Chapter 1

Introduction

1.1 Preliminaries

In the mathematical discipline of graph theory, a graph labeling is the assignment of labels to the edges or vertices or both of a graph. For many applications, the edges or vertices are given labels that are meaningful in the associated domain. The notion of labeling may be applied to all extensions and generalization of graphs. Most graph labelings trace their origins to labelings presented by Alex Rosa in his 1967 paper. Rosa identified three types of labelings, which he called, \( \alpha \)-labeling, \( \beta \)-labeling and \( \rho \)-labeling. \( \beta \)-labelings were later renamed as graceful labeling by S. W. Golomb and the name has been popular since.

A graceful labeling of a graph \( G \) with \( q \) edges and vertex set \( V \) is an injection \( f : V(G) \to \{0,1,2,\ldots,q\} \) with the property that the resulting edge labels are also distinct, where an edge incident with vertices \( u \) and \( v \) is assigned the label \( |f(u) - f(v)| \). A graph which admits a graceful labeling is called a graceful graph. Graceful graphs have several applications in coding theory, x-ray, crystallography,
radar communication, networks and radio astronomy. Various kinds of graphs are shown to be graceful, but only very few general results have been proved in this topic.

Harmonic graphs naturally arose in the study by Graham and Sloane of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph $G$ with $q$ edges to be harmonious if there is an injection $f$ from the vertices of $G$ to the group of integers modulo $q$ such that when each edge $xy$ is assigned the label $(f(x) + f(y))(mod \ q)$, the resulting edge labels are distinct. When $G$ is a tree, exactly one label may be used on two vertices. Lice and Zhang have proved that every graph is a subgraph of a harmonious graph.

Dr. E. Sampath Kumar$^{[3]}$ introduced the concept “Generalized Graph Structures”. The notion of graceful graph structure was introduced by Dr. R. B. Gnanajothi. She has defined graceful graph structure, k-structure graceful, structure graceful index of a graph and some results based on these definitions. In this thesis, focus is on generalized graph structures. This is quite useful in studying structures like graphs, signed graphs, semigraphs, graphs in which every edge is labeled, etc. Here we consider the graphs which are finite, simple and connected.

Based on these definitions and results given by Dr. E. Sampath Kumar and Dr. R. B. Gnanajothi, we extended our work in chapter 2 and prove that every nearly graceful graph $G$ gives rise to a graceful graph $H$ which is got by either adding an edge or by a vertex together with an edge to $G$. Also, we established that every almost graceful graph $G$ gives rise to a graceful graph $H$ which is got by either adding an edge or by a vertex together with an edge to $G$ and every pseudo graceful graph $G$ gives rise to a graceful graph $H$ which is got by either...
adding an edge or by a vertex together with an edge to G.

Graph decomposition problems play a vital role in the area of research in graph theory and combinatorics and further have numerous applications in various fields such as networking, engineering and DNA analysis. A decomposition of a graph G is a collection of its subgraphs such that every edge of G lies in exactly one member of the collection. Various types of decompositions have been introduced and studied by imposing conditions on the members of the decomposition. Harary introduced the notion of path cover which demands each member of the decomposition to be a path. Following Harary, several variations of decomposition have been introduced and extensively studied. Unrestricted path cover, cycle decomposition, path double cover, orthogonal double cover and graphoidal cover are some variations of decomposition. We motivated with this and decompose graphs into graceful subgraphs. Mostly we consider the ungraceful graphs into graceful subgraphs.

Many important families fail to be graceful. The complete graph $K_n$ is not graceful for $n > 4$. As there are so many labelings in Graph theory, graceful labeling plays a vital role. Also in many applications, people used complete graphs. Rosa’s concept of decomposing the complete graph into isomorphic sub graphs motivated us to find the structure graceful index of complete graphs. Moreover, we proved the structure graceful index of most of the families is less or equal to 2.

A generalization of combinatorial graphs that is different from hypergraphs is discussed here. The notion of a semigraph is a generalization of a graph while generalizing a structure, one naturally looks for one in which every concept or idea in the structure has a natural generalization. Semigraph is such a natural
generalization of graph, which resembles graph. The beauty of semigraphs lies in the variety of definitions/concepts, all of which coincide for graphs. This generalization is more closely related to the theory of graphs than that of Hypergraphs. The concept of magic labeling is also easily extended to semigraphs. But one can easily introduce graph structure concepts in semigraphs just like in semigraphs since the elements of an edge are arranged in an order.

Magic labelings were introduced by Sedlacek in 1963. Responding to a problem raised by Sedlacek, Stewart studied various ways to label the edges of a graph in the mid 60s. Stewart calls a connected graph semi-magic if there is a labeling of the edges with integers such that for each vertex \( v \) the sum of the labels of all edges incident with \( v \) is the same for all \( v \). A semi-magic labeling where the edges are labeled with distinct positive integers is called a magic labeling. As semigraph a generalized graph structure, the concept of magic labeling has been extended to semigraphs in a quite natural way. This concept is introduced in semigraph as, A semigraph \( G \) with \( p \) vertices is called magic if the vertices of \( G \) can be distinctly labeled 1 through \( p \) in such a way that when taking the sum of the vertex labels on each edge, the sums will be same.

The pioneering works on ‘Semigraphs’ by Dr. E. Sampath Kumar\(^4\) have already created much interest and enthusiasm among the graph theorists and the flow of development of this newly born idea is on the rising trend. The terminology and notations used here are in [2] and [3] unless otherwise specified. For obvious reason, all vertices on an edge of a semigraph are considered to be adjacent to one another. Accordingly, the vertices are divided into four types, namely, end vertices, middle vertices, middle-end vertices and isolated vertices.
The parameter, the capacity of a vertex similar to the degree of a vertex, was introduced and the problem of examining the graph structures in which sum of the vertex capacities is equal to sum of number of relations and total number of edges in the structure was posed in [3].

The rigorous study of dominating sets in graph theory began around 1960, even though the subject has historical roots dating back to 1862 when deJaenisch studied the problems of determining the minimum number of queens which are necessary to cover or dominate a $n \times n$ chessboard. In 1958, Berge defined the concept of the domination number of a graph, calling this as coefficient of External Stability. In 1962, Ore used the name dominating set and domination number for the same concept. In 1977 Cockayne and Hedetniemi made an interesting and extensive survey of the results known at that time about dominating sets in graphs. They have used the notation $\gamma(G)$ for the domination number of a graph, which has become very popular since then. The survey paper of Cockayne and Hedetniemi has generated lot of interest in the study of domination in graphs. In 1958, Claude Berge introduced the domination number of a graph. Domination can be a useful tool for determining certain business networks and making decisions. As Hedetniemi and Laskar(1990) note, the domination problem was studied from the 1950s onwards, but the rate of research on domination significantly increased in the mid-1970s. Business would benefit from the use of the concept of domination to strategically plan the location of their stores in order to reach the maximum amount of areas with minimal stores locations.

Domination in graphs has applications to several fields. Domination arises in facility location problems, where the number of facilities (e.g., hospitals, fire sta-
tions) is fixed and one attempts to minimize the distance that a person needs to travel to get to the closest facility. A similar problem occurs when the maximum distance to a facility is fixed and one attempts to minimize the number of facilities necessary so that everyone is serviced. Concepts from domination also appear in problems involving finding sets of representatives, in monitoring communication or electrical networks, and in land surveying (e.g., minimizing the number of places a surveyor must stand in order to take height measurements for an entire region).

At present, domination is considered to be one of the fundamental concepts in graph theory and its various applications to ad hoc networks, biological networks, distributed computing, social networks and web graphs partly explain the increased interest. Such applications usually aim to select a subset of nodes that will provide some definite service such that every node in the network is close to some node in the subset.

In chapter 2.....

As a by-product another family of graceful graphs has been identified. So in this chapter, we introduce a new graph $G_n$ related to our work. While decomposing the complete graph into graceful graphs, we defined graph $G_n$ and proved that $G_n$ graph is graceful for $n > 4$. The $G_n$ graph contains $V(G_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(G_n) = \{v_i v_i/i > 1\} \cup \{v_2 v_i/i > 2\} \cup \{v_3 v_i/i > 3\} \cup \{v_j v_n/5 \leq j \leq n - 1\}$. Using this $G_n$ graph, we decomposed $K_n$ into graceful subgraphs and derived that the structure graceful index of $K_n$ is 2 for $4 < n < 10$. Moreover, we derived one point union of $K_2$ and $K_6$ is not graceful. In this, we have analysed the various possibilities for the labeling of vertices in detail.
Also, we described that the structure graceful index of \(K_{10}\) is also 2. Moreover, we obtained the upper bound for the structure graceful index of \(K_n\), for \(n > 10\) in this chapter. In order to establish the upper bound, we derived one point union of \(K_m\) and \(K_{1,n}\) is graceful for \(2 < m < 7\). Also, we described that the structure graceful index of \(K_{10}\) is also 2. Moreover, we obtained the upper bound for the structure graceful index of \(K_n\), for \(n > 10\) in this chapter. In order to establish the upper bound, we also defined the graph \(G_{m,n}\) and derived that \(G_{m,n}\) is graceful for \(2 < m < 7\). Using this \(G_{m,n}\), we found the upper bound for \(K_n\).

In chapter 3.....

The structure harmonious index of a graph \(G\) is defined as the minimum \(k\) for which \(G\) is \(k\)-structure harmonious. In this chapter, we defined a new graph \(H_n\) and proved that \(H_n\) is harmonious for all \(n > 4\) and also we proved that the SHI of complete graphs for \(n > 4\) is less than or equal to \(m + 2\), for \(n > 4\), where \(m = \lfloor \frac{n-5}{3} \rfloor\). Also we defined the graph \(H_{m,n}\) and proved that \(H_{m,n}\) is harmonious for \(m = 3, 4, 5\) and \(H_{m,n}\) is harmonious when \(n \geq 4\). Also \(H_{m,n}\) is not harmonious for \(n \leq 3\). The structure harmonious index of many families of graphs are found to be less than or equal to 2.

Also results such as SHI of \(C_n\); SHI of \(F_n\); the six connected graphs with less than or equal to 5 nodes that are not harmonious; if \(G\) is harmonious, then the one-point union of an even number of copies of \(G\) using the vertex labeled zero as the shared point; triangular snakes \(\Delta_n\) with number of triangles congruent to \(2(mod\ 4)\) and \(C_m \times P_n\) were found to be less than or equal to 2.

We also derived \(C_{4,k}\) is harmonious, when \(k\) is odd. It is proved that \(SHI(C_{3}^{(t)}) = 2\), if \(t \equiv 2(mod\ 4)\). SHI of one point union of a triangle and \(C_n\) is 2, if \(n \neq 1(mod\ 4)\)
is also proved. Xu\textsuperscript{[19]} proved that all cycles with a chord are harmonious except \( C_6 \). Here we proved that SHI of \( C_6 \) with a chord is 2, if the chord divides \( C_6 \) into two paths \( P_2 \) and \( P_4 \) and is 3, if the chord divides \( C_6 \) into two paths \( P_3 \) and \( P_3 \). \( SHI(C_{n,k}) \leq 2 \), when \( k \geq 2 \) is also derived.

**In chapter 4.....**

In this chapter, we define standard graph structure of the semigraph \( G \) and denote it as \( SGS(G) \). We also prove the SGS of any semi graph \( G = (V, X) \) can’t be \( R_i \)-regular for any \( i, 1 \leq i \leq k \), where \( X = \{X_1, X_2, ..., X_k\} \), the standard graph structure of any semigraph is graceful, the structure graceful index of the consecutive adjacency graph \( G_{ca} \) of an edge complete \((p, 2)\) semi graph is 1, the structure graceful index of an end vertex graph of an edge complete \((p, 2)\) semigraph is \( \infty \), if \( p \geq 3 \) for type 1 and \( p > 4 \) for type 2 and type 3, the structure graceful index SGI of a zig-zag semigraph is \( \leq 2 \), the SGI of \( G_e \) for the Zig Zag semigraph is \( \leq 2 \), the SGI of the strongly complete semigraph is same as the SGI of the complete graph \( K_n \), \( K_n^{n-1} \) is graceful for all \( n \), \( K_n^n \) is graceful for all \( n \), the standard graph structure of any semigraph is harmonious, the structure harmonious index of the consecutive adjacency graph \( G_{ca} \) of an edge complete \((p, 2)\) semigraph is \( \leq 2 \), the SHI of consecutive adjacency graph of a zig-zag semigraph is \( \leq 2 \), the end vertex graph \( G_e \) of a zig-zag semigraph is harmonious. In this, we also dealt with magic labelings of edge complete \((p, 2)\) semigraphs with \( 4m \), \( 4m+1 \), \( 4m+2 \) and \( 4m+3 \) vertices.

**In chapter 5.....**

In this chapter, focus is on generalized graph structures. Here we introduce some properties of generalized graph structure and also answer the question im-
posed by Dr. E. Sampath Kumar. In this chapter, we made an attempt to answer the problem of examining the graph structures for which sum of vertex capacities is equal to sum of number of relations and the number of edges in the structure for some families of graphs viewed as graph structures.

A parameter, the capacity of a vertex similar to the degree of a vertex, was introduced and the problem of examining the graph structures in which sum of the vertex capacities is equal to sum of number of relations and total number of edges in the structure was posed in [3]. For any graph, there may be many structure representations. We consider only the standard graphs such as cycles and paths and see under which the above problem has a positive solution. The structure representation naturally arising out of chromatic partitioning is also dealt with.

In chapter 6.....

Domination in graphs has been an extensively researched branch of graph theory. The theory of domination has been the nucleus of research activity in graph theory in recent times. This is largely due to a variety of new parameters that can be developed from the basic definition of domination. In this chapter, using the definitions posed by Dr, E. Sampath Kumar\[^3\], we investigate under what conditions some familiar graphs, satisfy the domination parameters in graph structure.

**Definition 1.1.1** A graph $G$ is connected iff it has a spanning tree.

**Definition 1.1.2** A spanning tree of a graph $G$ is a spanning subgraph of $G$ that is a tree.

**Definition 1.1.3** Let $G$ be a graph. A subgraph $H$ of $G$ is called a spanning
Definition 1.1.4 A signed graph is one in which every edge has ‘+’ or ‘-’ sign.

Definition 1.1.5 A signed graph is balanced if, and only if, its vertex set \( V \) can be partitioned into two sets \( V_1 \) and \( V_2 \) such that each positive edge joins vertices of the same set, and each negative edge joins vertices of different sets.

Definition 1.1.6 A function \( f \) is called graceful labeling of a graph \( G \) if \( f: V(G) \rightarrow \mathbb{Z}_{q+1} \) is injective and the induced function \( f^*: E(G) \rightarrow \{1, 2, \ldots, q\} \) defined as \( f^*(e = uv) = |f(u) - f(v)| \) is bijection. A graph which admits graceful labeling is called a graceful graph.

Definition 1.1.7 A graph \( G \) with \( q \) edges is said to be harmonious if there is an injection \( f: V(G) \rightarrow \mathbb{Z}_q \) such that when each edge \( xy \) is assigned the label \( (f(x) + f(y))(\text{mod} q) \), the resulting edge labels are distinct.

Definition 1.1.8 A graph labeling \( f: E(G) \rightarrow \mathbb{Z} \) such that for any two vertices \( u, v \in V(G) \), we have \( \sum_{e \ni u} f(e) = \sum_{e \ni v} f(e) \) is called semi-magic. A semi-magic labeling \( f \) with \( f: E(G) \rightarrow \mathbb{N} \), an injection is a magic labeling.

Definition 1.1.9 A semigraph \( G \) is a pair \((V, X)\) where \( V \) is a non-empty set whose elements are called vertices of \( G \) and \( X \) is a set of \( n \)-tuples, called edges of \( G \), of distinct vertices, for various \( n \geq 2 \), satisfying the following conditions:

(i) Any two edges have at most one vertex in common.

(ii) Two edges \((u_1, u_2, \ldots, u_n)\) and \((v_1, v_2, \ldots, v_m)\) are considered to be equal if \( f \quad (a) \quad m = n \) and
(b) either $u_i = v_i$, $1 \leq i \leq n$ or $u_i = v_{n-i+1}$, $1 \leq i \leq n$.

Thus the edge $(u_1, u_2, \ldots, u_n)$ is same as the edge $(u_n, u_{n-1}, \ldots, u_1)$.

**Definition 1.1.10** A graph structure $G = (V; R_1, R_2, \ldots, R_k)$ consists of a non-empty set $V$ together with relations $R_1, R_2, \ldots, R_k$ on $V$ which are mutually disjoint such that each $R_i$, $1 \leq i \leq k$ is symmetric and irreflexive.

If $(u, v) \in R_i$ for some $i$, we call it an $R_i$-edge and is denoted by $uv$. One can represent a graph structure $G = (V; R_1, R_2, \ldots, R_k)$ in the plane just like a graph where each edge is labeled as $R_i$, $1 \leq i \leq k$.

**Example 1.1.11**

![Graph Structure](image)

The graph structure $G = (V; R_1, R_2, R_3)$ in this example has vertex set $V = \{v_1, v_2, v_3, v_4\}$. The $R_i$-edges, $i = 1, 2, 3$ in $G$ are as indicated in the figure.

**Example 1.1.12** Any graph $G$ can be regarded as a graph structure $G = (V; R_1, R_2)$, where $uv \in R_1$ iff $uv$ is an edge of $G$, and $uv \in R_2$ iff $u$ and $v$ are non-adjacent vertices in $G$. 
The graph structure $G = (V; R_1, R_2)$ in this example has vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $R_1 = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5\}$

$$R_2 = \{v_1v_3, v_1v_5, v_2v_4, v_2v_5, v_3v_1, v_3v_5\}.$$ 

**Definition 1.1.13** A graph structure $G = (V; R_1, R_2, \ldots, R_k)$ is said to be connected if the underlying graph of $G$ is connected.

**Definition 1.1.14** An $R_i$-path between two vertices $u$ and $v$ consists of only $R_i$-edges. An $R_i$-cycle consists of only $R_i$-edges.

**Definition 1.1.15** A set $S$ of vertices in a graph structure $G = (V; R_1, R_2, \ldots, R_k)$ is $R_i$-connected, for some $i$, if any two vertices in $S$ are connected by an $R_i$-path. A set of vertices may be $R_i$-connected for more than one $i$.

**Definition 1.1.16** A graph structure is totally edge disconnected if no two $R_i$-edges are adjacent. Then the graph structure has $R_i$-components.

**Definition 1.1.17** A dominating set for a graph $G = (V, E)$ is a subset $D$ of $V$ such that every vertex not in $D$ is adjacent to at least one member of $D$. The
domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for $G$. 