CHAPTER VI

MHD BOUNDARY LAYER FLOW DUE TO A ROTATING DISK WITH PERIODIC SUCTION AND HALL CURRENT
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6.1. Introduction

In the study of unsteady boundary layer flow with or without fluid rotation we come across interesting phenomena like flow separation, instability, turbulence, resonance and many other situations. The method of prevention of such phenomena has evoked great interest in a number of research workers. In the present times applications of the porous properties of wall are being made in various branches of engineering and technology. Suction has long been known as an important preventive to delay separation of boundary layer and instability of the flow. In some cases these undesirable flow phenomena are controlled by altering the geometry of the flow. For example, in Thornley problem (1969) resonance behaviour can be checked by the presence of another co-axial disk. The imposition of a magnetic field is often found to have a similar controlling effect.

Unsteady boundary layer flows with suction have some practical significance in boundary layer control systems. Stuart (1955) discussed the fluctuating flow over an infinite plate when the suction is constant but the free stream

velocity fluctuates about a steady non-zero mean. Watson (1958) developed the solution for an arbitrary free stream. Kelly (1965) initiated the study of the flow of a viscous fluid past a wall of infinite extent when a time dependent suction is applied to the wall. He has shown that the suction induces an unsteady flow parallel to the wall, and that if the suction varies periodically with time, the shape of the mean profile changes. He has further shown that no solution is possible for blowing, though the solution is possible for suction. Messiha (1966) made an attempt to examine the effect of variable suction velocity upon the skin friction and heat transfer in the boundary layer flow near an infinite flat plate. The problem was generalized by Puroshothaman (1977) by including coriolis force with $\varepsilon \ll 1$.

Gupta (1960b) and Kakutani (1961) investigated the hydromagnetic flow of an electrically conducting viscous fluid past an infinite porous plate in the presence of a transverse magnetic field. Gupta (1975) studied the MHD flow past an infinite porous plate with Hall current. His study was confined to the effect of suction (blowing) on non-rotating fluid model. It was seen that the solution exists for both suction and blowing at the plate. The flow and heat transfer in the hydromagnetic Ekman layer on a porous flat plate have been analysed by Mazumder et al (1976b). They have included Hall effects in their study. It was observed that asymptotic solution exists for suction as well as blowing at the plate. [When a vast expanse of viscous fluid
bounded by an infinite flat plate, rotates about an axis normal to the plate, a layer is formed near the plate such that the viscous and the Coriolis forces are of the same order of magnitude. This layer is usually known as the Ekman layer.

We can find quite a few papers in recent years on unsteady boundary layer flows for example those due to Debnath and Mukherjee (1973, 1974), Debnath (1973, 1974, 1975), Mazumder (1977) and Debnath and Senagupta (1977). All these studies are associated with boundary layer flows with rotation of the fluid bounded by a porous flat plate with uniform suction (or blowing) except the last one which concerns with variable suction (or blowing). These authors have devoted their attention to the analysis of the effects of rotation, suction (or blowing) and many other related aspects. The study of Debnath and Senagupta concerns with the hydrodynamic and hydromagnetic flows of a rotating fluid near an infinite porous wall which is in solid body rotation with the fluid under the additional assumption that the plate performs non-torsional oscillations of frequency \( \omega \). They observe in their study that there exists two boundary layers at the plate. Mazumder (1977) investigated the combined effects of rotation and Hall current on the hydromagnetic flow due to the non-torsional oscillations of a porous plate. The structure of the steady and unsteady velocity field and the associated multiple boundary layers are considered in this problem.
Thoruley (1968) studied in detail the occurrence of modified Stokes layer around a non-torsionally oscillating disk when both the fluid and the disk are in solid body rotation. It was observed that a modified Stokes layer is formed at the disk for all frequencies except for the resonance frequency which is equal to twice the angular velocity of rotation. Thoruley established that the resonance behaviour can be prevented (i) by the presence of a co-axial disk or (ii) if the disk rotates with a velocity of constant magnitude as the basic angular rotation. Puroshothaman (1979) considered the effect of variable suction or blowing on the unsteady boundary layer flow in a rotating fluid model. In this case both the fluid and the disk are in a state of rigid body rotation with constant angular velocity about an axis normal to the disk. It is found that meaningful solution exists for the case of suction, while resonance behaviour occurs in case of blowing, corresponding to certain values of the frequency of oscillation of the normal velocity and the angular velocity.

In the present study we have considered the hydro-magnetic boundary layer flow due to rotating viscous fluid system near a non-torsionally oscillating disk with periodic suction or blowing. The suction velocity at the disk is taken as \( w_0 (1 + \varepsilon A \cos \omega t) \). The velocity components \( u \) and \( v \) in the plane of the disk are given by

\[
u + iv = U \left[ c_1 + \varepsilon (a_1 \exp (i\omega t) + b \exp (-i\omega t)) \right]
\]
where \( U \) is a constant with the dimension of the velocity and \( a, b, c \) are complex constants. A Fourier expansion is made for the velocity components and the resulting coupled equations are solved. The possible influence of the magnetic field transverse to the disk and parallel to the axial direction is studied. Hall effect of the motion has also been considered. It is found that magnetic field totally eliminates the possibility of occurrence of resonance behaviour of the boundary layer, while the Hall effect has no additional contribution in the prevention of resonance.

6.2. Formulation of the problem

We consider the hydromagnetic axisymmetric flow of a semi-infinite expanse of an electrically conducting viscous fluid past an infinite porous disk at \( z = 0 \) with variable suction or blowing. Both the disk and the fluid are assumed to rotate as a solid body with constant angular velocity \( \Omega \) about an axis normal to the plane of the disk. The disk is also assumed to have a non-torsional motion of the type suggested by (6.1.1). We take cartesian coordinates \((x, y, z)\) such that \( z \)-axis is parallel to the axis of rotation. It is assumed that all flow variables except pressure are independent of \( x \) and \( y \). We assume that

(i) a uniform transverse magnetic field \( H_0 \) is acting parallel to the \( z \)-axis.
(ii) the disk is electrically nonconducting.

and

(iii) the induced magnetic field produced by the motion of the electrically conducting fluid is negligible in comparison to the imposed magnetic field.

From the Gauss divergence theorem namely

\[ \text{div} \mathbf{H} = 0 \quad (6.2.1) \]

one can find that the magnetic field component \( H_y \) is everywhere equal to the applied magnetic field i.e. \( H_y = H_0 \) constant everywhere in the flow. So \( \mathbf{H} = (0,0, H_0) \). The equation of conservation of electric charge \( \nabla \cdot \mathbf{J} = 0 \) gives \( J_z = \text{constant} \), where \( \mathbf{J} = (J_x, J_y, J_z) \). The constant is zero since \( J_z = 0 \) at the disk which is assumed to be electrically nonconducting. Thus \( J_z = 0 \) everywhere in the flow.

In this rotating frame of reference the unsteady motion of the fluid is governed by the equations

\[ \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} + 2 \Omega \times \mathbf{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{V} + \frac{\mu_0}{\rho} \mathbf{J} \times \mathbf{H} \quad (6.2.2) \]

\[ \text{div} \mathbf{V} = 0 \quad (6.2.3) \]

where \( \mathbf{V} = (u,v,w) \) is the velocity vector, \( p \) is the effective kinematic pressure, \( \rho \) is the density of the fluid, \( \nu \) is the kinematic viscosity of the fluid, \( \mu_0 \) is the magnetic permeability and \( \mathbf{H} \) is the unit vector along the \( z \)-axis.
Boundary conditions of the problem are

\[ z = 0 : \quad u + iv = U \left[ c + \varepsilon (a \exp(i\omega t) + b \exp(-i\omega t)) \right] \]

\[ w = w_0 (1 + \epsilon A \cos \omega t) \]

\[ z \to \infty \quad u, v \to 0 \]

where \( A \) is a constant, \( \varepsilon \) is a small positive number, \( \omega \) is the given frequency and \( a, b, c \) are as defined earlier in this chapter. \( w_0 < 0 \) and \( w_0 > 0 \) correspond to suction and injection respectively.

(6.2.3) gives

\[ \frac{\partial w}{\partial z} = 0 \]

which on integration gives by virtue of the assumption of the normal velocity at the disk

\[ w = w_0 (1 + \varepsilon A \cos \omega t) \quad \text{(6.2.5)} \]

The Lorentz force \( \mathbf{F} \) is given by

\[ \mathbf{F} = \mu_e \mathbf{j} \times \mathbf{H} = \mu_e (J_x, J_y, 0) \times (0, 0, H_0) \]

\[ = \mu_e H_0 (J_y \hat{i} - J_x \hat{j}) \quad \text{... (6.2.6)} \]

Making use of (6.2.5) and (6.2.6) in (6.2.2) the equations of motion in the rotating coordinate system are given by
When the strength of the magnetic field is very large the modified Ohm's law which includes Hall current is given by

\[ \frac{\partial u}{\partial t} + w_0 (1 + e A \cos t) \frac{\partial u}{\partial z} - 2 \alpha v = - \frac{1}{\mu_e} \frac{\partial P_e}{\partial x} + v \frac{\partial^2 u}{\partial y^2} - \frac{\mu_e H_0}{\mu} J_y \]

(6.2.7)

\[ \frac{\partial v}{\partial t} + w_0 (1 + e A \cos t) \frac{\partial v}{\partial z} + 2 \alpha u = - \frac{1}{\mu_e} \frac{\partial P_e}{\partial y} + v \frac{\partial^2 v}{\partial z^2} - \frac{\mu_e H_0}{\mu} J_x \]

(6.2.8)

\[ \frac{\partial w}{\partial t} = - \frac{1}{\mu_e} \frac{\partial P_e}{\partial z} \]

(6.2.9)

where the symbols have their usual notation.

In writing (6.2.10) the ion-slip, the thermo-electric effects and the electron pressure gradient are neglected. From equation (6.2.10) we get

\[ J_x + \frac{\omega_e \tau_e}{H_0} (J \times \vec{H}) = \sigma (E + \mu_e \vec{V} \times \vec{H}) \]

(6.2.10)

where (\(E_x, E_y, E_z\)) are the components of the electric field \(\vec{E}\).

Far away from the disk, the magnetic field is uniform and hence there is no electric current.

Thus \(J_x \to 0\) and \(J_y \to 0\) as \(z \to \infty\)

(6.2.12)
Using the conditions \( u \to 0, v \to 0 \) as \( z \to \infty \) from (6.2.4) and the condition (6.2.12) in (6.2.11) we get,

\[
E_x \to 0, \ E_y \to 0 \text{ as } z \to \infty \tag{6.2.13}
\]

The electric field is assumed to be zero as there is no external electric field and the electric field arising out of polarisation of charges is negligible. Hence \( E_x = 0 \) and \( E_y = 0 \) everywhere in the flow. Thus the components of the electric field given by (6.2.13) retain their values everywhere.

From (6.2.11) we get

\[
J_x = \frac{\sigma \mu_e H_0}{1 + m^2} (v + mu)
\]

\[
J_y = \frac{\sigma \mu_e H_0}{1 + m^2} (-u + mv)
\]

where \( m = \gamma_e \gamma_e \) is the Hall parameter.

The equations of motion (6.2.7) to (6.2.9) take the form

\[
\frac{\partial u}{\partial t} + w_0 (1 + \varepsilon \ A \cos \omega t) \frac{\partial u}{\partial z} = 2\omega v - \frac{1}{\tau} \frac{\partial \rho}{\partial x} + v \frac{\partial^2 u}{\partial z^2} + \frac{\mu_e H_0}{\eta} \frac{\sigma \mu_e H_0}{1 + m^2} (-v + mu)
\]

\[
\ldots \ldots \tag{6.2.15}
\]

\[
\frac{\partial v}{\partial t} + w_0 (1 + \varepsilon \ A \cos \omega t) \frac{\partial v}{\partial z} = -2\omega u - \frac{1}{\tau} \frac{\partial \rho}{\partial y} + v \frac{\partial^2 v}{\partial z^2}
\]

\[- \frac{\mu_e H_0}{\eta} \frac{\sigma \mu_e H_0}{1 + m^2} (v + mu)
\]

\[
\ldots \ldots \tag{6.2.16}
\]
\[ \frac{\partial v}{\partial t} = - \frac{1}{\rho} \frac{\partial p}{\partial z} \quad (6.2.17) \]

As there is no relative motion far away from the disk, we assume that there is no imposed pressure gradient along the x and y directions, so that

\[ \frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = 0 \quad (6.2.18) \]

Equation (6.2.17) gives the pressure gradient normal to the disk.

Writing \( q' = u + iv \) equation (6.2.15) and (6.2.16) and the boundary conditions (6.2.4) become

\[ \frac{\partial q'}{\partial t} + w_0 (1 + \varepsilon a \cos \omega t) \frac{\partial q'}{\partial z} + 2 \omega i q' = \frac{\sigma \mu^2 H^2}{\rho z^2} \frac{q''}{\rho (1 + i \varepsilon)} q' \]

and

\[ q'(0,t) = U \left[ c + \varepsilon \left( a \exp(i \omega t) + b \exp(-i \omega t) \right) \right] \]

\[ q'(+,t) = 0 . \]

We introduce the following transformations to nondimensionalise the equations (6.2.19) and (6.2.20)

\[ q = q'/U, \quad \eta = \frac{z |w_0|}{v}, \quad \zeta = \frac{v^2 t}{4 \nu}, \quad \lambda = \frac{4 v \omega \omega}{w_0} \]

\[ E = \frac{v \omega^2}{w_0}, \quad S = \frac{w_0}{|w_0|}, \quad M^2 = \frac{\sigma \mu^2 H^2}{\rho |w_0|^2} \frac{v}{w_0} \]

Then (6.2.19) and (6.2.20) become
\[
\frac{1}{4} \frac{\partial q}{\partial t} + S(1 + jA \cos \omega t) \frac{\partial q}{\partial \omega} + 2Ei q = \frac{\partial^2 q}{\partial \gamma^2} - \frac{n^2(1+im)}{1+m^2} q \quad \cdots (6.2.22)
\]

and
\[
q(0, \omega) = c + \varepsilon (ae^{-i\omega t} + be^{i\omega t}) \quad (6.2.23)
\]

\[
q(\infty, \omega) = 0
\]

6.3. Solution of the problem.

Following Puroshothaman (1979), we assume
\[
q(\gamma, \omega) = \sum_{-\infty}^{\infty} q(n)(\gamma) \exp \left( in\omega \right) \quad (6.3.1)
\]

Substituting (6.3.1) in (6.2.22) and (6.2.23) we obtain
\[
\left[ - \frac{\partial^2 q_n}{\partial \gamma^2} - S \left\{ \frac{\partial q_n}{\partial \gamma} + \frac{\varepsilon A}{2} \left( \frac{\partial q_{n-1}}{\partial \gamma} + \frac{\partial q_{n+1}}{\partial \gamma} \right) \right\} \right] \quad \cdots (6.3.2)
\]

\[
= - \left[ 2Ei + \frac{n^2}{1+m^2} (1+im) + \frac{1}{4} \ln \right] q_n = 0
\]

\[
q_0(0) = c, \quad q_1(0) = \varepsilon a, \quad q_{-1}(0) = \varepsilon b
\]

\[
q_n(0) = 0, \quad |n| > 1 \quad (6.3.3)
\]

\[
q_n(\infty) = 0 \text{ for all } n.
\]

Assuming \(\varepsilon \ll 1\), \(q_n\) is expanded in the form
\[
q_n = \sum_{j=0}^{\infty} \sum_{j=0}^{j} q_{nj}(\gamma) \quad (6.3.4)
\]
Putting (6.3.4) in (6.3.2) and equating the coefficients of $v^J$ on both sides we get

$$q^{''}_{nj} - \frac{S}{nj} q^{'}_{nj} - \left[ i(2E + \frac{n \sigma}{4}) + \frac{m^2}{1+m} (1+im) \right] q_{nj} = \frac{S}{2} \delta^J_s$$

$$(q^{'}_{n-1,j-1} + q^{'}_{n+1,j-1})$$

... (6.3.5)

Here a prime denotes differentiation with respect to $\gamma$.

The boundary conditions for $q_{nj}$ are

$q_{n0}(0) = c$, $q_{11}(0) = a$, $q_{-11}(0) = b$

$q_{nj}(0) = 0$ for all other $n$ and $j$ ...

$q_{nj}(\infty) = 0$ for all $n$ and $j$ ...

The solutions are

$q_{n0} = c \exp(m_0 \gamma)$ ...

$q_{nj} = 0$, $n \neq 0$ ...

$q_{11} = a \exp(m_1 \gamma) + \frac{2SA\alpha m_0}{\alpha} (\exp(m_0 \gamma) - \exp(m_1 \gamma))$ ...

$q_{-11} = b \exp(m_2 \gamma) - \frac{2SA\alpha m_0}{\alpha} (\exp(m_0 \gamma) - \exp(m_2 \gamma))$ ...

... (6.3.7) ...

... (6.3.8) ...

... (6.3.9) ...

... (6.3.10)
The shear stresses $\gamma_x$ and $\gamma_y$ at the disk are given by
\[
\begin{align*}
\frac{\zeta}{r} + \frac{1}{r} \zeta = q_0(0) + \varepsilon q_1(0) + \varepsilon q_{11} + o(\varepsilon^2)
\end{align*}
\]

which on simplication become

\[
\begin{align*}
&c m_0 + \varepsilon \exp(i\omega t) \left[ a m_1 + \frac{2Sa \text{c} m_0}{\alpha} (m_0 - m_1) \right] \\
&+ \varepsilon \exp(-i\omega t) \left[ b m_2 - \frac{2Sa \text{c} m_0}{\alpha} (m_0 - m_2) \right] + o(\varepsilon^2)
\end{align*}
\]

\[6.4. \quad \text{Discussion of the results:}
\]

The solutions (6.3.7) - (6.3.13) give the general features of the fluctuating boundary layers. The thickness of the boundary layer for the oscillation made \( m_n \) is \( O(\delta_n) \)

where \( \delta_n = \text{real part of} \left( \frac{v}{|w_0| m_n} \right) \). We also note that

\[
q_n \approx O(\varepsilon |n|).
\]

When \( 8\varepsilon = \alpha \), in case of suction, the observation that one of the boundary layers is independent of rotation is true in the magnetic case also.

The most important conclusion obviously is that magnetic field is an effective check on the occurrence of resonance in the case of blowing. In case of blowing, \( S = 1 \) and all the boundary layers are bounded even if \( 8\varepsilon = n\alpha \).

It is also true that when \( \alpha = 0 \), \( m_0 = m_1 = m_2 = \ldots \), so that there exists a single oscillation mode and consequently a single modified Ekman layer of thickness \( \delta_0 \). Though the
Hall parameter has no significant contribution on the boundary layer thickness and in checking or inducing resonance. However small, effectively controls the Ekman layers and does not allow any resonance behaviour to prevail either in case of suction or injection.