \( E(K_n \setminus G_n) \), then \( E(K_n \setminus G_n) \) into \( E(G_{n-4}) \) and \( E(K_{13} \setminus G_{13} \setminus G_{n-4}) \), then \( E(K_n \setminus G_n \setminus G_{n-4}) \) into \( E(G_{n-8}) \) and \( E(K_n \setminus G_n \setminus G_{n-4} \setminus G_{n-8}) \), then \( E(K_n \setminus G_n \setminus G_{n-4} \setminus G_{n-8}) \) into \( E(G_{n-12}) \) and \( E(K_n \setminus G_n \setminus G_{n-4} \setminus G_{n-8} \setminus G_{n-12}) \) and so on.

From this partition, in the last step we arrive the following cases:

1. \( E(G_9) \) and edges in one point union of \( K_5 \) and \( K_{1,\left\lceil \frac{n}{4} \right\rceil - 1} \),
   when \( n \equiv 1(\mod 4) \).

2. \( E(G_{10}) \) and edges in one point union of \( K_6 \) and \( K_{1,\left\lceil \frac{n}{4} \right\rceil - 1} \),
   when \( n \equiv 2(\mod 4) \).

3. \( E(G_7) \) and edges in one point union of \( K_3 \) and \( K_{1,\left\lceil \frac{n}{4} \right\rceil} \),
   when \( n \equiv 3(\mod 4) \).

4. \( E(G_8) \) and edges in one point union of \( K_4 \) and \( K_{1,\left\lceil \frac{n}{4} \right\rceil - 1} \),
   when \( n \equiv 0(\mod 4) \).

When \( n \equiv 1(\mod 4) \):

We have subgraphs which contain the edges of \( G_n, G_{n-4}, G_{n-8}, ..., G_9 \) and one point union of \( K_5 \) and \( K_{1,\left\lceil \frac{n}{4} \right\rceil - 1} \).

Let \( G_n^{(j)} = < A_j >, j = 1, 2, ..., m + 1 \) where \( m = \left\lfloor \frac{n-5}{4} \right\rfloor \).
Where \(< A_1 >\) is the subgraph of \(K_n\) induced by the edges in \(G_n\).
\(< A_2 >\) is the subgraph of \(K_n \setminus G_n\) induced by the edges in \(G_{n-4}\).

\(< A_m >\) is the subgraph of \(K_n \setminus G_n \setminus G_{n-4} \setminus G_9\) induced by the edges in \(G_9\).
\(< A_{m+1} >\) is the subgraph of \(K_n \setminus G_n \setminus G_{n-4} \setminus G_9\) induced by the edges in one point union of \(K_5\) and \(K_{1,\lfloor n/4 \rfloor - 1}\).

We have already proved that the graph \(G_n\) is graceful for \(n > 4\). Therefore \(G_n^{(j)}, j = 1, 2, ..., m\) is graceful. By case (iii), \(G_n^{(j)}, j = m + 1\) is graceful. Totally we have \(m + 1 = \lfloor \frac{n-5}{4} \rfloor + 1\) graceful subgraphs.
\[
\therefore SGI(K_n) \leq \left\lfloor \frac{n-5}{4} \right\rfloor + 1.
\]

When \( n \equiv 2(\text{mod } 4) \):

We have subgraphs which contain the edges of \(G_n, G_{n-4}, G_{n-8}, ..., G_{10}\) and one point union of \(K_6\) and \(K_{1,\lfloor n/4 \rfloor - 1}\).

Let \(G_n^{(j)} = \langle A_j \rangle, j = 1, 2, ..., m + 1\) where \(m = \lfloor \frac{n-6}{4} \rfloor\).
Where \(< A_1 >\) is the subgraph of \(K_n\) induced by the edges in \(G_n\).
\(< A_2 >\) is the subgraph of \(K_n \setminus G_n\) induced by the edges in \(G_{n-4}\).

\(< A_m >\) is the subgraph of \(K_n \setminus G_n \setminus G_{n-4} \setminus G_{14}\) induced by the edges in \(G_{10}\).
\(< A_{m+1} >\) is the subgraph of \(K_n \setminus G_n \setminus G_{n-4} \setminus G_{14}\) induced by the edges in one point union of \(K_6\) and \(K_{1,\lfloor n/4 \rfloor - 1}\).

Again \(G_n^{(j)} = \langle A_j \rangle, j = 1, 2, ..., m\) are graceful and by case(iv), \(G_n^{(j)}, j = m + 1\) is also graceful. We have \(m + 1 = \lfloor \frac{n-6}{4} \rfloor + 1\) graceful subgraphs.
\[
\therefore SGI(K_n) \leq \left\lfloor \frac{n-6}{4} \right\rfloor + 1.
\]
When \( n \equiv 3(\text{mod } 4) \):

We have subgraphs which contain the edges of \( G_n, G_{n-4}, G_{n-8}, \ldots, G_7 \) and one point union of \( K_3 \) and \( K_{1,\lfloor \frac{n}{4} \rfloor} \).

Let \( G_n^{(j)} = \langle A_j \rangle, j = 1, 2, \ldots, m + 2 \) where \( m = \left\lfloor \frac{n-7}{4} \right\rfloor \).

Where \( \langle A_1 \rangle \) is the subgraph of \( K_n \) induced by the edges in \( G_n \).

\( \langle A_2 \rangle \) is the subgraph of \( K_n \setminus G_n \) induced by the edges in \( G_{n-4} \).

\( \langle A_{m+1} \rangle \) is the subgraph of \( K_n \setminus G_n \setminus G_{n-4} \setminus \ldots \setminus G_{11} \) induced by the edges in \( G_7 \).

\( \langle A_{m+2} \rangle \) is the subgraph of \( K_n \setminus G_n \setminus G_{n-4} \setminus \ldots \setminus G_{11} \) induced by the edges in one point union of \( K_3 \) and \( K_{1,\lfloor \frac{n}{4} \rfloor} \).

Here also \( G_n^{(j)} = \langle A_j \rangle, j = 1, 2, \ldots, m + 1 \) are graceful and by case (i), \( G_n^{(j)} \), \( j = m + 2 \) is also graceful. And we have \( m + 2 = \left\lfloor \frac{n-6}{4} \right\rfloor + 2 \) graceful subgraphs.

\( \therefore SGI(K_n) \leq \left\lfloor \frac{n-7}{4} \right\rfloor + 2. \)

When \( n \equiv 0(\text{mod } 4) \):

In this form we have subgraphs which contain the edges of \( G_n, G_{n-4}, G_{n-8}, \ldots, G_8 \) and one point union of \( K_4 \) and \( K_{1,\lfloor \frac{n}{4} \rfloor} - 1 \).

Let \( G_n^{(j)} = \langle A_j \rangle, j = 1, 2, \ldots, m + 2 \) where \( m = \left\lfloor \frac{n-8}{4} \right\rfloor \).

Where \( \langle A_1 \rangle \) is the subgraph of \( K_n \) induced by the edges in \( G_n \).

\( \langle A_2 \rangle \) is the subgraph of \( K_n \setminus G_n \) induced by the edges in \( G_{n-4} \).

\( \langle A_{m+1} \rangle \) is the subgraph of \( K_n \setminus G_n \setminus G_{n-4} \setminus \ldots \setminus G_{12} \) induced by the edges in \( G_8 \).

\( \langle A_{m+2} \rangle \) is the subgraph of \( K_n \setminus G_n \setminus G_{n-4} \setminus \ldots \setminus G_{12} \) induced by the edges in
one point union of $K_4$ and $K_{1,\left\lfloor \frac{n}{4}\right\rfloor - 1}$.

Here also $G_n^{(j)} = < A_j >$, $j = 1, 2, \ldots, m + 1$ are graceful and by case(ii), $G_n^{(j)}$, $j = m + 2$ is also graceful. And we have $m + 2 = \left\lfloor \frac{n-6}{4} \right\rfloor + 2$ graceful subgraphs.

\[ \therefore SGI(K_n) \leq \left\lfloor \frac{n-8}{4} \right\rfloor + 2. \]

**Illustration 2.4.12**

The 3-structure graceful labelling of $K_{13}$ is shown below:

Here $13 \equiv 1(\text{mod } 4)$.

Hence by theorem 2.4.11, $SGI(K_{13}) = \left\lfloor \frac{n-5}{4} \right\rfloor + 1 = 2 + 1 = 3$.

For, partition the edges of $K_{13}$ into $E(G_{13})$ and $E(K_{13} \setminus G_{13})$. Then we have

\[ \text{Figure 2.10: } G_{13} \]

Partition the edges of $E(K_{13} \setminus G_{13})$ into $E(G_{n-4} = G_9)$ and $E((K_{13} \setminus G_{13}) \setminus G_9)$. 

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Figure 2.11: $G_9$

Figure 2.12: $K_{13} \setminus G_{13} \setminus G_9 = G_{5,2}$
Figure 2.13: 3-Structure graceful labeling of $K_{13}$

Hence $SGI(K_{13}) = 3$. 