3.1. Black holes and their thermodynamics

Black holes have eluded our understanding ever since the first one of their kind was discovered by Schwarzschild within the framework of General Relativity. However, over the last two decades and a half, it has been established that at least the uncharged non-rotating black holes are amenable to a thermodynamic description with its surface gravity interpreted as its temperature, known as the Hawking temperature, and one quarter of the area of its horizon (in appropriate units) — which may never diminish [70] — interpreted as its entropy, known as the Bekenstein-Hawking entropy [14]. Four laws governing the behavior of black holes, looked upon as thermodynamic systems, have been formulated [109, see [36] for an early and [183] for a recent review of black hole thermodynamics]. The classical thermodynamic analyses of black holes reveal that the Hawking temperature of the uncharged non-rotating black hole, namely the Schwarzschild black hole, is inversely proportional to its mass and the black hole has a negative specific heat thereby staking its stability. In view of the colossal gravitational attraction of a black hole, such oddities might be resolved through an understanding of its quantum behavior and the microscopic origin of its entropy.

A semiclassical consideration by treating quantum fields on a classical background geometry of a black hole leads to the prediction that the black hole emits thermal radiation [67] — known as Hawking radiation — as a perfect black-body, unlike in classical gravity, where it behaves as a perfect absorber. The mass of a black hole diminishes as it radiates, to the effect of increasing its Hawking temperature and thereby an escalated rate of radiation and this process leads to an inexorable end of the black hole. The black hole disappears by losing all its mass, leaving behind thermal radiation alone [68]. This phenomenon gives way to what is popularly known as the information-loss puzzle. The initial state of matter, which collapses to form a black hole, can be chosen to be a pure state of vanishing entropy (see, however, [119] for a recent critique of this viewpoint). The final state, however, is a thermal state with large entropy. In other words, the amount of information contained in the thermal
radiation left behind by the evaporated black hole is lesser than the information contained in the initial state which gave birth to it. Thus, information seems to have been lost on the disappearance of a black hole by evaporation. The information-loss puzzle asks, how to account for the missing entropy? Moreover, the purport of this catastrophic disappearance of the black hole is that there is no unitary time-evolution operator in a field theory on a black hole space-time, thus violating a basic tenet of quantum theory. Reconciliation of the life-style of a black hole with the infra-structure of quantum theory has been sought by hypothesizing that either the black hole is survived by a remnant of Planckian mass or that, albeit the black hole disappears completely leaving only thermal radiation, the radiation emanating from it remains in a pure state due to being correlated to the initial state of collapsing matter and radiation [65, and references therein]. These possibilities are justified by noting that after the mass of the evaporating black hole reduces to assume Planckian dimensions, the semi-classical considerations [68] fall short of depicting it anymore.

Thus it seems that the proper prediction of the fate of a black hole, and consequently the resolution of the information-loss puzzle, requires a coalition of quantum mechanics and gravity. This might be sought by altering the conventional quantum theory and/or the theory of gravity into a scheme of quantum gravity. Now that string theory is deemed to include quantum gravity, it is natural to expect an understanding of the phenomenon within the schemata of string theory.

Effective actions of various string theories possess black hole solutions in different dimensions. In two dimensions, for example, black hole solutions have been identified [113, 188, 115]. These are also amenable to a thermodynamic study [115, 120]. One can also study the process of formation and evaporation of a black hole in a semiclassical approximation within the scope of string theory. This was achieved with the study of the CGHS and RST models [27, 141]. These studies established the exigency of a full-fledged quantum theory of gravity to tackle the black hole problem.

Let us mention some of the significant features of these models. The CGHS model [27, see 65 for a review] is a toy model for studying the physics of black holes in two-dimensional string-dilaton gravity. The stationary black hole solutions of General relativity are eternal and therefore it is difficult to address the questions of their formation out of collapsing matter or their development during Hawking radiation within the scope of classical gravity. The CGHS model overcomes his shortcoming by patching the two-dimensional eternal black hole of dilaton gravity with the flat-space linear dilaton vacuum (LDV) by means of in-falling shock wave of matter. The collapse of matter leads to the formation of a horizon and one can trace the dynamic progression of the horizon within a semiclassical approximation until the horizon meets the space-time singularity, at which point the semiclassical approximation breaks down [48, 68].
Let us start with the classical action of two-dimensional string-dilaton gravity,

\[ S_{\text{classical}} = \frac{1}{2\pi} \int d^2x \sqrt{-g} e^{-2\phi} \left( \mathcal{R}(2) + 4(\nabla \phi)^2 + 4\lambda^2 \right), \]  

(3.1.1)
coupled to the action of the in-falling matter, given by

\[ S_{\text{matter}} = -\frac{1}{4\pi} \sum_{i=1}^{N} \int d^2x \sqrt{-g} (\nabla f_i)^2. \]  

(3.1.2)

In equations (3.1.1) and (3.1.2), \( g \) denotes the two-dimensional metric with Ricci scalar \( \mathcal{R}(2) \), \( \phi \) denotes the dilaton and \( \{ f_i ; i = 1, \cdots, N \} \) denotes a set of \( N \) number of massless scalar fields, that is the matter fields. The model described by the action \( S_{\text{classical}} + S_{\text{matter}} \) can be looked upon as the effective action for the radial modes propagating on the near-extreme magnetically charged black hole of four-dimensional dilaton gravity. The constant \( \lambda \) is inversely proportional to the magnetic charge of the four-dimensional black hole \([48]\). The classical action (3.1.1) possesses a time-independent or eternal black hole solution, which is expressed in the light-cone gauge as

\[ ds^2 = -e^{2\phi}dx^+dx^- + \frac{e^{-2\phi}}{\lambda^2}x^+x^-, \]  

where \( M \) is a parameter of the solution. We shall refer to the light-cone coordinates \( x^\pm \) as the Kruskal coordinates. The event horizon \((H)\) of the black hole is located at

\[ \lambda x^- = -\frac{M}{\lambda}. \]  

(3.1.5)

The mass of the black hole is given by the value of the dilaton field evaluated at the event horizon, \( M_o = \lambda/\pi e^{-2\phi_H} \). The Hawking temperature of the black hole is given by \( T_c = \frac{\lambda}{2\pi} \) and its Bekenstein-Hawking entropy is \( S = M_c/T_c = 2e^{-2\phi_H} \). The linear dilaton vacuum (LDV), corresponding to \( M = 0 \) in (3.1.3) and (3.1.4) constitutes another solution of the model. In describing the formation of the black hole (3.1.3) and (3.1.4) by collapse of in-falling matter, these two solutions are patched by an in-falling shock wave of \( \{ f_i \} \) moving in from the asymptotic past along the \( x^- \) direction. For one single scalar field \( f_1 \equiv f, N = 1 \), the matter action (3.1.2) contributes a stress-energy tensor

\[ \frac{1}{2} \partial_+ f \partial_- f = M \delta(x^+ - \frac{1}{\lambda}) \]  

(3.1.6)
to the equations of motion. The solution of the equations of motion from the coupled classical plus matter action is thus given by

\[ e^{-2\phi} = e^{-2\phi} = -\frac{M}{\lambda} (\lambda x^+ - 1) \Theta(x^+ - \frac{1}{\lambda}) - \lambda^2 x^+ x^- \]  

(3.1.7)
in the Kruskal coordinates. The equation (3.1.7) represents the formation of a black hole of mass \( M/\lambda \). For \( \lambda x^+ < 1 \), it is the LDV with a flat space-time, while for \( \lambda x^+ > 1 \), it is the eternal black hole solution (3.1.3) and (3.1.4). Now, the black hole starts radiating as soon as it comes into being. In
order to take into account the back-reaction on the background geometry due to Hawking radiation, one notes that although for a massless scalar field the trace of energy-momentum tensor vanishes classically, quantum mechanically, there is an one-loop anomaly relating the expectation value of the trace of the stress-energy tensor $T^\mu_\mu$ of the fields $f_i$ to $\mathcal{R}^{(2)}$, namely,

$$
\langle T^\mu_\mu \rangle = \frac{1}{24} \mathcal{R}^{(2)}
$$

for each massless scalar. In conformal gauge this means,

$$
\langle T^t_t \rangle = -\frac{N}{12} \partial_+ \partial_- \rho.
$$

Since the expectation value is evaluated with respect to the background metric corresponding to $\rho$, this relation introduces back-reaction on the space-time geometry due to changes in the stress-energy. The extra contribution can be shown to have arisen as a term in the equations of motion from the following non-local action:

$$
S_{\text{Liouville}} = -\frac{N}{96\pi} \int d^2 x \sqrt{-g} \mathcal{R}^{(2)} \Box^{-1} \mathcal{R}^{(2)}^{-1},
$$

where $\Box$ denotes the two-dimensional D'Alembertian. We shall refer to the model described by the action

$$
S_{\text{CGHS}} = S_{\text{classical}} + S_{\text{matter}} + S_{\text{Liouville}}
$$

as the CGHS model. This model is capable, in principle, of taking into account the effect of back-reaction in the system of gravity with collapsing matter. However, it is difficult to solve the CGHS model analytically. In order to preserve a global symmetry of the CGHS action and to make it analytically tractable, a variant of the CGHS model, namely the RST model, was introduced [141]. The action describing the RST model is given by

$$
S_{\text{RST}} = S_{\text{CGHS}} + S_{\text{counter}}
$$

where $S_{\text{counter}}$ is a counter-term given by

$$
S_{\text{counter}} = -\frac{N}{48\pi} \int d^2 x \sqrt{-g} \mathcal{R}^{(2)}.
$$

The RST model is solvable analytically in the limit of large $N$. Let us write down the solution of RST model pertinent to studying the formation and evaporation of black holes. It is given by [48],

$$
\chi(x^+, x^-) = \Omega(x^+, x^-) = -\lambda x^+ \left( x^- + \frac{1}{\lambda^2} P_+(x^+) \right) + \frac{1}{\lambda} \mathcal{M}(x^+) - \frac{1}{4} \ln \left( -4\lambda^2 x^+ x^- \right),
$$

(3.1.14)
in the large-$N$ limit, where $\mathcal{M}$ and $\mathcal{P}_+$ denote respectively the total energy measured at infinity and the total momentum that has fallen in from the asymptotic past till the retarded time $x^+$. The functions $\chi$ and $\Omega$ are defined in terms of the dilaton and the metric as:

$$\Omega = \frac{12}{N} e^{-2\phi} + \frac{1}{2} \phi + \frac{1}{4} \ln \frac{N}{48},$$

(3.1.15)

$$\chi = \frac{12}{N} e^{-2\phi} + \rho - \frac{1}{2} \phi - \frac{1}{4} \ln \frac{N}{3},$$

(3.1.16)

and we have chosen the Kruskal gauge condition, $\chi = \Omega$. The point at which $\partial_+ \Omega = 0$, is known as the apparent horizon of the black hole. Observers inside the apparent horizon are eventually attracted to the strong-coupling region of the space-time [48].

The solution (3.1.14) describes the formation, evolution and evaporation of the two-dimensional black hole. If the value of the function $\Omega$ goes below a critical value of $\frac{1}{3}$, then there exists no real value of $\phi$. The point at which $\Omega$ attains this critical value is inside the strong coupling region. This is interpreted as the origin of the radial coordinate, a boundary of the space-time. Till the boundary is time-like, one can put a reflecting boundary condition on the in-falling matter [141]. If the flux of energy of the incoming matter, defined as the rate of change of $\mathcal{M}$ with respect to the tortoise coordinate $\sigma^+$, defined as $\lambda x^+ = e^{\kappa \sigma^+}$, is less than a critical value of $\frac{1}{4} \lambda^2$, then the boundary always remains time-like and the incoming matter is reflected back from the boundary to the future null infinity. However, if the flux of energy is greater than the critical value, then there forms the apparent horizon, and behind it the boundary turns space-like. The scalar curvature $\mathcal{R}^{(2)}$ diverges: and a black hole is formed. After its formation, the black hole starts radiating and the apparent horizon evolves through a time-like trajectory and finally meets the space-time singularity. The semi-classical method works until this point. Beyond, the strong-coupling effects dominate, thus rendering the semi-classical description inadequate. There have been speculations about the fate of the black hole after the apparent horizon meets the singularity [48, 136] as well as on the possibility of avoiding this end by altering the boundary conditions from the reflecting one [166].

We conclude this discussion by emphasizing that in spite of being a toy model in two dimensions, the RST model is capable of describing the formation and evaporation of black holes analytically. It takes into account the back-reaction, a feature difficult to incorporate within General Relativity. Moreover, it has been shown that the progression of the apparent horizon is in conformity with the second law of black hole thermodynamics, entropy of the black hole increases during the evolution of the apparent horizon [118]. However, for a complete resolution of the information loss puzzle, one needs to incorporate the quantum effects in its entirety. In fact, it has been speculated that the space-time singularity might not occur in a quantum theory that can cope up with the strong coupling behavior of the string-dilaton gravity [48]. It might also be useful to consider the variants of the string-dilaton gravity. In fact, two-dimensional gravity has been studied earlier [84, 83, for example].
Some of these theories also possess black hole solutions. One of the major motivations for analyzing these two-dimensional toy black holes is to gain some insight into the nature of problems of more realistic black holes in the four-dimensional theory of gravity along with the quantum effects. The two-dimensional solutions can be obtained from higher dimensional ones, via some compactification scheme, say, and are simpler to deal with. We shall undertake a thermodynamic study of some two-dimensional black holes in the present chapter. There are several schemes to study the thermodynamic properties of black holes. Three major schemes are:

\( \Diamond \): The method of Euclidean action \([54, 55]\). This method treats the Euclideanized action corresponding to the black hole as the free energy and the periodicity of the Euclideanized time, which is introduced to avoid the conical singularity in the Euclidean metric, as the temperature of the black hole.

\( \Box \): The method of conical singularity \([167, 173, 125]\). This method functions by introducing a conical singularity in the metric as one varies the temperature defined as above.

\( \clubsuit \): The method of Noether charge \([181, 81, 82, 85]\). This method interprets the conserved Noether charge associated with the diffeomorphism invariance of the theory, evaluated on the bifurcate Killing horizon, to be defined later, as the entropy and the surface gravity of the black hole as the temperature. Equivalence of results obtained by this method and the earlier one has also been pointed out \([125]\).

Let us add that it is customary in the study of black holes to write the laws of thermodynamics in the asymptotic space-time \([55, 182, 109]\) similarly as in four dimensional gravity. For a static uncharged black hole in four dimensions, the Hawking temperature is

$$ T_c = \frac{1}{4\pi M_0}, $$

where \( M_0 \) denotes the mass of the black hole. This temperature is assumed to have been measured at the spatial infinity. Then Hawking's interpretation of the gravitational action as the entropy leads to the following asymptotically valid equation for this black hole:

$$ ST_c = \frac{M_0}{2}. \tag{3.1.17} $$

Similar laws also exist for other black holes. As mentioned above, if this equation is true for all \( M_0 \), then the specific heat of the black hole is negative; thus rendering the space-time unstable to radiation. It will lose mass with increasing temperature and is destined to meet with a catastrophic end. It has been pointed out \([25, 23]\), that the negativity of the specific heat of this black hole could be tracked down to ignoring the \( O\left(\frac{1}{r}\right) \) term in the action, where \( r \) denotes the spatial coordinate in a suitable chart. Retention of this term is tantamount to measuring the thermodynamic quantities at a finite \( r \). Therefore one needs a fourth scheme, a

\( \spadesuit \): finite-space formulation of black hole thermodynamics, which corresponds to measurement of thermodynamic quantities at a finite spatial position. Such a formulation in the
context of an effective two-dimensional string theory has been proposed in [56, see also [114, 24, 25, 82]].

In this chapter we shall discuss the thermodynamics of a class of black holes in two dimensions. First, we shall discuss the computation of the entropy using the Noether charge method. Then, motivated partly by the results of [25] in showing the connection between the stability of the four-dimensional black holes with the entropy at finite distance, we shall study various types of two-dimensional black holes. First, we explicitly show that the thermodynamic relations are well-defined for observations made from a finite distance. We then investigate the thermodynamics of a charged two-dimensional black hole with asymptotic properties similar to the ones mentioned above. In comparing our results with those obtained by other methods, e.g. the Noether charge method, we shall find that one has to make appropriate gauge transformations of the gauge potentials in order to obtain a consistent value of the entropy. We shall also study the specific heat of these solutions. An analysis taking into account the definition of the observed temperature shows that specific heat is positive for all the observers. Finally we shall compute thermodynamic quantities for some extremal black holes, and show that they have vanishing entropy.

3.2. Two-dimensional black holes

In string theory, the dilaton plays an important role in obtaining the black hole solutions and their physical behavior. For example, in two dimensions, pure Einstein gravity does not have any nontrivial solutions. But the introduction of dilaton gives rise to many interesting solutions, such as black holes [188, 113], cosmological universes [177, 52, 92] etc. The dilaton also plays an important role in defining the thermodynamics of the black holes [56]. The root of many of these new features is the non-minimal nature of the coupling of gravity and other fields to the dilaton. Therefore, the study of other non-minimally coupled scalars showing up in the low energy effective action of string theory is also of interest. Example of one such class of two-dimensional black holes have been found [26, see also [93, 106]]. These are derived from the compactification of the four-dimensional black holes in the presence of non-minimally coupled moduli fields in the extremal limit. These two-dimensional black holes asymptote to a space-time of constant negative curvature. This led to a vanishing temperature for the corresponding four-dimensional theory, except for the special case when the two-dimensional black hole corresponds to the one in string theory. However we shall show that an analogue of (3.1.17) is still valid in the two-dimensional theory. Introduction of an electric field in the low energy effective field theory ensuing from string theory has more exotic effects. In [110] the construction of such a black hole is given following the observation in [3] that the three-dimensional rotating BTZ black hole [11, 12, 5, 76, 90] might be interpreted as an electrically charged two-dimensional black hole. First let us describe the black holes whose thermodynamics we
shall study in later sections.

3.2.1. Uncharged black holes. Let us begin with a discussion of the class of two-dimensional black holes obtained in [26]. These emerge as the solutions of the two-dimensional action:

$$I = -\int_{\mathbb{M}} \sqrt{\gamma} e^{-2\phi} \left[ R^{(2)} + \frac{8k}{k-1} (\nabla \phi)^2 + \lambda^2 \right] - 2 \int_{\partial \mathbb{M}} \epsilon^{-2\phi} K, \quad (3.2.1)$$

where $K$ denotes the trace of the second fundamental form, $\partial \mathbb{M}$ denotes the one-dimensional boundary of the two-dimensional manifold $\mathbb{M}$ and $k$ is a parameter taking values $|k| \leq 1$. Here $\phi$ denotes the dilaton in two dimensions and $\lambda$ is a constant, playing the role of the cosmological constant. The action (3.2.1) reduces to the Jackiw-Teitelboim action [83, 84] for $k = 0$ and to the two-dimensional string effective action for $k = -1$ [113, 188, 56]. The equations of motion ensuing from (3.2.1) are:

$$R^{(2)} + \frac{8k}{k-1} (\nabla^2 \phi - (\nabla \phi)^2) + \lambda^2 = 0 \quad (3.2.2)$$

$$R^{(2)}_{ab} - \frac{1}{2} g_{ab} R^{(2)} - \frac{1}{2} g_{ab} \lambda^2 + 2 \nabla_a \nabla_b \phi - \frac{4}{1-k} \nabla_a \phi \nabla_b \phi$$

$$- 2 g_{ab} \left[ \nabla^2 \phi - \frac{2}{1-k} (\nabla \phi)^2 \right] = 0. \quad (3.2.3)$$

Here $g_{ab}$ denotes the metric of the space-time and $\nabla$ denotes the covariant derivative. Let us note that the second term in the action (3.2.1) is but a surface term and does not affect the equations of motion. However, it plays an important role in the thermodynamic considerations.

Let us consider the solutions of the equations (3.2.2) and (3.2.3). In view of the two-dimensional identity $R^{(2)}_{ab} = \frac{1}{2} g_{ab} R^{(2)}$, equation (3.2.3) implies the existence of an isometry. Let us define a vector

$$\zeta^a = e^{-\frac{2}{1-k} \phi} \nabla^a \phi. \quad (3.2.4)$$

Using the identity mentioned above, (3.2.3) can be rewritten as

$$\nabla_a \zeta_b + \nabla_b \zeta_a = g_{ab} \nabla^c \zeta_c. \quad (3.2.5)$$

Thus the vector $\zeta^a$ is a conformal Killing vector. Obviously, the definition (3.2.4) and hence the equation (3.2.5) is valid for $k \neq 1$. For the other special case of $k = -1$ this reduces to $\zeta^a = \nabla^a \phi$ [56]. The Hodge-dual of $\zeta^a$ is also a vector in two dimensions, namely, $\xi^a = * \zeta^a$. From (3.2.5) it follows that the dual vector is a Killing vector, obeying the Killing equation

$$\nabla_a \xi_b + \nabla_b \xi_a = 0. \quad (3.2.6)$$

In deriving this relation (3.2.6) one needs to use the property $\nabla_a \zeta_b = \nabla_b \zeta_a$ of the vector $\zeta^a$, which is an immediate consequence of (3.2.4). It also follows from the definitions that $\xi^a \zeta_a = e^{-\frac{2}{1-k} \phi} \zeta^a \nabla_a \phi = 0,$
which implies,

$$\xi^a \nabla_a \phi = 0,$$

(3.2.7)

provided the exponential factor is non-vanishing, which is true for $k \neq 1$. Now, since the vector $\xi^a$ is Hodge-dual to $\zeta^a$, for $\xi^a$ to be time-like, $\phi$ has to be a function of the spatial coordinate alone and in some local chart in which $t$ and $\sigma$ denote the temporal and the spatial coordinates, respectively, one is allowed to choose $\xi^a = (\frac{\sigma}{\sqrt{t}})^a$. Thus, the solutions of the equations of motion (3.2.2) and (3.2.3) can be written in the form:

$$ds^2 = -f(\sigma) dt^2 + d\sigma^2$$

(3.2.8)

where

$$f(\sigma) = \sinh^2(\gamma \sigma) \cosh^{2k}(\gamma \sigma)$$

(3.2.9)

$$e^{-2\phi} = e^{-2\phi_0} \cosh^{1-k}(\gamma \sigma),$$

(3.2.10)

with $\gamma = \frac{\Lambda}{\sqrt{2(1-k)}}$. These solutions are everywhere regular for any value of $k$ and have a Killing horizon at $\sigma = 0$, where $\xi^a \xi_a = 0$. For generic values of $k \neq -1$, these solutions asymptote to the anti-de Sitter (AdS) background with a linear dilaton for $\sigma \to \infty$, namely,

$$ds^2 = e^{2(k+1)\gamma \sigma} dt^2 + d\sigma^2,$$

(3.2.11)

$$\phi = \phi_0 + \frac{1}{2}(k-1)\gamma \sigma,$$

(3.2.12)

which constitute another class of solutions of the equations of motion. For the special case of $k = -1$, these describe the asymptotically flat stringy dilatonic black hole [188, 113].

3.2.2. Charged black holes. Let us now describe the charged version of the black holes mentioned above, for the special case of $k = 0$. An account of the thermodynamic behavior of the black holes for the case of $k = -1$, corresponding to the charged stringy black hole can be found in [56]. For $k = 0$, gauge fields are introduced [110] through the dimensional compactification of a three dimensional string effective action using a suggestion in [3]. The two-dimensional action in this case has the form,

$$I = - \int_{\mathbb{R}^2} \sqrt{g} e^{-2\Phi} [R^{(2)} + 2\lambda^2 - \frac{1}{4}e^{-4\Phi} F^2] - 2 \int_{\mathbb{R}} e^{-2\Phi} K,$$

(3.2.13)

where now $\Phi$ is a scalar field coming from the compactification and plays the role of dilaton for the two-dimensional action. Here $F$ denotes the electro-magnetic field strength corresponding to the gauge field $A$. The action (3.2.13) describes the Jackiw-Teitelboim theory with a gauge field. The
3. THE TRAPPING OF BLACK HOLES

equations of motion ensuing from this action are

\[ R_{ab}^{(2)} + 2 \nabla_a \nabla_b \Phi - 4 \nabla_a \Phi \nabla_b \Phi + \frac{1}{2} \epsilon^{-4\Phi} F_a^c F_b^c \]

\[ - g_{ab} \left[ \frac{1}{2} R^{(2)} + \lambda^2 + 2 \nabla^2 \Phi - 4 (\nabla \Phi)^2 - \frac{1}{8} \epsilon^{-4\Phi} F^2 \right] = 0, \]

(3.2.14)

\[ R^{(2)} + 2 \lambda^2 - \frac{3}{4} \epsilon^{-4\Phi} F^2 = 0, \]

(3.2.15)

\[ \partial_a (\sqrt{g} \epsilon^{-4\Phi} F^{ab}) = 0. \]

(3.2.16)

As in the previous case, the second term in (3.2.13) does not affect the equations of motion. In a local chart in which \( t \) and \( r \) denote the temporal and spatial directions respectively, the solution to the equations of motion take the following form,

\[ ds^2 = -(\lambda^2 r^2 - M + \frac{J^2}{4r^2}) dt^2 + (\lambda^2 r^2 - M + \frac{J^2}{4r^2})^{-1} dr^2, \]

(3.2.17)

\[ A_0 = - \frac{J}{2r^2}, \]

(3.2.18)

\[ e^{-2\Phi} = r, \]

(3.2.19)

where \( M \) and \( J \) are two parameters. An analysis as above shows that the solution can also be written in the form (3.2.8). In the non-extremal case, \( M^2 > \lambda^2 J^2 \), one can rewrite the solutions (3.2.17)–(3.2.19) by introducing the new spatial coordinate \( \rho \), defined as:

\[ r^2 = \frac{M + \sqrt{M^2 - \lambda^2 J^2 \cosh 2\lambda \rho}}{2\lambda^2} \]

(3.2.20)

by exploiting the fact that it admits a time-like Killing vector. Then we find,

\[ ds^2 = -G(\rho) dt^2 + d\rho^2, \]

(3.2.21)

where \( G(\rho) = \frac{1}{2} \frac{(M^2 - \lambda^2 J^2) \sinh^2 2\lambda \rho}{M + \sqrt{M^2 - \lambda^2 J^2 \cosh 2\lambda \rho}} \).

(3.2.22)

\( A_0 \) and \( e^{-2\Phi} \) are still given by (3.2.18)–(3.2.19) with \( r \) replaced from (3.2.20). In these coordinates the Killing horizon is at \( \rho = 0 \). We shall use this form later. Here let us mention that the parameter \( J \) in this solution gives the charge of this black hole. The metric has a curvature singularity at \( r = 0 \) for non-vanishing \( J \), as is seen from the Ricci scalar

\[ R = -2\lambda^2 - \frac{3J^2}{2r^4}. \]

(3.2.23)

It also goes asymptotically, \( r \to \infty \), to the AdS space-time.

Let us now consider the extremal limit of the solution (3.2.17)–(3.2.19), viz. \( M^2 = \lambda^2 J^2 \). It can be seen that the in the form (3.2.21) with \( G(\rho) \) given by (3.2.22) is singular in the limit \( M^2 \to \lambda^2 J^2 \), namely, \( G(\rho) = 0 \). This in fact can be tracked down to the fact that the transformation (3.2.20) itself is ill-defined in this limit, namely, \( r^2 \) becomes a constant. Thus, to have a consistent description of
the extremal black hole, we should change the coordinate to write the metric in the form (3.2.21) only after setting the charge equal to mass in (3.2.17)–(3.2.19). In that case the coordinate transformation is given by

\[ r^2 = \frac{M}{2\lambda^2} [1 + e^{2\lambda \rho}], \]  

(3.2.24)

where we have used the same notation \( \rho \) for the spatial coordinate as in the case of non-extremal black holes. We find

\[ ds^2 = -G(\rho)dt^2 + d\rho^2, \]  

(3.2.25)

where

\[ G(\rho) = \left( \frac{M}{2} \right) \frac{e^{4\lambda \rho}}{1 + e^{2\lambda \rho}}, \]  

(3.2.26)

where \( G \) in (3.2.22) and (3.2.26) are obviously different functions of the respective different coordinates \( \rho \). \( A_0 \) and \( e^{-2\lambda \phi} \) are still given by (3.2.18) and (3.2.19) with \( r \) replaced from (3.2.24). In the new coordinate system, \( (t, \rho) \), the horizon of the extremal black hole is at \( \rho = -\infty \) and the surface gravity, i.e. the Hawking temperature, turns out to be zero.

After describing the black hole solutions we are interested in, we shall now turn to analyzing their thermodynamic behavior.

### 3.3. Noether charge method for computing the entropy of a black hole

In this section we shall evaluate the entropy of the black holes discussed above using the Noether charge method [181, 85]. This method can be applied for computation of the entropy of a black hole possessing bifurcate Killing horizons in any diffeomorphism invariant theory in any space-time dimensions. The advantage of the technique over the conventional ones lies primarily in that the computation does not refer to the Hamiltonian of the theory; nor is it required to add a preferred surface term in the action (the second terms in (3.2.1) and (3.2.13), for example) and thirdly, the temporal coordinate need not be Euclideanized. Let us start by briefly recalling the scheme [181, 85, 81, see [183] for a recent review] in general.

We shall consider a space-time \( \mathcal{M}^D \) consisting of a \( D \)-dimensional manifold with a metric \( g_{ab} \) defined on it. Let \( L \) be a Lagrangian \( D \)-form of a theory defined on this space-time \( \mathcal{M}^D \), written in terms of a set of dynamical fields, including the metric and their first-order and second-order derivatives. Denoting these dynamical fields collectively by \( \psi \), let us consider the variation of \( L \) under a group of diffeomorphisms generated by a smooth vector field \( \xi^a \). The variations \( \delta \psi \) and \( \delta L \) of the fields \( \psi \) and the Lagrangian form \( L \), respectively, will be given by Lie derivatives
with respect to the vector field $\xi^a$, signifying their covariance:

$$
\delta \psi = L_\xi \psi \quad \text{and} \quad \delta L = L_\xi L = d(\xi L),
$$

(3.3.1)

where we have ignored the possibility of occurrence of an extra exact form appearing in the variation of the Lagrangian. The first variation of the Lagrangian form can be symbolically written as [181, 105, 81]

$$
\delta L = E \otimes \delta \psi + d\Theta,
$$

(3.3.2)

where the equations of motion for $\psi$ is $E = 0$, and $\otimes$ denotes a summation over the dynamical fields including relevant contractions of tensor indices. The equation of motion form $E$ is uniquely determined in (3.3.2) and this equation defines the $(D-1)$-form $\Theta$ (upto the addition of an exact form, which is inconsequential in the computation of entropy). The symplectic potential form $\Theta$ is constructed locally from the fields $\psi$, their variations $\delta \psi$ and their derivatives linear in $\delta \psi$.

One can now define the Noether current associated with the vector field $\xi^a$ as a $(D-1)$-form

$$
J = \Theta - \xi^a J_a,
$$

(3.3.3)

where $\Theta$ now is taken to be a function of $\psi$ and $\delta \psi = L_\xi \psi$. It follows from the definition of $J$, (3.3.3) and equations (3.3.1) and (3.3.2) that

$$
\delta J = d\Theta - \delta L = -E \otimes \delta \psi,
$$

(3.3.4)

which is zero when $E = 0$, i.e. when the equations of motion are satisfied. Thus, the Noether current $J$ is closed and we can write it at least locally in terms of a $(D-2)$-form $Q$ as,

$$
J = dQ,
$$

(3.3.5)

modulo the equations of motion, i.e. for the on-shell fields. The $(D-2)$-form $Q$ is referred to as the Noether charge or Noether potential associated with the vector field $\xi^a$. Equation (3.3.5) defines the Noether charge $Q$ uniquely upto the addition of a closed — and hence exact [180] — $(D-2)$-form. Examples of computations of the Noether current and Noether charge in General Relativity, dilatonic gravity and $D$-dimensional Lovelock gravity can be found in [81]. The Noether charge for a space-like hypersurface $\Sigma$ of $\mathbb{R}^D$, — a $(D-1)$-dimensional submanifold $\Sigma$ embedded in $\mathbb{R}^D$ — is given by the integral of $J$ over $\Sigma$, which in turn reduces to an integral of $Q$ over the boundary of $\Sigma$, viz. $\partial \Sigma$, by virtue of (3.3.5).

The entropy of a stationary black hole is defined in terms of the Noether charge for the Killing horizon of the black hole. Considering a smooth space-time $(\mathbb{R}^D, g_{ab})$, which admits a one-parameter group of isometries generated by a Killing vector field $\xi^a$, a Killing horizon of the space-time is defined to be a null hypersurface of $\mathbb{R}^D$ that is invariant under the action of the isometry group and on which
3.3. NOETHER CHARGE METHOD FOR COMPUTING THE ENTROPY OF A BLACK HOLE

the Killing vector is null: $\xi^a \xi_a = 0$. At this point let us recall that a bifurcate Killing horizon is comprised by the union of the Killing horizons. Moreover, if the horizon generators of a Killing horizon are geodesically complete to the past and if the surface gravity ($\kappa$), defined as

$$\frac{1}{2} \nabla^2 (\xi^a \xi_a) = -\kappa \xi^a,$$  \hspace{1cm} (3.3.6)

is non-vanishing, then the $(D - 1)$-dimensional Killing horizons intersect on a space-like $(D - 2)$-dimensional surface contained in the bifurcate Killing horizon, on which the horizon Killing vector vanishes, i.e. $\xi^a = 0$ [139]. This cross-section of all the Killing horizons is fixed under the Killing flow and is known as the bifurcation surface [139, 91]. Obviously, in a two-dimensional space-time the bifurcation surface reduces to a point.

Now, within the scheme of a theory that possesses a black hole solution with a bifurcate Killing horizon $\Sigma$, the entropy of the black hole is defined to be $2\pi$ times the Noether charge for the bifurcate Killing horizon. This is done by choosing the Killing vector field to be the generator of the Killing horizon, normalized to have unit surface gravity. Consequently, the Noether charge $Q$ is evaluated at the bifurcation surface $\partial \Sigma$,

$$S = 2\pi \int_{\Sigma} J = 2\pi \oint_{\partial \Sigma} Q.$$ \hspace{1cm} (3.3.7)

However, it can be proved that the same value for entropy is obtained by evaluating the Noether charge at any cross-section of the horizon [85].

Thus defined, the Noether charge $Q$ depends on the Killing filed and its derivatives. However, in actual computation one can dispense with an explicit reference to the Killing fields [181, 85]. Using the identity $\nabla_c \nabla_d \xi^a = R_{abcd} \xi^d$, satisfied by any Killing vector, where $R_{abcd}$ denote the Riemannian curvature tensor, one can eliminate the second and higher orders derivatives of $\xi^a$ in $Q$. The terms linear in $\xi^a$ do not contribute to the entropy, since $\xi^a$ vanishes on the bifurcation surface, while on the bifurcation surface one can write $\nabla_c \xi^a = e^a_{cd}$, the latter denoting the binormal to the bifurcation surface.

After these manipulations one can write down a general formula for the entropy of a black hole for a theory, whose Lagrangian $D$-form depends on the metric tensor and the Riemannian curvature of the space-time, as well as some matter fields $\phi_m$ and their first-order derivatives $\nabla_a \phi_m$:

$$L = L (g_{ab}, R_{abcd}, \phi_m, \nabla_a \phi_m).$$ \hspace{1cm} (3.3.8)

The expression for entropy (3.3.7) for a black hole solution of such a theory takes the following form [85, 118, 178]

$$S = -2\pi \oint_{\partial \Sigma} \left( \frac{\partial L}{\partial R_{abcd}} \right) \xi^a \xi^b,$$ \hspace{1cm} (3.3.9)
where $\epsilon_{ab} = \hat{\epsilon}_{ab}$ and $\tilde{\epsilon}$ denotes the induced volume form on the horizon cross-section.

Before proceeding to calculate the entropy of the black hole solutions mentioned earlier, it should be pointed out that the three ambiguities that enter the above construction of the Noether charge, namely, the possibility of adding an exact form to the Lagrangian $D$-form, a closed form to the symplectic form $\Theta$ and a closed form to the Noether charge $Q$ are of no consequence as far as the computation of entropy is concerned [85].

Let us now apply the technique described above to compute the entropies of the black holes (3.2.8) and (3.2.21) described in the previous section. The Lagrangian forms for the theories in question are evaluated from the first terms of the actions (3.2.1) and (3.2.13) respectively, ignoring the surface terms, as mentioned earlier. In both the cases the Lagrangian form has the form of (3.3.8) with the matter field $\phi_m$ being the dilaton $\phi$ in the action (3.2.1) and the gauge field $A$ in (3.2.13). Moreover, the black hole solutions (3.2.8) and (3.2.21) have bifurcation points at $\sigma = 0$ and $\rho = 0$, respectively, where the Killing vector $\xi^a$ vanishes. Thus one has to evaluate the Noether charge at these points in order to find the entropies of the respective black holes.

In view of the formula (3.3.9), only the curvature terms of the actions (3.2.1) and (3.2.13) contribute to the Noether charge. Thus, the entropy of the uncharged black hole (3.2.8) is given by

$$S = 4\pi e^{-2\sigma}|_{\sigma=0}$$

(3.3.10)

Similarly, for the charged black holes, (3.2.21), the entropy is given by

$$S = 4\pi e^{-2\phi}|_{\rho=0}$$

(3.3.11)

$$= \frac{2\sqrt{2}\pi}{\lambda} \left( M + \sqrt{M^2 - \lambda^2 J^2} \right)^{1/2}$$

(3.3.12)

(3.3.13)

In the next section we shall find that these expressions for the entropies of the two black holes are reproduced asymptotically in the finite-space formulation of black hole thermodynamics. We close this section by mentioning that the temperature of the black holes can be found by evaluating the surface gravity $\kappa$, defined in (3.3.6) and interpreting it as the temperature of the black hole, with the result that the temperatures of the black holes (3.2.8) and (3.2.21) are $\gamma/2\pi$ and $\frac{\sqrt{2}}{2\pi} \left( M + \sqrt{M^2 - \lambda^2 J^2} \right)^{1/2}$, respectively. From the knowledge of temperature and entropy, one can find out other thermodynamic quantities as well.

### 3.4. Thermodynamics of black holes in a finite-space formulation

In this section we shall study the thermodynamics of the black holes (3.2.8) and (3.2.21) in the finite-space formulation [56, 103]. In the finite-space formulation, the thermodynamic quantities are computed locally, corresponding to a black hole trapped inside a box of finite spatial extent, with the
observer positioned at the wall of the box. In this sense, this is a generalization of the Euclidean approach to the thermodynamics of black holes. Thus, the discussion of thermodynamics in this formulation begins with the definition of free energy derived from the evaluation of the Euclideanized action for the black holes, given by

$$\mathcal{F} = \frac{I}{\beta},$$  \hspace{1cm} (3.4.1)\

where $I$ is the Euclideanized action evaluated for the metric of the space-time under consideration and $\beta$ is the inverse temperature. Since we are concerned with the thermodynamics of the black holes at a finite spatial separation, i.e., at a finite value of the spatial coordinate that enters the metric, we have to evaluate the action with the boundary contribution on a space-like slice. In other words, the measurements are made at the wall of a box which is treated as the boundary of the space-time under consideration. This is possible if the solutions admit a time-like Killing vector $\xi^a = (\frac{\partial}{\partial t})^a$. As discussed earlier, the black holes (3.2.8) and (3.2.21) does meet this criterion. The proper periodicity of the Euclideanized time at a fixed value of the spatial coordinate is interpreted as the local temperature $T_w = \beta^{-1}$. The conserved dilaton current $j_a = \epsilon^a_b \nabla_b e^{-2\phi}$ defines another thermodynamic potential

$$\mathcal{D} = \int j$$  \hspace{1cm} (3.4.2)\

where $\Sigma$ is a space-like hypersurface bounded by the wall of the box. Here we note a direct consequence of the non-minimal coupling of the dilaton in two dimensions. In four dimensions, the dilaton field can be decoupled from the curvature term by rescaling the metric, thus going over to the Einstein frame. As a result, it becomes the part of a general matter action and does not affect the thermodynamics of the black hole [56, 89]. Further, the dilaton charge $\mathcal{D}$ in equation (3.4.2) is essentially the value of the dilaton field $e^{-2\phi}$ on the boundary, which is a scalar. Consequently this quantity is a measurable one, in contrast to the coordinates which parametrize it [56]. We should point out that, one can replace the $e^{-2\phi}$ in the definition of $j_a$ by any arbitrary function of $\phi$ and still the current will be conserved. However, the corresponding dilaton charge will be a function of the one that we are considering. Thus there will not be any new conserved quantum number associated with it. We shall therefore treat $\mathcal{F}$ as a function of the two thermodynamic quantities $T_w$ and $\mathcal{D}$. Then by the first law of thermodynamics, one can define the entropy $S$ and the dilaton potential $\psi$ as:

$$S = - \left[ \frac{\partial \mathcal{F}}{\partial T_w} \right]_{\mathcal{D}} \hspace{0.5cm} \psi = - \left[ \frac{\partial \mathcal{F}}{\partial \mathcal{D}} \right]_{T_w}.$$  \hspace{1cm} (3.4.3)\

But it is not the Helmholtz free energy, rather its Legendre transform, $\mathcal{E} = \mathcal{F} + ST_w$, that defines the non-available energy. This corresponds to the mass of the space-time, provided it exists, as the
limiting value of the difference between the energies for the black hole and its asymptotic background solution at spatial infinity.

3.4.1. Thermodynamics of the uncharged black holes. We shall study the thermodynamics of the solutions (3.2.8)–(3.2.10) for generic $k$. The special case of $k = -1$ is dealt with at length in [56]. Using the dilaton equation (3.2.2), the action (3.2.1) can be evaluated for a generic $k$ to yield

$$I = -2 \int_{\partial M} e^{-2\phi} \left( K - \frac{4k}{k-1} n^a \nabla_a \phi \right)$$

(3.4.4)

for the on-shell dilaton, where $n^a$ denotes the unit outward normal on $\partial M$. For our choice of boundary, $n^a = (0, \frac{1}{\sqrt{g_{11}}})$, and $K = \frac{\partial_t \ln \sqrt{|g_{00}|}}{\sqrt{g_{11}}}$, with suffixes 0 and 1 referring to the temporal and spatial coordinates respectively. And the action has the form

$$I = \int_{\partial M} \sqrt{1 - \frac{1}{g_{11}}} e^{-2\phi} \left( \frac{1}{2} \frac{\partial_t g_{00}}{g_{00}} - \frac{4k}{k-1} \partial_t \phi \right).$$

(3.4.5)

Then, by defining a new variable, $x = \gamma \sigma$, the Helmholtz free energy for the solution (3.2.8) can be written as

$$F = IT_w$$

$$= -2\gamma T_w e^{-2\phi} \left[ \cosh^2 x - k \sinh^2 x \right]$$

(3.4.7)

where $T_c$ denotes the proper periodicity of the Euclidean time at the horizon and is related to that at the wall, $T_w$, by Tolman's formula,

$$T_w = \frac{T_c}{\sqrt{g_{00}}}$$

(3.4.8)

The Euclideanized metric corresponding to (3.2.8) has a conical singularity as $\sigma \to 0$, unless $r \equiv \imath$ it has a periodicity $T_c = \frac{\gamma}{2\pi}$. Thus,

$$T_w = \frac{T_c}{\sinh x \cosh x}$$

(3.4.9)

and the dilaton charge for this black hole is

$$D = e^{-2\phi_0} \cosh^{1-k} x.$$  

(3.4.10)

Then using (3.4.7) and (3.4.10) we find

$$F = -2\gamma D (\coth x - k \tanh x).$$

(3.4.11)

In (3.4.11), we have eliminated the constant $e^{-2\phi_0}$ in favor of the dilaton charge $D$, the basic principle being, that one should not keep arbitrary parameters in the description of the thermodynamic
3.4. THERMODYNAMICS OF BLACK HOLES IN A FINITE-SPACE FORMULATION

quantities except those which appear in the action itself. The coordinate $x$ is kept as an implicit variable defined by (3.4.9) [56]. Since both $\mathcal{F}$ and $T_w$ depend implicitly on $x$, we can write $S$ as

$$S = - \left[ \frac{\partial \mathcal{F}}{\partial x} \right]_D \left[ \frac{dT_w}{dx} \right]^{-1}.$$  \hfill (3.4.12)

This yields, on inserting the respective expressions,

$$S = 4\pi D \cosh^{k-1} x = 4\pi e^{-2\phi_0}.$$ \hfill (3.4.13)

The black hole energy as defined by the Legendre transform of $\mathcal{F}$ is given by

$$\mathcal{E}_{BH} = -2(1 - k)T \tanh x$$ \hfill (3.4.14)

All the thermodynamic quantities listed above go over to those for the string black hole by choosing $k = -1$ [56]. In fact, the entropy is a constant of $x$ for all values of $k$ including the case of the stringy black hole. We however notice some important differences between the situations $k = -1$ and $k \neq -1$. Unlike the case of $k = -1$, the solutions (3.2.8)-(3.2.10) asymptote to the AdS linear dilaton vacuum for general $k$. As a result $T_w$ vanishes asymptotically as $e^{-(k+1)x}$, while $D$ goes to infinity as $e^{(k+1)x}$. The energy of the black hole is to be computed with reference to the AdS linear dilaton vacuum defined by (3.2.11) and (3.2.12). The free energy (3.4.6) becomes,

$$\mathcal{F}_{AdS} = -2\lambda(1 - k)D,$$ \hfill (3.4.15)

which implies $S_{AdS} = 0$ and $\mathcal{E}_{AdS} = -2\lambda(1 - k)D$. Then defining $M_0 \equiv \mathcal{E}_{BH} - \mathcal{E}_{AdS}$ we obtain

$$M_0 = 2\gamma D(1 - k)[1 - \tanh x].$$ \hfill (3.4.16)

An analogue of (3.1.17) at finite $x$ was written in [56] for $k = -1$ and has the form

$$S = \frac{M_0}{T_c} \left( 1 - \frac{M_0}{16\pi D T_c} \right)^{(1-k)/2}.$$ \hfill (3.4.17)

It can be verified that the above equation generalizes to

$$S = 4\pi D \frac{M_0}{2\pi D T_c(1 - k)} \left( 1 - \frac{M_0}{8\pi D T_c(1 - k)} \right)^{(1-k)/2}.$$ \hfill (3.4.18)

for a generic value of $k$. Note that unlike the case of asymptotically flat metric, the quantity $M_0$ for a general $k$ vanishes in the limit $x \to \infty$, by the Tolman redshift factor, as $M_0 \sim M_{ADM} e^{-(k+1)x}$, where $M_{ADM} = (1 - k)\frac{2\pi}{\gamma} e^{-2\phi_0}$ denotes the ADM-mass of the black hole. Equation (3.4.18) is one of the main results of this section. Also, one can verify that, in the asymptotic limit,

$$S T_c = \frac{2 M_{ADM}}{1 - k},$$ \hfill (3.4.19)

which is precisely the relation given in [26].
3.4.2. Thermodynamics of the charged black holes. Let us now investigate the thermodynamics of the charged version of the black hole corresponding to $k = 0$. For the case of $k = -1$, the charged black hole solution and its thermodynamics is discussed in [115] and [56] respectively. We study the thermodynamics of these black hole solutions for observations done from finite distances. In this case, the use of the equations of motion (3.2.14)–(3.2.16) implies the following value of the classical action (3.2.13):

$$I = - \int_{\partial \Omega} \left[ n^a F_{ab} A^b e^{-\phi} + 2 K e^{-2\phi} \right]. \quad (3.4.20)$$

The free energy is obtained by the evaluation of (3.4.20). We note however that there is an ambiguity in the evaluation of (3.4.20) due to the freedom of a constant shift in the gauge potential: $A_a \rightarrow A_a + \text{constant}$, in the equations of motion. Constant shifts have been applied earlier [89, 54] in the evaluation of the classical actions in order to avoid divergence in the gauge potential at the horizon. In our case, on the other hand, $A_a$ is well-defined at $\rho = 0$. But, as we shall see later, this shift is needed for the consistency of the present method of computation with the Noether's charge prescription.

Once again the temperature is given by the periodicity of the proper time in a local inertial frame around $\rho$ and satisfies the relation:

$$T_\omega = \sqrt{2} T_c \frac{[M + \sqrt{M^2 - \lambda^2 J^2} \cosh 2x]}{\sqrt{M^2 - \lambda^2 J^2} \sinh 2x}, \quad (3.4.21)$$

where $x = \lambda \rho$, and

$$T_c = \frac{\sqrt{2}\lambda}{2\pi} \frac{\sqrt{M^2 - \lambda^2 J^2}}{(M + \sqrt{M^2 - \lambda^2 J^2})^{1/2}} \quad (3.4.22)$$

is the proper periodicity at the horizon. The dilaton charge is now given by

$$D = \left[ \frac{M}{2\lambda^2} (1 + \chi \cosh 2x) \right]^{1/2}, \quad (3.4.23)$$

where we have defined a parameter $\chi = \sqrt{1 - \left(\frac{\lambda J}{M}\right)^2}$ with $\chi^2 > 0$, i.e. $\chi$ is real. One can the evaluate (3.4.20), with a shift in the gauge potential $A_\mu \rightarrow A_\mu(\rho) - A_\mu(\rho = 0)$, and the free energy is

$$F = -2\lambda D \coth x. \quad (3.4.24)$$

The form of equation (3.4.24) needs some qualification. As in (3.4.11), an implicit variable $x$ has been used in writing them. However, the thermodynamic variables are only the dilaton charge ($D$), the local temperature ($T_\omega$) and the electric charge ($Q$), defined as $Q = -\frac{1}{2} e^{-\phi} e_{ab} F^{ab}$ evaluated at...
the boundary. To show that the free energy can be written purely in terms of $\lambda$ and thermodynamic variables $T_w$, $Q$ and $D$, it suffices to record the following relations:

$$Q = 2\lambda^2 \chi \sqrt{1 - \chi^2 \sinh 2x} \over (1 + \chi \cosh 2x)^2, \tag{3.4.25}$$

and

$$X = \frac{\pi^2 T_w^2 \sinh^2 2x - \lambda^2}{\lambda^2 \cosh 2x - \pi^2 T_w^2 \sinh^2 2x}. \tag{3.4.26}$$

Since the free energy $F$ in (3.4.24) does not depend explicitly on $Q$ and $\chi$, entropy is once again computed using equation (3.4.12) and can be written as

$$S = -\frac{4\lambda D \coth x}{T_w} \left[ \frac{\chi(1 + \chi \cosh 2x)}{1 - \chi^2 - (1 + \chi \cosh 2x)^2} \right]. \tag{3.4.27}$$

The consistency of this procedure is provided by the fact that

$$\frac{d\chi}{dT_w} \equiv \frac{\partial \chi}{\partial T_w} + \frac{\partial \chi}{\partial x} \left( \frac{dT_w}{dx} \right)^{-1} = 0. \tag{3.4.28}$$

As a result, in differentiating with respect to $T_w$ and $x$, $\chi$ is taken as a constant. In the same manner as above, $S$ can also be thought to be a function of thermodynamic variables only. In the limit, $x \rightarrow \infty$, corresponding to the asymptotic infinity, we find the value of entropy approaches

$$\lim_{x \rightarrow \infty} S = \frac{2\sqrt{2\pi}}{\lambda} \left[ M + \sqrt{M^2 - \lambda^2 J^2} \right]^{1/2}. \tag{3.4.29}$$

We note that the expression (3.4.29) is in agreement with the ones derived by the Noether charge prescription (3.3.12) and other conventional ones [50, 181, 81, 118]. At this point let us re-stress the crucial role of the choice of gauge in deriving (3.4.29) for this comparison.

We now come to the stability analysis of the black holes through the evaluation of the specific heat. The space-time is thermodynamically stable to radiation provided the specific heat is positive. It is noted that, at least in those cases, in which $S$ and $T_w$ are asymptotically constants of $x$, say, $S_0$ and $T_0$, respectively, an equation of the type (3.1.17) is satisfied and the specific heat,

$$C = T_0 \left( \frac{dS_0}{dT_0} \right), \tag{3.4.30}$$

is negative, viz, $-S_0$. For the case under consideration, however, it is naive to conclude from this that the black hole is unstable.

We now compute the specific heat in the present formulation and establish the stability of the black hole solutions. The specific heat is now given by the formula:

$$C_D \equiv T_w \left[ \left( \frac{\partial S}{\partial T_w} \right)_{T_w, D} + \left( \frac{\partial S}{\partial x} \right)_{T_w, D} \left( \frac{dT_w}{dx} \right)^{-1} \right]. \tag{3.4.31}$$
3. THE TRAPPING OF BLACK HOLES

Figure 3.3. Specific heat ($C_{DQ}$) against the spatial coordinate ($x$) for the black hole solution (3.2.17) with $M = \lambda = 1$. Curves are labelled by different values of $\chi$.

NB: The origin on the x-axis is shifted. The curves start at $x = 0$

For the uncharged black holes (3.2.8)–(3.2.10), using the entropy (3.4.13), we obtain the specific heat as:

$$C_D = 4\pi(1 - k) \frac{e^{-2\phi_0}}{k + \coth^2 z},$$  \hspace{1cm} (3.4.32)

which is positive for all $|k| < 1$. Therefore one concludes that these black holes are stable. For $k = -1$, on the other hand, $C_D$ is infinite in the asymptotic limit. This conforms to the observations made earlier [48].

For the charged black hole, the specific heat for a constant $D$ and $Q$ is found to be:

$$C_{DQ} = \frac{8\pi\chi}{\lambda} \sqrt{\frac{M}{2}} (1 + \chi) \frac{\cosh^2 z(1 + \chi \cosh 2z)}{[(1 - \chi^2) - (1 + \chi \cosh 2z)^2]^2} \left[(1 - \chi^2) - (1 + \chi \cosh 2z)^2 + 2\chi(1 + \chi \cosh 2z)ight]$$

$$-2\chi^2 \sinh^2 2\pi \frac{(1 - \chi^2) + (1 + \chi \cosh 2z)^2}{(1 - \chi^2) - (1 + \chi \cosh 2z)^2},$$  \hspace{1cm} (3.4.33)

where once again we have used the constancy of $\chi$.

We have plotted $C_{DQ}$ as a function of $z$ in Figure 3.3, for certain values of $\chi$ and found that it is positive throughout. Its asymptotic value is same as that of entropy, $S$ in (3.4.29). In the other limit, $z \to 0$, the specific heat vanishes as $\sim z^2$. It is now interesting to note that for $z$ close to zero we also have $C_{DQ} \sim T_w z^2$. As is known that a power law dependence of specific heat on temperature is a signature of the presence of massless modes in a theory. Its significance in our context, in the light of masslessness of the dilaton, should be interesting to analyze.
3.5. Extremal black holes

Let us now study the thermodynamics of two-dimensional charged extremal black holes in the finite space formulation [104]. The results for the charged black hole derived in the previous section apply only to non-extremal black holes, namely, $\chi > 0$. Although the expression for entropy (3.4.27) is well-defined in the limit $\chi \to 0$, this is not quite the correct value of the asymptotic entropy for the extremal black hole, with $\chi = 0$, because, as mentioned earlier, the extremal black hole solution (3.2.25) does not follow from the non-extremal solution (3.2.18) by simply taking the limit $\chi \to 0$. Thus, the extremal black hole demands a special treatment. In fact, even in four dimensions, the non-rotating charged black hole, namely, the Reissner-Nordström black hole, has a non-vanishing entropy — due to having a finite area of the horizon — even in the extremal limit (viz. charge $\to$ mass), although the temperature tends to zero. As a result, although the extremal limit is known to be unattainable due to the cosmic censorship [129], there is no purely thermodynamic way to establish this. In thermodynamics, unattainability of absolute zero by adiabatic processes follows from Nernst’s postulate of the vanishing of entropy in this limit [28, see [184], however, for a discussion of the Nernst’s law in the context of black hole thermodynamics]. It has been advocated [69, 108, 176] that, thermodynamically, an extremal black hole is thermodynamically a different object than its non-extremal counterpart and a consistent treatment does show that its entropy vanishes. It will be interesting to see how the considerations of [69] translate to the cases treated here. In this section we apply the method of the previous section to some extremal black holes and show that the entropy in these cases is also zero, which corroborates the results of [69]. However in the present formulation the local entropy itself turns out to be zero. In this regard, the extremal black holes are thermodynamically similar to the linear dilaton vacuum solutions. We also compute the energy of the extreme black holes from thermodynamic considerations and reproduce the ADM-mass. We deal with two extremal black hole solutions. One of these is an asymptotically flat solution for the Heterotic string in two dimensions [115], and the other black hole considered is (3.2.25).

Note that since $T_c = 0$ in the cases at hand, due to a vanishing surface gravity $\kappa$, the thermodynamic quantities are to be interpreted as limiting values. For example, the free energy of the extremal black hole will be computed as

$$F = \lim_{T \to 0} \frac{T}{\beta}.$$  

(3.5.1)

We shall, for convenience of writing, not mention the limits explicitly in the following. An alternative thermodynamic scheme is developed in [120] by subtracting infinities in the action. The present formulation sweeps these infinities under the carpet as a divergent contribution to the chemical potential $\psi$.

For the sake of comparison, we shall start with the discussions of the charged two-dimensional
black hole in the Heterotic string theory [115]. These emerge as the solutions to the action:

\[ \mathcal{I} = - \int \sqrt{g} e^{-2\phi} \left[ \mathcal{R}^{(2)} + 4(\nabla \phi)^2 + \lambda^2 - \frac{1}{2} F^2 \right] \ - 2 \int e^{-2\phi} K, \]  

(3.5.2)

in the same notations as in § 3.2. A black hole solution of this theory can be written as:

\[ ds^2 = -G(r)dt^2 + \frac{1}{G(r)}dr^2, \]  

(3.5.3)

where \( G(r) = 1 - 2me^{-\lambda r} + q^2 e^{-2\lambda r} \)

(3.5.4)

\[ e^{-2\phi} = e^{-2\phi_0} e^{\lambda r} \]  

(3.5.5)

\[ A_0 = \sqrt{2q} e^{-\lambda r}, \]  

(3.5.6)

in a chart \((t, r)\). Thermodynamics of this black hole has been discussed in [56] for the non-extreme case. The solution can be written in the form (3.2.21) as:

\[ ds^2 = - \frac{(1 - \frac{2m}{r}) \sinh^2 \lambda \rho}{(1 + \sqrt{1 - \frac{2m}{r} \cosh \lambda \rho})^2} dt^2 + d\rho^2, \]  

(3.5.7)

\[ \phi = \phi_0 - \frac{1}{2} \log \left[ m + \sqrt{m^2 - q^2 \cosh \lambda \rho} \right], \]  

(3.5.8)

\[ A_0 = \frac{\sqrt{2q}}{m + \sqrt{m^2 - q^2 \cosh \lambda \rho}}, \]  

(3.5.9)

and the asymptotic entropy for the black hole is

\[ S = \frac{\pi m}{\lambda} \left[ 1 + (1 - \frac{q^2}{m^2})^{1/2} \right], \]  

(3.5.10)

which reduces to \( \frac{\pi m}{\lambda} \) in the extremal limit, viz. \( \frac{q^2}{m^2} \rightarrow 1 \). Another prescription to get non-zero asymptotic entropy in the extremal limit is also suggested in [120]. However, let us note that the transformation between the two coordinates \( r \) and \( \rho \), viz.

\[ r = \frac{1}{\lambda} \log \left( m + \sqrt{m^2 - q^2 \cosh \lambda \rho} \right) \]  

(3.5.11)

is valid only for \( q < m \), and the metric (3.5.7) is not defined in the extremal limit. For a consistent description, therefore, one should start from (3.5.3) with \( q \) set equal to \( m \) in the first place, and then transform the coordinates:

\[ r = \frac{1}{\lambda} \log \left( m + e^{\lambda(\rho - \rho_0)} \right), \]  

(3.5.12)

where \( \rho_0 \) is a choice of integration constant. Starting anew with the coordinate transformation is in conformity with [69, see also [176]] where it is maintained that the extremal and non-extremal black holes are different objects. Note that, the extremal black hole has a different topology than the non-extreme one. The horizon is now situated at \( \rho = -\infty \) unlike the non-extreme case, where horizon was at \( \rho = 0 \). Now both the black hole and the asymptotic space are described using the
We have retained the constant $\rho_0$ in the above expressions. We will however see later on that thermodynamics is independent of $\rho_0$. The extreme black hole also asymptotes to the flat space with linear dilaton

\begin{align*}
  ds^2 &= -dt^2 + d\rho^2, \\
  \phi &= \phi_0 - \frac{1}{2} \lambda (\rho - \rho_0),
\end{align*}

(3.5.17) (3.5.18)

We shall now evaluate the Euclidean action for the black hole (3.5.3)-(3.5.6). It takes the form

\begin{align*}
  I = \frac{1}{2\pi} \int_{\partial M} e^{-2\phi} \left( \frac{1}{2} \partial_\gamma g_{00} - 2 \partial_1 \phi \right) 
\end{align*}

(3.5.19)

The dilaton charge is found to be

\begin{align*}
  D = e^{-2\phi_0} (m + e^z), \quad \text{where} \quad z = \lambda (\rho - \rho_0).
\end{align*}

(3.5.20)

The free energy of this black hole is then obtained by dividing the action by the local temperature and takes the form

\begin{align*}
  \mathcal{F} = -2\lambda D.
\end{align*}

(3.5.21)

We note that the free energy depends only on one thermodynamic quantity, namely, the dilaton charge. Hence the entropy, as defined in (3.4.3), is identically zero. It also implies that the thermal energy $\mathcal{E} = \mathcal{F} = -2\lambda D$. Calculating the free energy in the similar way for the flat space linear dilaton vacuum one finds the following thermodynamic quantities:

\begin{align*}
  D_{\text{fs}} &= e^{-2\phi_0} e^z, \\
  \mathcal{F} &= -2\lambda D_{\text{fs}}, \\
  S &= 0, \\
  \mathcal{E}_{\text{fs}} &= -2\lambda D_{\text{fs}}.
\end{align*}

(3.5.22) (3.5.23) (3.5.24) (3.5.25)
Hence the mass of the black hole can be obtained as the asymptotic value of the energy difference of the two space-times as

$$M_0 = \lim_{\varepsilon \to 0} (\varepsilon - E_b) = m e^{-2\Phi},$$

which is in fact the ADM-mass of the black hole. Note that unlike the non-extremal cases, there is no change of variable involved in calculating $M_0$ due to their description by the same charts in this case. Interestingly, we also see the similarity in the expressions for free energies of the black hole and the linear dilaton vacuum solutions.

Finally, let us find out the thermodynamic quantities for the extremal black hole (3.2.17)-(3.2.19). The dilaton charge $D$ for this black hole is given by

$$V = \text{Evaluating (3.4.20), with a shift in the gauge potential $A,\hat{A,p} = A(p = -\infty)$ the free energy once again takes the form}

$$F = -2\lambda D.$$

Hence the entropy vanishes again. The thermal energy of the black hole is to be computed with reference to the AdS linear dilaton vacuum defined by

$$ds^2 = \frac{M}{2\lambda^2} e^{2\Phi} dt^2 + d\rho^2,$$

$$\Phi = -\frac{1}{2} \log \sqrt{\frac{M}{2\lambda^2} - \frac{1}{2} \lambda \rho}.$$

The free energy becomes $F_{AdS} = -2\lambda D_{AdS}$, which implies $S_{AdS} = 0$ and $E_{AdS} = -2\lambda D$. Then the mass $M_0 \equiv E - E_{AdS}$ is given by

$$M_0 = \left(\frac{m}{2}\right)^{1/2} e^{-\varepsilon}$$

This gives the correct ADM-mass of the black hole, $\frac{m}{2}$, by taking into account the redshift factor.

To summarize, in this chapter we have investigated black hole thermodynamics at finite distances for a class of black holes. We have compared the expression for entropy computed in this scheme with the one calculated by the Noether charge technique and shown that they are in agreement in the asymptotic limit. We have also argued for the thermodynamic stability of these black holes by calculating the specific heat. Similar results in higher-dimensional theories have been noticed earlier [24]. Let us point out again the role of the gauge freedom in the consistent evaluation of thermodynamic potentials. We have further shown that the entropy for the extremal black holes
vanishes in two dimensions. In fact, we found that the entropy vanishes locally as could be expected from the vanishing of the local temperature. It will be interesting to understand these results from a microscopic point of view. Also, two-dimensional Heterotic string theory have a certain kind of charged black hole solution [60] different than the ones discussed here. It will be interesting to investigate the thermodynamics of these solutions in the present framework. The duality invariance of thermodynamic quantities have been shown for black holes in string theory [77]. Whether these results are still valid for observations at finite distances is worth addressing. One can possibly generalize the results of this chapter to include a dilaton potential. We expect that in that case the nature of specific heat will differ from the results presented here, the reason being the absence of a massless mode. Finally, it will be interesting to find out a connection between the finite-space formalism [56] and the Noether charge method similarly as the connection between the approach of [25] and the Noether charge method has been established [82]. This might give rise to a means of dealing with higher curvature gravity within the present framework, as the method of Noether charge is applicable to such theories [86].
A RETROSPECTION

In this thesis, we have studied some aspects of duality symmetries in string theory and some of its consequences. We have also considered M-theory and found out its connection to the other five string theories as the second topic. The third topic of our discussion has been some aspects of the thermodynamic analysis of two-dimensional black holes.

In Chapter 1 we have shown that the low-energy effective action of Heterotic string theory enjoys a symmetry under $O(d, d + 16)$ transformations, when compactified on a $d$-torus, down to $D = 10 - d$ dimensions. Resorting to the world-sheet picture we have shown that only a subgroup of this symmetry group, namely $O(d, d + 16; \mathbb{Z})$, is a symmetry of the full string theory and this discrete symmetry group has been named T-duality. When compactified to four dimensions on a four-torus, the theory has another symmetry, under the group $SL(2; \mathbb{R})$, and once again, a subgroup $SL(2; \mathbb{Z})$ of this is the symmetry of the full theory [156]. This goes by the name S-duality. In three dimensions, the symmetry group of the low-energy effective action is $O(8, 24; \mathbb{R})$, which contains the direct product of the groups $O(7, 23; \mathbb{R})$ and $SL(2; \mathbb{R})$, as a proper subgroup [158]. A subgroup $O(8, 24; \mathbb{Z})$ of this is a symmetry of the full theory and goes by the name of U-duality. We have discussed the analysis of the symmetry structure of stationary axisymmetric Einstein equations in four dimensions, which lead to unravelling an infinite dimensional symmetry group of the theory, namely, the Geroch group. Borrowing this analysis we have studied the Heterotic string theory in two dimensions. We have identified the counterpart of the Ehlers transformation in the two-dimensional theory in absence of the moduli fields. Including the moduli fields, the symmetry of the low-energy theory is $O(8, 24)$ and the symmetry of the full theory is again its discrete subgroup $O(8, 24; \mathbb{Z})$ [159]. Finally, we have commented on the structure of symmetries in Type-I and Type-II theories.

In Chapter 2, we have discussed some aspects of M-theory. This is a conjectured theory, not known yet, that does not give in to a perturbative formulation in terms of the string coupling. We have used the description of this theory as one whose low-energy behavior is governed by the eleven-dimensional
non-chiral $N = 1$ supergravity theory. We have shown that this theory, when compactified on a circle, leads to the Type-IIA theory, and can be thought of as the strong-coupling limit of Type-IIA theory. Compactified on an orbifold $S^1/Z_2$, along with a reversal of the sign of the three-form field, this M-theory yields the $E_8 \times E_8$ Heterotic string theory. On a torus $T^2$ this gives Type-IIB theory. The connection of M-theory to Type-I theory has been discussed. The "derivation" of the duality between Heterotic and Type-I theory from M-theory is also pointed out. We have seen in these considerations that, for compactifications of M-theory on orbifolds, considerations of cancellation of gravitational anomaly plays a crucial role. We have thus considered compactifications of M-theory in six and two dimensions, where this consideration is non-trivial. We have considered M-theory on orbifolds of $T^5$ and $K3 \times S^1$ down to six dimensions and found out the spectrum of the resulting theories. On these spaces one obtains six-dimensional $N = 1$ chiral supergravity theories with more than one — namely nine — tensor multiplets and different number of vector multiplets. Finally, we have considered compactifications of M-theory on orbifolds of $T^5$, $K3 \times T^5$ and $K3 \times K3 \times S^1$ down to two dimensions. These lead to models with $(16, 0)$, $(8, 0)$ and $(4, 0)$ supersymmetries in two dimensions, respectively.

The final topic we have discussed in Chapter 3 is thermodynamics of some two-dimensional black holes. To motivate the consideration of two-dimensional black holes, we have reviewed some aspects of the CGHS and RST models. We have then studied some black hole solutions which are solutions of two-dimensional gravitational actions with non-minimally coupled scalar fields, similar to the dilaton in string theory. We have computed the entropy of these black holes by using the Noether charge method, where the entropy is given as the Noether charge corresponding to diffeomorphism invariance of the action, evaluated on the bifurcate Killing horizon of the black hole, or on any other cross-section of the Killing horizon. We have then studied the thermodynamics of the black holes, which are generally asymptotically AdS, in a finite-space formulation. We find that the laws of black hole thermodynamics can be cast in a form that reproduces the conventional ones in the asymptotic limit, although the thermodynamic quantities may vanish or diverge in that limit. Moreover, we find that the specific heat of the black holes is positive at any spatial section of the space-time, thereby signifying the stability of such black holes against Hawking radiation. We also find that the extreme black holes have vanishing entropy, if the entropy is calculated after taking the extremal limit.