PART II

QUANTUM MECHANICS IN CURVED SPACE-TIME
§ 1. Introduction and Notations:- Many interesting approaches have been made to explain the gravitational phenomena within the framework of special theory of relativity with the help of spin 2 particles (Wentzel (1949) p. 204, Madhav Rao (1950), Gupta (1952) and (1957)). There have been also some different approaches to explain the same by functional treatment (Deser (1957)) or by a drastic revision of our basic concepts (Misner (1957)). But, since these methods have met with only limited success, we have here attempted an alternative and more conventional procedure to consider this problem. We deduce the equations of elementary particles and their interactions in curved spacetime very much within the frame-work of general relativity.

To obtain these equations, the postulate, that our quantum mechanical equations and expressions are given at any point by what we already have in flat spacetime in terms of the local geodesic coordinates at that point, seems helpful. This method, however, defines an affine relationship different from what we usually take.

It is here assumed that the curvature is introduced by the energy-momentum tensor by the equation

\[ G_{\mu\nu} = \kappa J_{\mu\nu} \]  

as in the general theory of relativity.

It has not been possible to solve these equations explicitly, but in certain specific cases such
as that of a space conformal to a flat space-time, the form of these equations in interesting in the sense that the mass term is replaced by an invariant space-time function.

However, in the linear approximation when we assume that the corrections due to gravitation are small, it is possible to give explicitly the way in which they occur, but since these do not throw any new light about the exact effects of the gravitational field at very small distances at which they are expected to be important (Landau (1955)), the solutions in these approximations have not been calculated.

We shall take \( \mathbf{x} = (x^1, x^2, x^3) \) to represent the space coordinates and \( x^0 \) to represent the time coordinate in curved space-time. We throughout take natural units so that \( h = c = 1 \). The metric of the flat Minkowski space is taken as \( \delta_{\mu\nu} \) where \( \delta_{11} = \delta_{22} = \delta_{33} = -\delta_{00} = 1 \) and \( \delta_{\mu\nu} = 0 \) when \( \mu \neq \nu \). When the space is almost flat, we take the metric of the curved space-time as

\[
\eta^{\mu\nu} = \delta^{\mu\nu} + \theta^{\mu\nu},
\]

\[
\theta_{\mu\nu} = \delta_{\mu\nu} - \theta_{\mu\nu}
\]

(2a)

(2b)

where \( \theta_{\mu\nu} \) are small, and the above result corresponds to the linear approximation. Clearly, in this approximation,

\[ \theta_{\mu\nu} = \delta_{\mu\kappa} \delta_{\nu\lambda} \theta^{\kappa\lambda} \]

and thus \( \theta_{\mu\nu} \) behaves as tensors in raising and lowering indices in flat space-time. We shall always raise and lower
the indices with the metric of the flat space-time when we shall take the linear approximation.

We shall now briefly recollect some of the important results of Riemannian Geometry that we may utilise (vide e.g. Eisenhart (1949)).

§ 2. Christoffel Symbols, Curvature Tensor and Parallelism

In a metric space, Christoffel symbols of the first and second kinds are respectively defined as

\[ [i, j, k] = \frac{1}{2} \left[ \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right] \]

and

\[ \{ \ell, k \} = g ^{\ell} _{i} [i, j, k] \]

\[ = \frac{1}{2} g ^{\ell} _{i} \left( \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) \]

Under a coordinate transformation, the transformation of the Christoffel symbols of the second kind is given by

\[ \{ \ell, k \} \frac{\partial x^\ell}{\partial x^\lambda} = \{ l, j \} \frac{\partial x^i}{\partial x'^\mu} \frac{\partial x^j}{\partial x'^\nu} + \frac{\partial^2 x^\ell}{\partial x'^\mu \partial x'^\nu} \]

where the prime denotes the new coordinates and the Christoffel symbols in the new coordinates. The curvature tensor is defined as

\[ R ^{\ell} _{ijk} = \{ \ell, [k], j \} + \{ \ell, m \} \{ m, i, k \} \]
In the above, the comma denotes the partial derivative with respect to the corresponding coordinate variable, and \([k \ldots j]\) means we are to take first \(k \ldots j\) as written, and then subtract the quantity with \(k\) and \(j\) interchanged. We shall also subsequently adopt the above notation when brevity demands it.

The quantity (5) is the (mixed) curvature tensor since the vanishing of this tensor is the necessary and sufficient condition that we can obtain a coordinate system of the space in which the metric tensor is a constant i.e. that the space be flat.

By contracting the above tensor, we define the Ricci tensor given as

\[ R_{ij} = R_k{}^{i}{}_{j}{}^{k} \quad . \tag{6} \]

We can easily see that this tensor is symmetric in the indices \(i\) and \(j\).

With the help of the Christoffel symbols of the second kind, we can obtain covariant differential coefficients of a contravariant or covariant vector. Denoting these covariant derivatives by a semicolon, we have,

\[ F^\mu{}_{;\nu} = F^\mu{}_{,\nu} + \{_{\lambda}{}^\mu_{\lambda}{}_{\mu}\} F^\lambda \quad . \tag{7a} \]

and

\[ F_{\mu;\nu} = F_{\mu,\nu} - \{_{\mu}{}_{\nu}{}_{\lambda}\} F^\lambda \quad . \tag{7b} \]
With the help of Christoffel symbols, we can also write down the covariant differential coefficients of tensors of higher rank.

In such a space, the equation of the geodesic is

$$\frac{d^2 x^\iota}{ds^2} + \left\{ \iota, j, k \right\} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0,$$

where $s$ denotes the invariant arc length parameter of the curve. Geodesics are the "straightest" lines in curved space.

We can also use Christoffel symbols to define parallel displacement of a vector along a curve. For this purpose, we first define infinitesimal parallel displacement. Let us have a vector $F^i$ defined at a point $P$ with coordinates $x^i$, and let $Q$ be a neighbouring point with coordinates $x^i + dx^i$. Then we assume that due to a parallel displacement of the vector $F^i$, the change in any component of this vector is linear in the components of this vector and in the displacement $dx^i$. Then we can write

$$dF^i = -L^i_{jk} F^j dx^k.$$

$L^i_{jk}$ are called the components of the affine relationship of the space. The negative sign in equation (9) is a matter of convention. Now, let us consider parallel displacement along the curve $x=x(t)$. Then equation (9) gives us,
Equation (10) gives us the value of the vector at any point of the curve as a result of parallel displacement along the curve when we know the value of the vector at some point of the curve.

If the affine connection is symmetric, i.e.,

\[ L^i_{jk} = L^i_{kj}, \]

then, in a metric space, applying that the length of a vector is not changed by a parallel displacement, we can prove that

\[ L^i_{jk} = \{ i \}^{j} \}_{k}, \quad (11) \]

and thus we find that the symmetric affine connection is given by the Christoffel symbols of the second kind.

\[ \xi^i = \left( \frac{dx^i}{ds} \right)_P \]

Thus, on the curve, with \( s \) as the arc length measured from the point \( P \),

\[ x^i = x^i_P + \xi^i s + (1/2!) \left( \frac{\partial^2 x^i}{\partial s^2} \right)_P s^2 + (1/3!) \left( \frac{\partial^3 x^i}{\partial s^3} \right)_P s^3 \]

+ \ldots \ldots \ldots \ldots \}
Then, by repeated use of equation (2), we have,

\[ x^i = x_p^i + \xi^i s - (1/2!) \left\{ \frac{\partial y^j}{\partial x^k} \right\}_p \xi^j \xi^k s^2 - \frac{1}{3!} \left( \Gamma^i_{jkl} \right)_p \xi^j \xi^k \xi^l s^3 - \ldots \]  

(12)

where \( \Gamma^i_{jkl} \) are determined in terms of the Christoffel symbols at the point \( P \) and its derivatives at that point.

Let us substitute \( \bar{x}_p^i = \xi^i s \).

Then equation (12) gives us,

\[ x^i = x_p^i + \bar{x}_p^i - (1/2!) \left\{ \frac{\partial y^j}{\partial x^k} \right\}_p \bar{x}_p^j \bar{x}_p^k - (1/3!) \left( \Gamma^i_{jkl} \right)_p \bar{x}_p^j \bar{x}_p^k \bar{x}_p^l - \ldots \]  

(13)

In equation (13), \( \bar{x}_p^i \) are the Riemannian coordinates corresponding to the point \( P \) and the original coordinate system \( x^i \). With the bar denoting the corresponding quantities in the Riemannian coordinates, we know,

\[ \left\{ \frac{\partial y^j}{\partial x^k} \right\}_p = (\Gamma^i_{jkl} \ldots)_p = 0 \]  

(14)

It is well-known that equation (13) possesses an inverse in a sufficiently small neighbourhood of \( P \). Hence, whatever equation we may write down in terms of Riemannian
coordinates at \( P \), we can transform it to the more general coordinate frame.

After obtaining Riemannian coordinates, we have further to make a linear transformation with constant coefficients to obtain (local) normal coordinates at \( P \), the differential invariant form for which is the same as that of flat space-time.

The characteristic feature of these coordinates is that the ordinary derivative here is equivalent to the covariant derivative. The same property can be maintained if, instead of Riemannian coordinates, we take geodesic coordinates given as (with the prime distinguishing these coordinates and the corresponding quantities in these coordinates),

\[
x^i = x_p^i + \left( \frac{1}{i!} \right) \{ \frac{1}{j} \}^k_p x^j_p x^k_p - \left( \frac{1}{3i!} \right) C^i_{jkl} x_p^j x_p^k x_p^l - \ldots
\]

(15)

The coefficients \( C^i_{jkl} \ldots \) are here quite arbitrary. Also, \( \{ 1 \}^i_{jk} = 0 \), which gives us the equivalence of ordinary derivatives and the covariant derivatives for such a coordinate system.

\section*{4. Distant Parallelism and Absolute normal coordinates:}

Equation (10) above gives us a parallel displacement of a vector for two distant points
which depends on the curve joining the two points. This will be independent of the path only when the space is flat. We now wish to define distant parallelism in curved space that does not depend on the path. For this purpose, we assume the following postulates about the structure of the space which is of dimension $n$ (Thomas (1934) p. 16).

A) There exists a unique affine connection for the determination of the affine properties of the space.

B) At each point $P$ of the space there is determined a configuration consisting of $n$ independent vectors issuing from $P$.

C) Corresponding vectors in the configurations determined at two points $P$ and $Q$ of the space are parallel.

D) The components $t^i_j$ of the vectors (where $j$ denotes the system of $n$ vectors and, $i$, the components of the individual vectors), determining the configuration at any point $P$ of the space, are analytic functions of the coordinates.

It will be presently seen that $t^i_j$ define the local coordinate system at any point $P$. When we denote by $T_i^j$ the inverse of $t^i_j$, we have,

$$t^i_j T^j_k = \delta^i_k ; \quad t^i_j T_j^i = \delta^j_k . \quad (16)$$

By postulate C, since the fundamental $c$ vectors $t^i_j$ ($i$ denotes the variable components) are parallel, we must have for any curve the equation

$$\frac{dt^i_j}{ds} + \Delta^i_{kl} t^k_j \frac{dx^l}{ds} = 0 ,$$
which gives us,

$$t_{j,k}^i + \Delta_{lk}^i t_{j}^l = 0.$$ 

In the above, $\Delta_{lk}^i$ are the components of the affine connection. Hence, by equation (16) we obtain,

$$\Delta_{lk}^i = -t_{j,k}^i T_{l}^j = t_{j}^i T_{l}^j k.$$ \hspace{1cm} (17)

We can obtain the paths in a space of distant parallelism by the above affine connection. Such paths, i.e. 'curves which are generated by continuously displacing a vector parallel to itself along its own direction', are solutions of the system of equations

$$\frac{d^2x^i}{ds^2} + \Delta_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} = 0,$$

which can be written as

$$\frac{d^2x^i}{ds^2} + \Lambda_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} = 0.$$ \hspace{1cm} (18)

with

$$\Lambda_{jk}^i = \frac{1}{2} (\Delta_{jk}^i + \Delta_{kj}^i)$$ \hspace{1cm} (18')

In order to associate normal coordinates in this space of distant parallelism, we have to assume further the following postulates, (Thomas (1934) p. 88).

A') With each point of the space of distant parallelism there is associated a normal coordinate system $X^i$ having its origin at that point.
B') The coordinate axes of the normal system $X^i$ at each point $P$ of the space are tangent to the directions determined by the fundamental vectors at $P$, the coordinates $X^i$ being so chosen that the conditions

$$\frac{dx^i}{dX^j} = t^i_j$$

are satisfied at $P$.

C') The paths in equation (18) above which pass through the origin of the normal system $X^i$ have the form

$$x^1 = \xi^1 s$$

where $\xi^1$ are constants.

We can now obtain an equation similar to equation (13) for these normal coordinates. This will be

$$x^i = x^i_p + t^i_j(P) X^j - (1/3!) H^i_{jkl}(P) X^j X^k - (1/3!) H^i_{jkl}(P) X^j X^k X^l - \cdots \quad (19)$$

where the coefficients $H^i_{jkl}(P)$ are determined by equation (18).

It may be seen that so long as the fundamental vectors $t^i_j$ remain the same, under a coordinate transformation of $x^i$, these normal coordinates do not change. This is the reason why we call these coordinates absolute normal coordinates. These normal coordinates will however transform the same way when we consider a transformation of the fundamental vectors $t^i$. 

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§ 1. General Theory:

Einstein's gravitational equation is given as

\[ T_{\mu\nu} = \kappa \mathcal{J}_{\mu\nu} \]  

(1)

where

\[ T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \]  

(2)

with \( R \) as the contracted Ricci tensor. In equation (1), \( \mathcal{J}_{\mu\nu} \) is the energy momentum tensor and \( \kappa \) is a universal constant. When we take the linear approximation given by equations (2), Section-1, and when the coordinate condition

\[ \frac{\partial \theta_{\mu\nu}}{\partial x_{\nu}} - \frac{1}{2} \frac{\partial^2 \theta}{\partial x^\mu \partial x^\nu} = 0 \]  

(3)

is satisfied, we have

\[ G_{\mu\nu} = -\frac{1}{2} (\Box \theta_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} \Box \theta) \]  

(4)

In equations (3) and (4) we have used the notation \( \theta = \delta_{\mu\nu} \theta_{\mu\nu} \) and \( \Box = \delta_{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} \). Hence we obtain from equation (1)

\[ \Box \theta_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} \Box \theta = -2\kappa \mathcal{J}_{\mu\nu} \]  

(5)

Equations (3) and (5) will give us the value of \( \theta_{\mu\nu} \) when we know the value of \( \mathcal{J}_{\mu\nu} \) for any classical matter field.

When we assume that quantised fields are
included in our description, we replace equation (5) by
\[ \Box \theta_{\mu \nu} - \frac{1}{2} \delta_{\mu \nu} \Box \theta = -2 \kappa \langle \mathcal{J}_{\mu \nu} \rangle \] (5')

where \( \langle \mathcal{J}_{\mu \nu} \rangle \) is the expectation value of the energy-momentum operator for our dynamical system.

We now take \( \theta_{\mu \nu} \) to be known, and consider the equations for elementary particles in such a case.

Bhabha (1949) has given a set of linear equations for elementary particle-fields of nonzero rest mass as
\[ (\alpha^\mu \frac{\partial}{\partial x^\mu} + m) \psi = 0 \] (6)

where the \( \alpha \)-matrices satisfy different algebraic rules for different particles in flat space-time. We assume tentatively that in this relationship of these matrices, the \( \delta^{\mu \nu} \) of the flat space is changed to \( q^{\mu \nu} \), and equations (6) remain the same in form. A justification of this procedure will be seen later on.

Thus for a Dirac particle, we should have, replacing \( \alpha \)-matrices by \( \gamma \)-s,
\[ \gamma'^{\mu} \gamma'^{\nu} + \gamma'^{\nu} \gamma'^{\mu} = 2 q^{\mu \nu} \] (7)

where the primed quantities here and henceforward indicate the corresponding changed quantities in curved space-time.

Clearly, the relationship (7) is satisfied in the first order in \( \theta^{\mu \nu} \) if we take
\[ \gamma'^{\mu} = \gamma^{\mu} + \frac{1}{2} \theta^{\mu \nu} \gamma_{\nu} \] (8)
where
\[ \gamma_\nu = \delta_{\nu\lambda} \gamma^\lambda \]  \hspace{1cm} (8')

Again, we can see by direct verification that to the same order in \( \theta^{\mu \nu} \), the changed Duffin-Kemmer relationship
\[ \beta^\mu \gamma^\nu \beta^\lambda + \beta^\nu \gamma^\lambda \beta^\mu = g^{\mu \nu} \beta^\lambda + g^{\nu \lambda} \beta^\mu \]  \hspace{1cm} (9)

is satisfied when we define \( \beta^\mu \) in terms of the Duffin-Kemmer matrices in a way similar to equation (8) for \( \theta^\mu \) in terms of \( \gamma^\mu \). This is also obvious when we adopt for the \( \beta \) -matrices the representation (Corson (1953) p.41)
\[ \beta^\mu = \frac{1}{2} (I \times \gamma^\mu + \gamma^\mu \times I) \]

where the \( \gamma \)-s are the above mentioned Dirac matrices and the cross denotes direct product.

The discussion above includes particles of spin \( \frac{1}{2}, 0 \) and 1, which are physically most important. Hence in equation (6) we take the general changed \( \alpha \)-matrices as
\[ \alpha'^\mu = \alpha^\mu + \frac{1}{2} \theta^{\mu \nu} \alpha^\nu \]  \hspace{1cm} (10)

a form justifiable for at least the above set of particles. Equation (6) now changes to
\[ (\alpha'^\mu \frac{\partial}{\partial x^\mu} + m) \psi'(x) = 0 \]  \hspace{1cm} (11)

In equation (11) we substitute \( \psi'(x) = \psi(x) + \delta \psi(x) \), where \( \psi(x) \) is the solution of equation (6) and \( \delta \psi(x) \) is a small correction to this field. Then we obtain the
In order to solve equation (12), we require the Green's functions $G(x,x')$ with

$$ (\alpha^\mu \frac{\partial}{\partial x^\mu} + m) G(x,x') = - \delta (x-x'), \quad (13) $$

which gives us,

$$ \delta \psi(x) = (1/2) \int G(x,x') \theta^{\mu\nu}(x') \frac{\partial \psi(x')}{\partial x^\mu} d^4x'. \quad (14) $$

The relationship (10) for curved space-time is seen to satisfy the requirements of the algebra of the $\sigma$-matrices in a formal way for Dirac and Duffin-Kemmer cases. We can obtain to the same approximation the set of equations (11) from an entirely different consideration that is physically more satisfactory, and hence may be taken to embrace all other equations of the same type.

At any point $0$ in space-time having coordinates $x_0$ we choose local coordinate system such that we have, at $0$, with $X^\alpha$ denoting these coordinates,

$$ \delta_{\mu\nu} dX_0^\mu dX_0^\nu = g_{\mu\nu}(x_0) dx^\mu dx^\nu \quad (15) $$

which gives

$$ \delta_{\mu\nu} = t^\lambda_\mu(0) t^\kappa_\nu(0) g_{\lambda\kappa}(x_0), \quad (15') $$

where

$$ dX_0^\mu = T^\mu_\nu(0) dx^\nu, \quad dx^\mu = t^\mu_\nu(0) dX_0^\nu. \quad (15'') $$
give us the relationship between the lengths of the infinitesimal measuring rods and intervals of time as viewed by an observer in the local frame of reference at 0 and an observer in the general frame of reference. It is easily seen that the equations \((15')\) are satisfied when we take

\[
t^\lambda_\mu(0) = \delta^\lambda_\mu + \frac{1}{2} \theta^\lambda_\mu(x_0)
\]

in the linear approximation.

Now we assume that the original set of equations \((6)\) are satisfied at the point 0 in terms of the local coordinates at 0. Thus at this point we have,

\[
(\alpha^\mu \frac{\partial}{\partial x^\mu} + m) \psi' = 0
\]

which in terms of original coordinates becomes

\[
(\alpha^\mu t^\lambda_\mu(0) \frac{\partial}{\partial x^\lambda} + m) \psi' = 0
\]

Hence, by equation \((16)\), we obtain an equation identical with equation \((11)\) when we note that

\[
\theta^\lambda_\mu(x_0)\alpha^\mu = \theta^\lambda_\mu(x_0) \alpha^\mu.
\]

Since 0 is any point in space-time, the above physical assumption gives us the same set of equations as obtained earlier in a formal manner. This leads us to the statement of a general postulate:

**Postulate A.** Our quantum mechanical equations and expressions at any point are given by what we already have in flat space-time in terms of the local coordinates at that
point.

The above postulate corresponds to the fact that in general relativity the space is locally flat.

§2. Equations in a space conformal to a flat space-time

In this case we take the metric tensor as (Eisenhart (1949) p. 89)

\[ q^\mu\nu(x) = \delta^\mu\nu \Lambda(x) \quad (19) \]

Hence,

\[ q_{\mu\nu}(x) = \delta_{\mu\nu} \left( \Lambda(x) \right)^{-1}. \]

In the above, \( \Lambda(x) = 1 + \lambda(x) \) is an invariant spacetime function, and \( \lambda(x) \) can be taken as small when we take the linear approximation. Now, for equations (15') and (15''), we can take,

\[ t^\mu\nu(x) = \delta^\mu\nu \left( \Lambda(x) \right)^{\frac{1}{2}} \]

and

\[ T^\mu\nu(x) = \delta^\mu\nu \left( \Lambda(x) \right)^{-\frac{1}{2}} \quad (20) \]

such that, by equation (20), equation (17) reduces to

\[ (\alpha^\mu \frac{\partial}{\partial x^\mu} + \frac{m}{\Lambda(x)}) \Psi'(x) = 0 \quad (21). \]

Equation (21) has the interesting feature that in such a space the equation for the elementary particle is slightly changed in the term involving the rest-mass of the particle, which is now replaced by a space-time function.

In order to solve for \( \Lambda(x) \), we note
that here the coordinate condition (3) is not satisfied.
A direct evaluation of \[ G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} q_{\mu\nu} R \] gives us the result that
\[
G_{\mu\nu} = (\Lambda(x))^{-1} \left( -\frac{\partial^2 \Lambda(x)}{\partial x^\mu \partial x^\nu} + \delta_{\mu\nu} \Box \Lambda(x) \right) \\
+ (\Lambda(x))^{-2} \left( \frac{1}{2} \frac{\partial \Lambda}{\partial x^\lambda} \frac{\partial \Lambda}{\partial x^\nu} - \frac{5}{4} \delta_{\mu\nu} \delta^{\lambda_k} \frac{\partial \Lambda}{\partial x^\lambda} \frac{\partial \Lambda}{\partial x^\kappa} \right).
\] (22)

Here we are interested in a dynamical system that gives rise to a curved space conformal to a flat one. This is possible only if the equations (1) are satisfied. We wish to evaluate \( \Lambda(x) \) in this particular case, and for this purpose it is sufficient to evaluate
\[
q^{\mu\nu} G_{\mu\nu} \equiv -R = \kappa q^{\mu\nu} \mathcal{F}_{\mu\nu} 
\]
which gives rise to the equation
\[
3 \Box \Lambda(x) - \frac{q}{2} (\Lambda(x))^{-1} \delta^{\lambda_k} \delta_{\nu} \frac{\partial \Lambda}{\partial x^\lambda} \frac{\partial \Lambda}{\partial x^\kappa} = \kappa q^{\mu\nu} \mathcal{F}_{\mu\nu}.
\] (23)

It may be noted here that \( \mathcal{F}_{\mu\nu} \) must have certain symmetry so that when we obtain the solution of equation (23), each one of equations (1) should also be satisfied.

Equation (23) can be written down in a slightly better form by substitution \( \Lambda(x) = \exp(\alpha(x)) \).
On the right hand side of the above equation, since \( \phi_{\mu \nu} \) is a scalar, at any point we can take this as equal to \( \delta_{\mu \nu} J_{\mu \nu} (0; \text{local}) \), and by postulate A, we can write down the usual expression for \( J_{\mu \nu} (0; \text{local}) \) of flat space. Hence equation (23) after simplification becomes

\[
\exp(\alpha(x)) \delta_{\mu \nu} \left[ \frac{\partial^2 \alpha(x)}{\partial x^\mu \partial x^\nu} - \frac{1}{2} \frac{\partial \alpha}{\partial x^\mu} \frac{\partial \alpha}{\partial x^\nu} \right] = \left( \frac{\kappa}{3} \right) \delta_{\mu \nu} J_{\mu \nu} (0; \text{local}). \tag{24}
\]

In the linear approximation, however, the above equation attains the very simple form

\[
\square \lambda(x) = \left( \frac{\kappa}{3} \right) \delta_{\mu \nu} J_{\mu \nu} (\text{local}), \tag{25}
\]

which can be solved easily once we know the value of \( \lambda(x) \) in terms of the local coordinates for any particle-field.

In this approximation, equation (21) simplifies to

\[
(\alpha^\mu \frac{\partial}{\partial x^\mu} + m) \Psi'(x) = \frac{1}{2} m \lambda(x) \Psi'(x)
\]

and for the correction term, equation (12) becomes

\[
(\alpha^\mu \frac{\partial}{\partial x^\mu} + m) \delta \Psi(x) = \frac{1}{2} m \lambda(x) \Psi(x), \tag{26}
\]

which does not involve the derivative of flat space-time wave-function, as was the case for the more general type of space.
In particular, for the Dirac particle, we can take in the local coordinates,

\[ \mathcal{J}_\mu^\nu (0, \text{local}) = -\frac{i}{2} \left( \overline{\psi} \gamma_\mu \frac{\partial \psi}{\partial x_\mu} - \frac{\partial \overline{\psi}}{\partial x_\mu} \gamma_\mu \psi \right) \]

such that, applying equations of motion, we get,

\[ \delta^{\mu^\nu} \mathcal{J}_\mu^\nu (0, \text{local}) = m \overline{\psi} \psi. \]

(27)

On the right hand side of the above equation, \( \overline{\psi} \psi \) can be interpreted as the probability density of finding the particle in space-time volume around the point \( 0 \) in terms of the local coordinates at that point. We represent this invariant function by \( \mathcal{Q}(x) \). Once we know its value, we may solve the exact equation (24) or the approximate one (25). Thus, equation (25) becomes

\[ \Box \mathcal{Q}(x) = m(\kappa/3) \mathcal{Q}(x) \]

(28)

where \( \mathcal{Q}(x) \sqrt{-\det \mathcal{g}_{\mu\nu}(x)} \, d^4x \) indicates the probability of finding the particle in the space-time volume \( d^4x \).

In the above, throughout we have assumed that the fields are not second quantised. In the latter case, however, we are to take \( \mathcal{J}_\mu^\nu \) wherever we have \( \mathcal{J}_\mu^\nu \).

\[ \S \ 3. \ Conditions \ of \ Covariance: \]

The method of deducing equation (11) from the equation (17) with the postulate stated is very interesting because it gives us a method for writing down the equations in an exact manner and solving them in some special cases. But, for purposes of
covariance, we must rather take normal coordinates (Section-1, § 3) in which case, not only is the metric of the form (15) at the point, but the ordinary derivatives are identical with the covariant derivatives, so that the equation (17) or any similar equation or expression is covariant under a coordinate transformation when it is covariant under rotation in a flat space.

For this purpose, we recall the coordinate transformation (12) of Section-1, which has a unique inverse so long as we confine our attention to a domain characterised by the fact that through any point of the domain and through the point P, only a single geodesic can pass. Hence, if the above transformation is a bi-unique transformation throughout space-time, we must have as necessary condition that, space-time world must have infinite extension with absolute past and absolute future. However, when we are interested in this transformation for deducing properties at P or in its immediate neighbourhood, the above assumption of the equation referred to always having an inverse may be taken for granted.

In the postulate A, we must always take the (local) normal coordinates or the (local) geodesic coordinates at any point, in which case not only is the metric of the form required, but the Christoffel symbols vanish at that point in this particular coordinate system. With this restriction (Postulate A!) the covariance of postulate A for any equation is established.

Let us assume that at the point 0 we have
obtained the local coordinates satisfying the above requirements such that equations (15") are satisfied. Now, considering equation (1) at the point 0, we can adopt the usual value of $\mathcal{G}_{\mu \nu}$ in the local coordinate system by the postulate $A'$, in terms of the wave-function and the derivatives of the fields in the same coordinate system,

$$\mathcal{G}_{\mu \nu}(x_0) = T^\lambda_{\mu}(0) T^\nu_\kappa(0) \mathcal{G}^\kappa_{\lambda}(0; \text{local geodesic}) \quad (29)$$

and thus we obtain the equation

$$G_{\mu \nu}(x_0) = \kappa T^\lambda_{\mu}(0) T^\nu_\kappa(0) \mathcal{G}^\kappa_{\lambda}(0; \text{local geodesic}). \quad (30)$$

In the above equation, we may regard $G_{\mu \nu}(x_0)$ as a function of the matrix $\| T^\lambda_{\mu}(x_0) \|$ by the equation (15'), such that equation (30) becomes a coupled differential equation and may be solved for a given energy-momentum tensor. Then, the equations of the elementary particles can be written down by using postulate $A'$ and finally substituting

$$\frac{\partial}{\partial x_0^\mu} \bigg|_0 = t^\lambda_{\mu}(x_0) \frac{\partial}{\partial x^\lambda} \bigg|_0 \quad (31)$$

Also, we can convert any tensor occurring in the local frame of reference to the general frame of reference by using $\| T^\lambda_{\mu}(x_0) \|$ or $\| t^\lambda_{\mu}(x_0) \|$.

Equation (30) is however, in any case, very much complicated and is almost impossible to solve. On the other hand, if we assume to know the metric of the space,
in order to find out the equations of the elementary particles, we are still to find out $t^\mu(x_p)$ and $T^\mu(x_p)$ from equation (13) of Section-1, and using these and their derivatives, we can generalise any result of flat space to curved space-time. The calculations in all these cases are very complicated. For example, even for the simple transformation,

$$x^i = x^i_p + x^i_p - (1/2) \left\{ \left( \begin{array}{c} i \\ j \end{array} \right) \right\}_p x^j_p x^k_p,$$  \hspace{1cm} (32)$$

it is not possible to write down the inverse transformation explicitly. Only the solution in the linear approximation is easy, but this coincides with the conventional results and does not give us any new information.

Also, when we take into account the necessity of covariance, the above treatment for the space conformal to a flat space-time is seen to be invalid, and a more detailed analysis in the line mentioned is necessary.

§ 4. Second Quantisation:-- Previously, in the linear approximation, we have expressed in equation (14) the change in the wave-function of any particle-field when the metric of the space is known. We shall assume that the flat space wave-function $\psi(x)$ remains the same. This then gives us a quantisation of the total wave-function. But this process is unique only when we can express the curved space field operator in terms of the flat space one unambiguously. In the linear approximation, we can write the commutator
or the anti-commutator of the field as
\[
i S'(x, y) \equiv \left[ \Psi'(x), \overline{\Psi}'(y) \right]
\]
\[
= \left[ \Psi(x), \overline{\Psi}(y) \right] + \left[ \delta \Psi(x), \overline{\Psi}(y) \right] + \left[ \Psi(x), \delta \overline{\Psi}(y) \right]
\]
(33)

with
\[
\left[ \Psi(x), \overline{\Psi}(y) \right] = i S(x - y)
\]
(34)
taken according to Udgaonkar (1952). We are to take the corrections according to equation (14).

The Green's functions of equation (13) for the general particle fields have been given by Gupta (1955) and in the physically more important cases, by Schwinger (1951). Similar to equation (33), we can also write down the expression for the propagator as
\[
\Psi'(x) \overline{\Psi}'(y)
\]
\[
= \Psi(x) \overline{\Psi}(y) + \delta \Psi(x) \overline{\Psi}(y) + \Psi(x) \delta \overline{\Psi}(y)
\]
(35)

where dots denote contractions in the sense of Wick (1950).

The above statements relate to two aspects of the problem. Firstly, we may have an external gravitational field, where the metric is known. Then, the above results give us second quantisation directly. On the other hand, we may wish to include the gravitational effect of the dynamical system under consideration. Then, we must first find out the metric by taking \( < \gamma_{\mu\nu} > \) for our dynamical system and then proceed as before.
§ 5. Interacting Fields:

To consider interaction, we first note that Tomonaga equation has to be now taken as

\[
i \frac{\delta \Psi'(\sigma)}{\delta \sigma(x) \sqrt{-g(x)}} = H_I(x) \Psi'(\sigma)
\]  

(36)

In the above equation, \( g(x) = \text{det} g_{\mu\nu}(x) \), such that \( \sqrt{-g(x)} \, d^4x \) is an invariant volume element, \( H_I(x) \) is the invariant interaction Hamiltonian density and \( \Psi'(\sigma) \) is the state vector functional on the space-like surface \( \sigma \). This equation is consistent only if the integrability condition is satisfied. For example, in absence of derivatives in \( H_I(x) \) we must have,

\[
[H_I(x), H_I(x')] = 0
\]

whenever \( x \) and \( x' \) are separated by a space-like interval.

Writing \( \Psi'(\sigma) = U'(\sigma) \Psi'(-\infty) \), and proceeding according to the standard method of Dyson, we obtain,

\[
U'(\infty) = 1 + \sum_{n=1}^{\infty} S_n'
\]  

(37)

where

\[
S_n' = (-i)^n/(n!) \int d^4x_1 \ldots d^4x_n \, P(\sqrt{-g(x_1)} H_I(x_1) \ldots \sqrt{-g(x_n)} H_I(x_n))
\]  

(38)

Thus, while evaluating the P-bracket above, besides the usual contractions, there will be extra terms.
arising out of $\delta \psi(x)$ or $\delta \overline{\psi}(x)$ that might be present in $H_1(x)$ in the linear approximation. These will involve the integrals of the differential coefficients of the flat space propagators. However, we can use the equation (14) in the linear approximation only if the integrability condition is satisfied.

§ 6. Gravitational Corrections for the Meson and the Electromagnetic Fields:— We shall first consider the corrections to a meson field since here the equations are explicitly in the form (6), whereas for the electromagnetic field, we are to replace $m$ in the equation (6) by a suitable singular matrix. We shall take the case of a scalar meson, since this case is simple and since we can directly go over to pseudoscalar mesons that are important for the nuclear forces. The meson field will be described by a 5x5 irreducible representation of the $\beta$-matrices satisfying Duffin-Kemmer rules. The representation we choose is (Kemmer (1939))

$$
\beta^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \beta^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
\beta^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \beta^0 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \chi = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{bmatrix}.
$$
The $\beta^\mu$ above are Duffin-Kemmer matrices of flat space-time and $\chi$ is the corresponding wave-function. The changed wave-function in the linear approximation thus becomes

$$\chi' = \chi + \frac{1}{2} \int g(x,x') \theta^\nu(x') \beta_\nu \frac{\partial \chi(x')}{\partial x^\mu} dx', \quad (39)$$

where $g(x,x')$ is the Green's function given by Gupta (1955).

We know that in our representation $\chi_5$ is a scalar and satisfies Klein-Gordon equation. We shall explicitly deduce this well-known result here as follows:

We first note that

$$\langle \beta^\mu \beta^\nu \rangle_\alpha = \delta^\mu_\alpha \delta^\nu_\alpha \quad (40)$$

Let us now consider the equation

$$(\beta^\mu)^\lambda_\tau \frac{\partial \chi_\tau}{\partial x^\mu} + m \chi_\lambda = 0 \quad (41)$$

which also means

$$(\beta^\mu)^5_\lambda \frac{\partial \chi_\lambda}{\partial x^\mu} + m \chi_5 = 0 \quad (42)$$

Substituting the value of $\chi_\lambda$ from equation (41) in equation (42), we get,

$$- (\beta^\mu)^5_\lambda (\beta^\nu)^\lambda_\tau \frac{\partial^2 \chi_\tau}{\partial x^\mu \partial x^\nu} + m^2 \chi_5 = 0,$$

which, by equation (40), becomes,

$$(\Box - m^2) \chi_5 = 0 \quad (43)$$
We have explicitly written down equations (40) to (42) since the corresponding equations are useful to obtain that of \( \mathcal{K}_5' \). Proceeding in the same way as above for the set of equations in curved space (where \( \beta^1 \nu \) are now functions of \( x \)), the equation for \( \mathcal{K}_5' \) becomes

\[
-(\beta''^\mu)^{\lambda} (\beta''^\nu)^{\lambda} \frac{\partial^2 K_{\nu}^i}{\partial x^\mu \partial x^\nu} - (\beta''^\mu)^{\lambda} \frac{\partial}{\partial x^\mu} (\beta''^\nu)^{\lambda} \frac{\partial K_{\nu}^i}{\partial x^\nu} + m^2 K_{\nu}^i = 0 ,
\]

which in the linear approximation reduces to

\[
(\gamma^\mu^\nu + \theta^\mu^\nu) \frac{\partial^2 K_{\nu}^i}{\partial x^\mu \partial x^\nu} + \frac{1}{2} \frac{\partial \theta^\mu^\nu}{\partial x^\mu} \frac{\partial K_{\nu}^i}{\partial x^\nu} - m^2 K_{\nu}^i = 0 .
\]

We could also obtain the above equation by using the postulate A or A'. Then we have,

\[
(\gamma^\mu^\nu \frac{\partial}{\partial X^\mu} \frac{\partial}{\partial X^\nu} - m^2) K_{\nu}^i = 0 .
\]

But, in the linear approximation, we can take, by equation (16),

\[
\frac{\partial}{\partial X^\mu} = \frac{\partial}{\partial x^\mu} + \frac{1}{2} \theta^\lambda^\nu \frac{\partial}{\partial x^\lambda}
\]

and hence the changed D'Alembertian is given as

\[
\square' \equiv \square \text{ (local )}
\]

\[
= (\gamma^\mu^\nu + \theta^\mu^\nu) \frac{\partial^2}{\partial x^\mu \partial x^\nu} + \frac{1}{2} \frac{\partial \theta^\mu^\nu}{\partial x^\mu} \frac{\partial}{\partial x^\nu} ,
\]

(47)
which is the operator acting on $X^5$ in equation (46) instead of $q$ alone and thus is identical with equation (45).

The exact expression for the D'Alembertian obtained in the above manner is

$$\square' \equiv \square \text{(local geodesic)}$$

$$= \delta^{\mu\nu}(t^\lambda_{\mu} \frac{\partial}{\partial x^\lambda})(t^\nu_{\nu} \frac{\partial}{\partial x^\nu})$$

$$= g^{\mu\nu} \left[ \frac{\partial^2}{\partial x^\mu \partial x^\nu} - t^\lambda_{\mu} \frac{\partial}{\partial x^\lambda} (T^\nu_{\lambda}) \frac{\partial}{\partial x^\nu} \right] , \quad (48)$$

where we have used equations similar to (15') for $g^{\mu\nu}$ and that $\frac{\partial}{\partial x^\lambda}(t^\nu_{\nu} T^\nu_{\lambda}) = 0$.

Here we note a certain similarity of the right hand side of equation (48) to the invariant differential parameter given as (Eisonhart (1949) p. 41)

$$g^{\mu\nu} \left[ \frac{\partial^2}{\partial x^\mu \partial x^\nu} - \{ \frac{\lambda}{\mu} \} \frac{\partial}{\partial x^\lambda} \right] . \quad (49)$$

The expression (49) is the same as right hand side of equation (48) provided that

$$\frac{\partial}{\partial x^\mu}(T^i_{\nu}) = \frac{\partial}{\partial x^\nu}(T^i_{\mu}) \quad (50)$$

and

$$\frac{\partial}{\partial x^\mu}(t^i_{\nu}) = \frac{\partial}{\partial x^\nu}(t^i_{\mu}) , \quad (51)$$
so that we have
\[ T^i_{\alpha} \left\{ \gamma^\mu_{\nu} \right\} = \frac{\partial}{\partial x^\mu} ( T^i_{\nu} ) \] (52)
and
\[ t^k_{\mu} t^j_{\nu} \left\{ t^i_{\mu} \right\} = -\frac{\partial}{\partial x^k} ( t^i_{\nu} ) \] (53)

In deducing equations (52) and (53), we have used that the Christoffel symbols vanish in the local geodesic coordinate system. But it must be noted that equations (50) and (51) mean that the equations for the \( X^i \) coordinates are integrable, and hence that the space is in fact a flat one.

To write down the equation of the electromagnetic field in the Duffin-Kemmer form, we take the representation (Corson (1953) p. 41)
\[ \beta^\mu = \frac{1}{2} ( I \sqrt{\gamma} \sqrt{\gamma} + \gamma^\mu \gamma^\ell I ) \] (54)
and consider first the flat space equation
\[ (\beta^\mu \frac{\partial}{\partial x^\mu} + \gamma ) \psi = 0 \] (55)
where \( \psi \) is a sixteen component column vector and \( \gamma \) is a singular matrix given as
\[ \gamma = \text{diag} \left[ 1,1,0,0,1,1,0,0,0,0,1,1,0,0,1,1 \right] . \]

We note that in equation (55) and subsequently, we are to remember the difference in notation because of the choice of the flat space metric - i.e. \( \gamma^\mu \) (Corson) = \( -i \gamma^\mu \) (ours);
otherwise the notation here is the same as that of Corson. For the curved space-time, however, we are to replace \( \beta^\mu \) by \( \gamma^\mu \), which is equivalent to replacing \( \gamma^\mu \) by \( \gamma'^\mu \) in equation (54). When we write \( \Psi \) as a square matrix \( \Psi'' \), equation (55) takes the form

\[
\frac{1}{2} \left( \gamma^\mu \frac{\partial \Psi''}{\partial x^\mu} + \frac{\partial \Psi''}{\partial x^\mu} \gamma'^\mu \right) + (\Psi \Psi'') = 0
\] (56)

in flat space-time, and

\[
\frac{1}{2} \left( \gamma'^\mu \frac{\partial \Psi'''}{\partial x^\mu} + \frac{\partial \Psi'''}{\partial x^\mu} \gamma'^\mu \right) + (\Psi \Psi''') = 0
\] (56')

in curved space-time. In equation (56'), an interpretation similar to that of \( \Psi'' \) is to be taken; i.e.

\[
\Psi''' = -\frac{1}{4} \left[ \frac{i}{2} G_{\mu\nu} \gamma_5 \gamma'^\mu \gamma'^\nu + i \gamma_\mu \gamma_5 \gamma'^\mu
\right.
\]

\[
\left. + \gamma'^\mu \gamma'^\nu - i\varphi' - q' \gamma_5 \right] \epsilon^0 .
\] (57)

It is to be noted that in equation (57)

\[
y_5 = -\frac{\sqrt{g(x)}}{4!} \epsilon_{\mu\nu\chi\kappa} \gamma'^\mu \gamma'^\nu \gamma_\chi \gamma_\kappa
\]

remains the same as in flat space-time.

As before, we now explicitly obtain the equation (56') by using the flat space equations in the local geodesic coordinates. For this purpose, we first note that
\[
\frac{\partial}{\partial x^\sigma} \left( G_{\mu \nu} \text{ (local geodesic) } \gamma^5 \gamma^\mu \gamma^\nu \right) \gamma^\sigma \\
= t^\alpha_\sigma \frac{\partial}{\partial x^\sigma} \left( t^\lambda_\mu t^\kappa_\nu G^\prime_{\lambda \kappa} \gamma^5 \gamma^\mu \gamma^\nu \right) \gamma^\sigma \\
= \frac{\partial}{\partial x^\sigma} \left( G^\prime_{\lambda \kappa} \gamma^5 \gamma^\lambda \gamma^\kappa \right) \gamma^{\prime \sigma} \\
\] 

(58)

where we have made use of the relationship

\[
\gamma^{\prime \mu} = t^M_\nu \gamma^\nu \\
\] 

(59)

which are exact equations instead of approximations (8) or (10), and which agree with the fact that the $\gamma^\mu$ should transform like tensors. Thus, when we write equation (56) in local geodesic coordinates, other terms of it that will arise in addition to the left hand side of equation (58), will also transform in a similar way to the general coordinate system. Hence, taking equation (55) to be true in the local coordinate system, and using equation (59) where necessary, the same equation (56') is obtained. This also shows that the interpretation (57) is the appropriate one.

With this equivalence of the two procedures in mind, we can write down the electro-magnetic equations in curved space-time. Here, we use the electromagnetic equations of flat space

\[
\delta^\nu_\sigma \frac{\partial}{\partial x^\sigma} G_{\mu \nu} = 0
\]

and

\[
G_{\mu \nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \\
\]

(60)

and the Postulate A'. In the linear approximation, the
equations thus obtained may be solved in the manner already mentioned with the help of the corresponding Green's functions.

§ 7. Affine Relationship: It has already been pointed out that the differential operators (48) and (49) are equivalent provided the equations (50) and (51) are satisfied, which are not valid for a curved space-time. But a glance at equations (48) and (49) suggests that we may take an alternative affine relationship

$$\Delta^\alpha_{\mu \nu} = t^\alpha_{\alpha} T^\alpha_{\mu \nu}$$

(61)

where the comma denotes differentiation with respect to the corresponding space-time variable. When $T^\alpha_{\mu}$ exists only as a space-time function and not as a differential coefficient of an (integrable) flat space coordinate system $x^\alpha$, $T^\alpha_{\mu \nu}$ and $T^\alpha_{\nu \mu}$ will generally be different, and thus $\Delta^\alpha_{\mu \nu}$ in equation (61) is not symmetric in $\mu$ and $\nu$.

This affine relationship was chosen by Einstein (1928) in his description of unitary field theory. We see that this affine relationship is a 'natural' one, since it has been derived with a fairly physical assumption, and defines the 'natural' derivatives of a contravariant or covariant tensor just as the ordinary covariant derivatives are defined with the usual affine relationship. This is easily seen from the fact that
\[ F^\gamma_{\alpha\beta} \equiv t^\gamma_{\alpha} T^\beta_{\nu} F^\nu_{\gamma} \text{(local geodesic)} \]

\[ = t^\gamma_{\alpha} T^\beta_{\nu} \frac{\partial}{\partial x^\nu} (F^\alpha_{\gamma}) \]

\[ = t^\gamma_{\alpha} T^\beta_{\nu} \gamma_{\nu}^\mu \frac{\partial}{\partial x^k} (T^\alpha_{\mu} F^\nu) \]

\[ = F^\gamma_{\alpha\nu} + \Delta^\gamma_{\nu\mu} F^\mu \quad (62) \]

In the above and henceforward the semicolon denotes the natural derivative defined in the above manner as opposed to equations (7) of Section-1, and this again demonstrates why we should take \( \Delta^\gamma_{\nu\mu} \) as the affine relationship.

Green (1958a) has started with an affine relationship

\[ \frac{1}{2} (\gamma^\mu_{\nu} + \gamma^\nu_{\mu} \gamma^\rho_{\lambda}) \quad (63) \]

on consideration of its transformation property, which is demonstrated to be the same as that of an affine relationship (equation (4), Section-1). But, use of equation (59) for our case gives us that the expression (63) reduces to

\[ \frac{1}{2} t^\gamma_{\alpha} T^\beta_{\mu\nu} (\gamma_{\alpha}^\nu \gamma_{\beta}^\mu + \gamma_{\beta}^\nu \gamma_{\alpha}^\mu) \]

\[ = t^\gamma_{\alpha} T^\alpha_{\mu\nu} \]

and thus is identical with the affine relationship defined by equation (61). Hence we obtain, that when the affine relationship of Green is a numerical multiple of the unit
matrix, as it must be when the curved space \( g^{\mu\nu} \) are
given by equation (59), this affine relationship is the
'natural' one. We also find that the necessary and suffi­
cient condition imposed by Green for his affine relation­
ship to be a numerical multiple of the unit matrix, i.e.:

\[
\gamma'^{\mu} = \gamma'^{\mu},\gamma - \Delta^\lambda \mu^\nu \gamma'^{\lambda},
\]

\[
= \frac{\partial}{\partial x^\nu} (T_\alpha^\mu \gamma^\nu) - t_\alpha^\lambda T_\alpha^\mu \gamma^\nu \gamma'^{\lambda},
\]

\[
= 0 \text{ , (64)}
\]
is automatically satisfied.

A parallel displacement of a vector \( F^\mu \)
with respect to any curve \( x = x(t) \) for our affine rela­
tionship is given by equation (10), Section-1, written as

\[
\frac{dF^\mu}{dt} + \Delta^\mu \nu \lambda F^\nu \frac{dx^\lambda}{dt} = 0 \text{ .}
\]

Such a displacement is independent of the path when the
system of equations

\[
\frac{\partial F^\mu}{\partial x^\lambda} + \Delta^\mu \nu \lambda F^\nu = 0 \text{ . (65)}
\]

have a solution. For this, we need the integrability condi­
tion

\[
\frac{\partial^2 F^\mu}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 F^\mu}{\partial x^\nu \partial x^\lambda} = 0 \text{ . (66)}
\]

Simplification of the left hand side of equation (66) by
equation (65) gives us (we refer to the notation following equation (5), Section-1):

\[
\frac{\partial^2 F^{\mu}_{\nu\kappa}}{\partial x^\nu \partial x^\kappa} - \frac{\partial^2 F^{\mu}_{\nu\kappa}}{\partial x^\kappa \partial x^\nu} = (\Delta^{\mu}_{\nu[x,\kappa]} + \Delta^{\kappa}_{\nu[x,\mu]} \Delta^{\mu}_{\alpha \kappa}) F^{\nu}.
\]  

(67)

Thus, we have, equation (65) is integrable when

\[
\Delta^{\mu}_{\nu[x,\kappa]} + \Delta^{\kappa}_{\nu[x,\mu]} \Delta^{\mu}_{\alpha \kappa} = 0.
\]  

(68)

We shall presently see that equation (68) is true. This is proved easily by using equation (64) (Green (1958a)). Since ordinary derivatives are commutative, we have,

\[
\gamma'_{\nu,\lambda,\kappa} - \gamma'_{\nu,\kappa,\lambda} = 0.
\]  

(69)

But by equation (64),

\[
\gamma'_{\nu,\lambda,\kappa} = (\Delta^{\mu}_{\nu\lambda,\kappa} + \Delta^{\kappa}_{\nu\lambda} \Delta^{\mu}_{\alpha \kappa}) \gamma'_{\mu}.
\]  

(70)

Since the \(\gamma\)-matrices are linearly independent, equations (69) and (70) prove the result (68). Hence we have by equation (65), the parallel displacement of any vector here is independent of the path of displacement. With the Christoffel symbols of the second kind as the affine relationship, this is true only when the space is flat. However, equation (68) does not put any such restriction on the space.
§ 8. Discussions: In the above, we have considered in some detail the effect of taking Postulate A, i.e., assuming that our equations are the same as obtained in the framework of special theory of relativity when we consider the local coordinates at any point. As has already been mentioned in § 3, this procedure is not covariant under the transformations of general theory of relativity, since the ordinary derivatives occur in our equations, and these are still not covariantly defined in the local coordinate system. For this reason, we have further restricted that we should choose such local coordinates, that the Christoffel symbols at the point where the local coordinates are taken, should also vanish. But, with this restriction, it has been impossible to solve any case exactly when we already assume to know the metric of the space, even for the simple example of the space that is conformal to a flat space-time.

The most interesting result that has been obtained, however, is the definition of an unsymmetric affine relationship from the curved space D'Alembertian with the help of the Postulate A' which enables us, in its turn, to define distant parallelism of vectors and also natural derivatives. The advantages of these concepts originally introduced by Einstein (1928), have been particularly emphasized by Green in his recent work (1958 a & b ).

We however, note that the definition of the above affine relationship presupposes certain structure of the space that has been mentioned in Section-1, § 4.
Such an assumption, in fact, is necessary in order that we may be able to apply Postulate $A'$. Consistent with the above definition of the affine relationship, the most preferred system of local geodesic coordinates of Postulate $A'$ would be the absolute normal coordinates that have been quoted in § 4 of Section 1. However, we can speak of the advantage or even the possibility of the present scheme only when we apply it to some definite examples. For this purpose, we need the fundamental vectors that have been defined in Section 1 and have been used throughout the present work. Unfortunately, these are quite complicated to evaluate for any reasonable type of space, and we may have to resort to some type of approximation for any conclusions to be drawn.
CONCLUSIONS, PART- II.

We shall here briefly enumerate the findings and conclusions of Part-II:

1) We find that the Postulate A, i.e., that the expressions and equations of the elementary particles in a local frame of reference at any point remain the same as the flat space ones, is useful to obtain equations of elementary particles in curved space-time. The equations obtained by this method in the linear approximation coincide with the linear approximation otherwise taken.

2) With the Postulate A quoted above, the equations of elementary particles in a space conformal to a flat space can be written down without the linear approximation. In such equations the mass term is replaced by an invariant space-time function.

3) Postulate A is not always covariant for transformations of general relativity. Covariance here can be maintained by specifying the local coordinates of Postulate A to local geodesic coordinates (Postulate A').

4) Postulate A' defines the unsymmetric affine relationship often referred to as the 'natural' affine relationship. This requires certain assumptions regarding the structure of the space, and the local geodesic coordinates quoted above should be specified more appropriately to only absolute normal coordinates.
REFERENCES, PART-II.


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