CHAPTER 6 - AN INVENTORY MODEL WITH
ALLOWABLE SHORTAGE USING TRAPEZOIDAL FUZZY NUMBERS
An Inventory Model with Allowable Shortage Using Trapezoidal Fuzzy Numbers

6.1. Introduction:

Inventory control is very important field for both real world applications and also research purpose. In earlier stage the uncertainties of inventory models are treated as randomness and are handled by using probability theory.

The first quantitative treatment of inventory was the simple EOQ model. This model was developed by Harris [34] and Wilson [91] and they are investigated in academics and industries. Later Hadley [33] analyzed many inventory systems.

In this chapter we are developing an inventory model using trapezoidal fuzzy number for holding cost, ordering cost and shortage cost. Signed distance method is used for defuzzification. Due to irregularities or physical properties of the material all the time we cannot take parameters as variables. For this situation we apply fuzzy concepts. Shortage is allowed and it is completely backlogged.

An algorithm is developed to find the optimal order quantity and also for minimizing the total cost. Sensitivity analysis is carried out through the numerical examples.
6.2. Notations

S – Maximum order level

T – Length of the plan

TC – Total cost for the period [0, T]

$T\tilde{C}$ – Fuzzy total cost for the period [0, T]

F(Q) – De-fuzzified total cost for [0, T]

$F(Q)^*$ – Minimum de-fuzzified total cost for [0, T]

$Q_0^*$ – Optimal order quantity

6.3. Assumptions

In this chapter, the following assumptions are considered:

(i) Total demand is considered as constant.

(ii) Time of plan is constant.

(iii) Shortage is allowed and it is completely backlogged.

(iv) Holding cost, ordering cost and shortage cost are fuzzy in nature.

6.4. Inventory model in Crisp sense

First, we deal an inventory model with shortages, in crisp environment. The economic lot size is obtained by the following equation:

$$Q = \sqrt{2DA} \sqrt{\frac{HT + s}{HTs}}$$

-----(6.1)
Maximum order level is

\[ S = \frac{Qs}{HT + s} \]  

\[ \text{---(6.2)} \]

The total cost for the period \([0, T]\) is

\[ TC = \frac{HTs^2}{2Q} + \frac{s(Q - S)^2}{2Q} + \frac{AD}{Q} \]  

\[ \text{---(6.3)} \]

Substituting (6.2) in (6.3) and simplifying we get,

\[ TC = \frac{HTQs^2}{2(HT + s)^2} + \frac{sQ(1 - \frac{s}{HT + s})^2}{2} + \frac{AD}{Q} \]  

\[ \text{---(6.4)} \]

The optimum \(Q^*\) and \(TC^*\) can be obtained by equating the first order partial derivatives of \(TC\) to zero and solving the resulting equations,

Optimal order quantity

\[ Q^* = \sqrt{2DA} \sqrt{\frac{HT + s}{HTs}} \]  

\[ \text{---(6.5)} \]

Minimum total cost

\[ TC^* = \sqrt{2DA} \sqrt{\frac{HTs}{HT + s}} \]  

\[ \text{---(6.6)} \]
6.5. Inventory model in Fuzzy sense

We consider the model in fuzzy environment. Since the ordering cost, holding cost and shortage cost are fuzzy in nature, we represent them by trapezoidal fuzzy numbers.

Let \( \tilde{A} \) : fuzzy ordering cost per order

\( \tilde{H} \) : fuzzy carrying or holding cost per unit quantity per unit time

\( \tilde{s} \) : fuzzy shortage cost per unit quantity per unit time

The total demand and time of plan are considered as constants. Now we fuzzify total cost given in (6.4), the fuzzy total cost is given by,

\[
T\tilde{C} = \frac{\tilde{H}TQ\tilde{s}^2}{2(\tilde{H}T + \tilde{s})^2} + \frac{\tilde{s}Q(1 - \frac{\tilde{s}}{\tilde{H}T + \frac{\tilde{s}}{2}})^2}{2} + \frac{\tilde{A}D}{Q} 
\]

\[ \text{--------}(6.7) \]

Our aim is to apply signed distance method to defuzzify the total cost and then obtain the optimal order quantity by using simple calculus techniques.

Suppose \( \tilde{A} = (a_1, a_2, a_3, a_4) \quad \tilde{H} = (h_1, h_2, h_3, h_4) \quad \tilde{s} = (s_1, s_2, s_3, s_4) \) are trapezoidal fuzzy numbers, in LR form where \( 0 < s < H \) and all are positive numbers then from (6.7) we have,
Figure 6.1: Trapezoidal fuzzy number

\[ T\tilde{C} = \frac{(h_1, h_2, h_3, h_4) \otimes TQ \otimes (s_1, s_2, s_3, s_4)^2}{2 \otimes ((h_1, h_2, h_3, h_4)T \oplus (s_1, s_2, s_3, s_4))^2} \]

\[ \oplus \frac{(s_1, s_2, s_3, s_4) \otimes Q \otimes (1 - \frac{(s_1, s_2, s_3, s_4)}{(h_1, h_2, h_3, h_4)T \oplus (s_1, s_2, s_3, s_4)})^2}{2} \]

\[ \oplus \frac{(a_1, a_2, a_3, a_4) \otimes D}{Q} \]

\[ = [a, b, c, d] \text{ (say)} \]
Now $A_L(\alpha) = a + (b - a) \alpha$

$$A_L(\alpha) = \frac{h_1s_1^2TQ}{2(h_4T + s_4)^2} + \frac{s_4Q(1 - \frac{s_4}{h_1T + s_1})^2}{2} + a_1D\frac{Q}{Q}$$

$$+ \left\{ \left( \frac{h_1s_1^2}{(h_3T + s_4)^2} - \frac{h_1s_1^2}{(h_4T + s_4)^2} \right) \frac{TQ}{2} \right\} \alpha$$

$---(6.10)$

And $A_R(\alpha) = d - (d - c)\alpha$

$$A_R(\alpha) = \frac{h_1s_1^2TQ}{2(h_1T + s_1)^2} + \frac{s_4Q(1 - \frac{s_4}{h_1T + s_1})^2}{2} +$$

$$+ a_4D\frac{Q}{Q}$$

$$\left\{ \left( \frac{h_1s_1^2}{(h_3T + s_1)^2} - \frac{h_1s_1^2}{(h_2T + s_1)^2} \right) \frac{TQ}{2} \right\} \alpha$$

$---(6.11)$
Defuzzifying $T\tilde{C}$ by using signed distance method we get,

$$d(T\tilde{C}(\tilde{A}, \tilde{H}, \tilde{S}), 0) = \frac{1}{2} \left[ A_z(\alpha) + A_R(\alpha) \right] d\alpha$$

$$TC = \frac{1}{2} \int_0^1 \left[ \frac{h_1 s_i^2 TQ}{2(h_4 T + s_4)^2} + \frac{s_1 Q(1 - \frac{s_i}{h_4 T + s_1})^2}{2} + \frac{a_1 D}{Q} \right] + \left[ \frac{h_2 s_2^2}{(h_4 T + s_3)^2} - \frac{h_1 s_1^2}{(h_4 T + s_4)^2} \right] \frac{TQ}{2}$$

$$+ \left[ s_2(1 - \frac{s_3}{h_2 T + s_2})^2 \right] + \left[ -s_1(1 - \frac{s_4}{h_4 T + s_1})^2 \right] \frac{Q}{2} + \alpha \left[ \frac{s_4 Q(1 - \frac{s_i}{h_4 T + s_4})^2}{2} \right]$$

$$+ \frac{a_4 D}{Q} + \left[ \frac{h_4 s_i^2}{(h_4 T + s_4)^2} - \frac{h_5 s_3^2}{(h_5 T + s_2)^2} \right] \frac{TQ}{2}$$

$$+ \left[ s_4(1 - \frac{s_i}{h_4 T + s_4})^2 \right] + \left[ -s_3(1 - \frac{s_2}{h_3 T + s_3})^2 \right] \frac{Q}{2} + \alpha \left[ \frac{a_4 - a_3 D}{Q} \right]$$
Integrating and simplifying we get,

\[
TC = \frac{D}{4Q} \left[ a_1 + a_2 + a_3 + a_4 \right] + \frac{TQ}{8} \left[ \frac{h_1 s_1^2}{(h_1 T + s_1)^2} + \frac{h_2 s_2^2}{(h_1 T + s_2)^2} \right] + \frac{Q}{8} \left[ s_1 (1 - \frac{s_4}{h_1 T + s_1})^2 + s_2 (1 - \frac{s_3}{h_1 T + s_2})^2 \right] + \frac{Q}{8} \left[ s_3 (1 - \frac{s_2}{h_1 T + s_3})^2 + s_4 (1 - \frac{s_1}{h_1 T + s_4})^2 \right] = F(Q) \text{ (say)}
\]

\[\text{----(6.12)}\]

**Computation of } Q_D^* \text{ at which } F(Q) \text{ is minimum:**

\[ F(Q) \text{ is minimum when } \frac{dF(Q)}{dQ} = 0, \text{ and where } \frac{d^2F(Q)}{dQ^2} > 0 \]

Now, \( \frac{dF(Q)}{dQ} = 0 \) gives the economic order quantity as:

\[
Q_D^* = \sqrt[4]{\frac{2D[a_1 + a_2 + a_3 + a_4]}{T} \left[ \frac{h_1 s_1^2}{(h_1 T + s_1)^2} + \frac{h_2 s_2^2}{(h_1 T + s_2)^2} \right] \left[ s_1 (1 - \frac{s_4}{h_1 T + s_1})^2 + s_2 (1 - \frac{s_3}{h_1 T + s_2})^2 \right] + \frac{Q}{8} \left[ s_3 (1 - \frac{s_2}{h_1 T + s_3})^2 + s_4 (1 - \frac{s_1}{h_1 T + s_4})^2 \right]}
\]

\[\text{---- (6.13)}\]
Also, at $Q = Q_D^*$, we have $\frac{d^2 F(Q)}{dQ^2} > 0$

This shows that $F(Q)$ is minimum at $Q = Q_D^*$. And from (6.12),

$$F(Q)^* = \frac{D}{4Q_D^*} \left[a_1 + a_2 + a_3 + a_4\right] + \frac{TQ_D^*}{8} \left[\frac{h_1 s_1^2}{(h_1 T + s_1)^2} + \frac{h_2 s_2^2}{(h_2 T + s_2)^2} + \frac{h_3 s_3^2}{(h_3 T + s_3)^2} + \frac{h_4 s_4^2}{(h_4 T + s_4)^2}\right]$$

$$+ \frac{Q_D^*}{8} \left[s_1 \left(1 - \frac{s_4}{h_1 T + s_1}\right)^2 + s_2 \left(1 - \frac{s_3}{h_2 T + s_2}\right)^2\right]$$

$$+ s_3 \left(1 - \frac{s_2}{h_3 T + s_3}\right)^2 + s_4 \left(1 - \frac{s_1}{h_4 T + s_4}\right)^2$$

$$------(6.14)$$

6.6. Algorithm:

**Step 1:** Calculate total cost for the crisp model for the given crisp values of $A$, $H$, $s$, $T$ and $D$.

**Step 2:** Now, determine fuzzy total cost using fuzzy arithmetic operations on fuzzy holding cost, fuzzy ordering cost and fuzzy shortage cost, taken as trapezoidal fuzzy numbers.
Step 3: Use signed distance method for defuzzification. Then find fuzzy optimal order quantity which can be obtained by putting the first derivative of $F(Q)$ equal to zero and where the second derivative is positive at $Q = Q_{D^*}$.

6.7. Numerical examples:

Example 1:

Crisp model:

Let $A = \text{Rs. 20/- per unit}$, $H = \text{Rs. 12/- per unit}$, $D = 500$ unit,

$T = 6$ days, $s = \text{Rs. 6/- per unit}$.

Then $Q^* = 60.0999$ units

$TC = \text{Rs. 332.82}$.

Fuzzy model:

Let $D = 500$ unit, $T = 6$ days, $\tilde{A} = (15, 19, 21, 24)$,

$\tilde{H} = (8, 11, 13, 16), \tilde{s} = (3, 4, 5, 6)$.

Then $Q^* = 65.8385$ units

$F(Q)^* = \text{Rs. 299.98}$.
Table 6.1. Sensitivity Analysis on demand rate

<table>
<thead>
<tr>
<th>S. No</th>
<th>Demand (D)</th>
<th>For $\hat{A} = (15, 19, 21, 24)$</th>
<th>For $\hat{A} = (15, 18, 22, 24)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$Q^*$</td>
<td>$F(Q^*)$</td>
</tr>
<tr>
<td>1</td>
<td>450</td>
<td>62.4599</td>
<td>284.5853</td>
</tr>
<tr>
<td>2</td>
<td>475</td>
<td>64.171</td>
<td>292.384</td>
</tr>
<tr>
<td>3</td>
<td>500</td>
<td>65.8385</td>
<td>299.98</td>
</tr>
<tr>
<td>4</td>
<td>525</td>
<td>67.464</td>
<td>307.388</td>
</tr>
<tr>
<td>5</td>
<td>550</td>
<td>69.052</td>
<td>314.618</td>
</tr>
</tbody>
</table>

From the above table we observed that:

(i) The economic order quantity obtained by signed distance method is closer to crisp economic order quantity.

(ii) Total cost obtained by signed distance method is less than crisp total cost.

(iii) For different values of ordering quantity by changing middle two spreads, the economic order quantity remains fixed. Same is true for total cost.

6.8. Conclusion:

In this model we have used signed distance method for defuzzifying the holding cost, ordering cost and shortage cost. These costs are taken as trapezoidal fuzzy numbers. We conclude that for an EOQ model if holding cost, ordering cost
and shortage cost are expressed as trapezoidal fuzzy numbers, then the results obtained are much better than the case of triangular fuzzy numbers used in the model proposed by Dutta and Pavan Kumar [18]. Finally we conclude that even though we are allowing shortage for an EOQ model the total cost is much lesser than the model proposed by Dutta and Pavan Kumar [18]. Numerical examples are given to illustrate this model.