CHAPTER - III

PROPERTIES OF CERTAIN ANALYTIC FUNCTIONS

INVOLVING RUSEHEWYH DERIVATIVES

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3.1 Let H denote the class of functions of the form

\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k \]

which are analytic in the unit disk \( E = \{ z : |z| < 1 \} \).

Let \( S^*(\alpha) \) and \( C(\alpha) \), \( 0 \leq \alpha < 1 \), denote the subclasses of functions in \( S \) which are respectively, starlike of order \( \alpha \) and convex of order \( \alpha \).

In [29], Goel and Sohi have studied the class of functions \( f(z) \in H \) satisfying

\[ \text{Re}\left\{ \frac{D^{n+1}f(z)}{z} \right\} > \alpha, \ z \in E, \ n \in \mathbb{N}_0 = \{ 0, 1, 2, \ldots \} \]

where \( 0 \leq \alpha < 1 \) and \( D^nf(z) \) is the Ruscheweyh's derivative of order \( n \) (c.f. section 1.4).

Aouf [4], further generalized this class of functions by introducing the class \( V_n(A, B, \alpha) \). Thus, a function \( f(z) \in H \) is said to be in the class \( V_n(A, B, \alpha) \) if for \(-1 \leq B < A \leq 1\), and \( 0 \leq \alpha < 1 \) the condition

\[ \frac{D^{n+1}f(z)}{z} < \frac{1 + \{ B + (A-B)(1-\alpha) \} z}{1 + Bz}, \ z \in E, \ n \in \mathbb{N}_0 \]

holds. He showed that \( V_{n+1}(A, B, \alpha) \subseteq V_n(A, B, \alpha) \) for all \( n \in \mathbb{N}_0 \).
Ponnusamy and Karunakaran [91] proved that if 
\[ f(z) \in B_1(\alpha, \mu), \text{ that is, } \Re \left\{ z^{1-\mu} f(z)/(f(z))^{1-\mu} \right\} > \alpha \quad (z \in E) \],
then
\[ \Re \left( \frac{f(z)}{z} \right)^\mu > \frac{2\mu \alpha + 1}{2\mu + 1} \quad (z \in E). \]

Libera [53] showed that if \( M(z) \) and \( N(z) \) are analytic in the unit disc \( E \), \( M(0) = N(0) = 0 \) and \( N(z) \) maps \( E \) onto a multisheeted domain with respect to the origin then 
\[ \Re (M'(z)/N'(z)) > \beta \quad \text{implies} \quad \Re (M(z)/N(z)) > \beta \quad \text{for} \quad \beta = 0 \]
and MacGregor [57] for \( \beta \) real. Various generalization of this result can be found in the literature. Recently, Ponnusamy and Karunakaran [91] found some conditions on the function \( M(z) \) and \( N(z) \) so that \( \Re (M'(z)/N'(z)) > \beta \)
implies \( \Re (M(z)/N(z)) > \beta' > \beta \), \( z \in E \).

In this chapter we consider two subclasses of \( p \)-valent analytic functions, namely, \( T_{n,p}(A,B,\lambda) \) and \( B_{n,p}(\alpha, \beta, \mu, \lambda) \) involving Ruscheweyh derivatives. In section 3.2, we propose to give some applications of Briot-Bouquet differential subordination which would not only improve and sharpen many of the earlier results contained in [17, 63], but would also give rise to a number of new results for other subclasses as well. In section 3.3, we use the Briot-Bouquet differential subordination to the investigation of Libera and Bernadi transformations for functions belonging to the class \( T_{n,p}(A,B,\lambda) \), and obtain sharp results. Finally, in section 3.4, we apply the same Briot-Bouquet differential subordination technique to
another class $B_{n,p}(\alpha, \beta, \mu, \lambda)$ and obtain some interesting results.

3.2. In this section, we introduce the class $T_{n,p}(A, B, \lambda)$ and $B_{n,p}(\alpha, \beta, \mu, \lambda)$ and give some Lemmas which are needed to establish our main results.

**Definition 3.2.1.** Let $A, B$ and $\lambda$ be arbitrary fixed real numbers such that $-1 < B < A < 1$ and $\lambda > 0$. Let $n$ be any integer greater than $-p$. A function $f(z) \in H_p$ is said to be in the class $T_{n,p}(A, B, \lambda)$ if it satisfies the condition

$$J_{n,p}(f; \lambda) < \frac{1 + Az}{1 + Bz} \quad (z \in E)$$

where

$$J_{n,p}(f; \lambda) = (1-\lambda) \frac{D^{n+p-1} f(z)}{z^p} + \lambda \frac{D^{n+p} f(z)}{z^p}.$$ 

Here $D^{n+p-1} f(z)$ denotes the Ruscheweyh's derivative of order $(n+p-1)$ (c.f. section 1.4).

It is readily seen that $T_{n,1}(1-2\alpha, -1, 1)$ is the class consisting of functions $f(z) \in H$ which satisfies

$$\text{Re} \left\{ D^{n+p} f(z)/z \right\} > \alpha \quad \text{in} \quad E. \quad \text{This class was introduced and studied by Goel and Sohi [29], whereas } T_{0,1}(1-2\alpha, -1, 1) \quad \text{is the class studied by Owa, Obradovic and Nunokawa [80].}$$

Further, it is clear that $T_{0,1}(A, B, 1) \equiv P'(A, B)$ is the class
studied by Obradovic [75]. We denote $T_{n,1}(A,B,\lambda)$ by $T_n(A,B,\lambda)$.

**Definition 3.2.2.** Let $\alpha, \beta$ and $\mu$ be real numbers such that $0 < \alpha < 1$, $0 < \beta \leq 1$ and $\mu > 0$. Let $n$ be any integer greater than $-p$ and $\lambda$ be a complex number such that $\text{Re}(\lambda) > 0$. A function $f(z) \in H_p$ is said to be in the class $B_{n,p}(\alpha,\beta,\mu,\lambda)$ if it satisfies

$$
(3.2.2) \quad \text{Re}\left\{ \left(1-\lambda\right)\left(\frac{D^{n+p-1} f(z)}{D^{n+p} g(z)}\right)^\mu + \lambda\left(\frac{D^{n+p} f(z)}{D^{n+p} g(z)}\right)^{\mu-1} \right\} > \alpha \quad (z \in E)
$$

where $g(z) \in H_p$ and satisfies the condition

$$
\text{Re}\left\{ \frac{D^{n+p-1} g(z)}{D^{n+p} g(z)} \right\} > \beta \quad (z \in E).
$$

We note that $B_{0,1}(\alpha,1,\mu,1) \equiv B_1(\alpha,\mu)$, the subclass of Bazilevic functions of the type $\alpha(0 \leq \alpha < 1)$ (c.f. section 1.4) and $B_{0,1}(\alpha,\beta,\mu,\lambda)$ is the class studied by Ponnusamy and Karunakaran [91]. We, further, observe that $B_{n,p}(\alpha,1,1,\lambda) \equiv T_{n,p}(1-2\alpha,-1,\lambda)$.

We need the following Lemmas.

**Lemma 3.2.1** [35]. If $p(z) = 1+p_1z + p_2z^2 + \ldots$ is analytic in $E$ and $h(z)$ is a convex function in $E$ with $h(0) = 1$ and $\gamma$ is a complex constant such that $\text{Re}(\gamma) > 0$, then

...
(3.2.3) \[ p(z) + \frac{zp'(z)}{\gamma} < h(z) \]

implies

\[ p(z) < \gamma z^{-\gamma} \int_0^z t^{-\gamma-1} h(t) dt = q(z) < h(z) \]

and \( q(z) \) is best dominant.

The following Lemma is due to Nunokawa [73].

**Lemma 3.2.2.** Let \( f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \) be analytic in \( E \). If there exists a \((p - m + 1)\)-valent starlike function \( g(z) = z + \sum_{k=p-m+2}^{\infty} a_k z^k \) in \( E \) such that

\[
\operatorname{Re}\left\{ \frac{zf'(m)(z)}{g(z)} \right\} > 0, \quad z \in E
\]

then \( f(z) \) is \( p \)-valent in \( E \).

For \( a, b, c \) real numbers other than \( 0, -1, -2, \ldots \), the hypergeometric series

\[ F(a,b;c;z) = 1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \ldots \]

represents an analytic functions in \( E \) [87]. The following identities are well known [87].

**Lemma 3.2.3.** For \( a, b, c \) real numbers other than \( 0, -1, -2, \ldots \) and \( c > b > 0 \), we have
\[
\int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} \, dt = \frac{\Gamma(b) \Gamma(c-b)}{\Gamma(c)} F(a, b; c; z)
\]

(3.2.5) \[ F(a, b; c; z) = (1-z)^{-a} F(a, c-b; c; -\frac{z}{z-1}) \]

(3.2.6) \[ F(1,1;2;z) = -z^{-1} \ln(1-z) \]

(3.2.7) \[ C(c-1)(z-1)F(a, b; c-1; z) + \\
+ c[c-1-(2c-a-b-1)z]F(a, b; c; z) + \\
+ (c-a)(c-b)zF(a, b; c+1; z) = 0 \]

**Lemma 3.2.4.** For any real number \( d \neq 0 \), we have

(3.2.8) \[ F(1,1;2; \frac{dz}{dz+1}) = \frac{(1+dz)\ln(1+dz)}{dz+1} \]

(3.2.9) \[ F(1,1;3; \frac{dz}{dz+1}) = \frac{2(1+dz)}{dz+1} \left[ 1 - \frac{\ln(1+dz)}{dz} \right] \]

(3.2.10) \[ F(1,1;4; \frac{dz}{dz+1}) = \frac{3(1+dz)}{2(dz)^2} \left[ 2 \cdot \ln(1+dz) - dz(2-dz) \right] \]

(3.2.11) \[ F(1,1;5; \frac{dz}{dz+1}) = \frac{2(1+dz)}{3(dz)^3} \left[ \frac{(2dz)^2-3dz+6}{3} - \frac{2\ln(1+dz)}{dz} \right] \]

The proof of Lemma 3.2.4 follows from the identities (3.2.6) and (3.2.7).
We now prove

**Theorem 3.2.1.** Let the function $f(z)$ defined by

(1.2.34) be in the class $T_{n,p}(A,B,a)$. If $n+p > \lambda > 0$, then

(3.2.12) \[ z^{\lambda} f(z) < q(z) < \frac{1 + Az}{1 + Bz}, \quad z \in E \]

where

\[
q(z) = (1+Bz)^{-1}\\left[ F(1,1;1 + \frac{n+p}{\lambda}; \frac{Bz}{Bz+1}) + \right.
\]
\[
+ \frac{(n+p)Az}{n+p+\lambda} F(1,1;2 + \frac{n+p}{\lambda}; \frac{Bz}{Bz+1}) \]

and $q(z)$ is the best dominant. Furthermore,

(3.2.13) \[ \Re \left\{ \frac{D^{n+p-1} f(z)}{z^p} \right\} > \phi \]

where

\[
\phi = (1-B)^{-1}\left[ F(1,1;1 + \frac{n+p}{\lambda}; \frac{B}{B-1}) - \right.
\]
\[
- \frac{(n+p)A}{n+p+\lambda} F(1,1;2 + \frac{n+p}{\lambda}; \frac{B}{B-1}) \].

**Proof.** Since for $n > -p$

(3.2.14) \[ D^{n+p-1} f(z) = z^p + \sum_{k=1}^{\infty} \frac{\Gamma(n+p+k)}{\Gamma(n+p)k!} a_{p+k} z^{p+k} \]

we have

(3.2.14) \[ z(D^{n+p-1} f(z)) = (n+p)D^{n+p} f(z) - nD^{n+p-1} f(z). \]
Let \( p(z) = \frac{D^{n+p-1} f(z)}{z^P} \). Then \( p(z) \) is analytic in \( E \) with \(| p(0) = 1 \) and as \( f(z) \in T_{n_p}(A,B,\lambda) \), (3.2.1) coupled with (3.2.14) yields

\[
p(z) + \frac{(\lambda/n+p)z p'(z)}{z^P} = J_{n_p}(f;\lambda) < \frac{1+Az}{1+Bz}, \quad z \in E.
\]

Thus by using Lemma 3.2.1 for \( \lambda = (n+p)/A \), we deduce that

\[
D^{n+p-1} \frac{f(z)}{z^P} < \frac{n+p}{(n+p)\lambda} z^P \frac{n+p}{(n+p)\lambda} -1 \int_0^1 \frac{(1+Az)}{(1+Bz)} dt = q(z), \quad \text{(say)}.
\]

Now the function \( q(z) \) can be rewritten as

\[
q(z) = (\frac{n+p}{\lambda}) \int_0^1 s^\lambda \frac{1}{1+Bs} \left( \frac{1+Az}{1+Bz} \right) ds
\]

\[
= \frac{1}{\lambda} \int_0^1 s^\lambda (1+Bs) ds + A(n+p)z \int_0^1 s^\lambda (1+Bs) ds
\]

\[
= (1+Bz)^{-1} \left[ F(1,1; 1 + \frac{n+p}{\lambda}; \frac{Bz}{Bz+1}) + \frac{(n+p)Az}{n+p + \lambda} F(1,1; 2 + \frac{n+p}{\lambda}; \frac{Bz}{Bz+1}) \right].
\]

The last expression follows by using the identities (3.2.4) and (3.2.5). This completes the proof of (3.2.12).
Next to prove (3.2.13), it suffices to show that

\[(3.2.15) \inf \{q(z)\} = q(-1), \quad |z| < 1\]

Since for \(-1 \leq B < A \leq 1\), \((1 + Az)/(1+Bz)\) is convex (univalent) in \(E\), we have for \(|z| \leq r < 1\),

\[(3.2.16) \quad \Re \left( \frac{1 + Az}{1 + Bz} \right) \geq \frac{1 - Ar}{1 - Br}.

Setting

\[g(s,z) = \frac{1 + Asz}{1 + Bs} \quad 0 \leq s \leq 1, \quad z \in E\]

and

\[d\mu(s) = s^{n+p} (\frac{n+p}{\Lambda}) \, ds\]

which is a positive measure on \([0, 1]\), we get

\[q(z) = \int_{0}^{1} g(s,z) d\mu(s)\]

so that

\[\Re \{q(z)\} = \int_{0}^{1} \Re \left( \frac{1 + Asz}{1 + Bs} \right) d\mu(s)\]

\[\geq \int_{0}^{1} \frac{1 - Asr}{1 - Br} d\mu(s) = q(-r), \quad |z| \leq r < 1.\]

Now, letting \(r \rightarrow 1^-\) in the above inequality, we obtain

\[\Re \{q(z)\} \geq q(-1), \quad z \in E\]
which implies (3.2.15). Hence the theorem.

Putting \( p = 1 \) in Theorem 3.2.1, we obtain

**Corollary 3.2.1.** Let \( f(z) \in T_n(A, B, \lambda) \) and \( n+1 > \lambda > 0 \).

Then

\[
(3.2.17) \quad \frac{D^n f(z)}{z} < q(z) = (1+Bz)^{-1} \left[ F(1, 1; 1 + \frac{n+1}{\lambda}; \frac{Bz}{Bz+1}) + \right.
\]
\[
\left. + \frac{(n+1)A}{n+1 + \lambda} F(1, 1; 2 + \frac{n+1}{\lambda}; \frac{Bz}{Bz+1}) \right]
\]
\[
< \frac{1+A z}{1+Bz}, \quad z \in E
\]

and \( q(z) \) is the best dominant. Furthermore,

\[
(3.2.18) \quad \text{Re} \left\{ \frac{D^n f(z)}{z} \right\} > \mathcal{C},
\]

where

\[
\mathcal{C} = (1-B)^{-1} \left[ F(1, 1; 1 + \frac{n+1}{\lambda}; \frac{B}{B-1}) - \right.
\]
\[
\left. - \frac{(n+1)A}{n+1 + \lambda} F(1, 1; 2 + \frac{n+1}{\lambda}; \frac{B}{B-1}) \right].
\]

In the case \( \lambda = 1 \) and \( n = 0 \), Corollary 3.2.1 yields the following:

**Corollary 3.2.2.** Let \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in P^n(A, B) \),

then

\[
(3.2.19) \quad \frac{f(z)}{z} < q(z) = \left\{ \begin{array}{ll}
\frac{A}{B} + (1 - \frac{1}{B}) \ln(1+Bz), & B \neq 0 \\
1 + \frac{A}{B} z, & B = 0
\end{array} \right.
\]
and \( q(z) \) is the best dominant. Further,

\[
\text{Re}\left( \frac{f(z)}{z} \right) \geq \begin{cases} 
\frac{A}{B} r - \frac{1}{B} (1 - \frac{A}{B}) \ln(1-Br), & B \neq 0 \\
1 - \frac{A}{2} r^2, & B = 0.
\end{cases}
\]

The function \( q(z) \) defined above shows that the estimate (3.2.20) is sharp.

The proof of Corollary 3.2.2 follows by letting \( n = 0 \) and \( \lambda = 1 \) in Corollary 3.2.1 followed by using the identities (3.2.8) and (3.2.9). This result was also obtained by Obradovic [75].

Remark. It is readily seen from Corollary 3.2.2 (Equation 3.2.19) that if \( f(z) \in P'(A,B) \), then for \( |z| = r < 1 \)

\[
|f(z)| \geq \begin{cases} 
\frac{A}{B} r - \frac{1}{B} (1 - \frac{A}{B}) \ln(1-Br), & B \neq 0 \\
r - \frac{A}{2} r^2, & B = 0,
\end{cases}
\]

\[
|f(z)| \leq \begin{cases} 
\frac{A}{B} r + \frac{1}{B} (1 - \frac{A}{B}) \ln(1+Br), & B \neq 0 \\
r + \frac{A}{2} r^2, & B = 0.
\end{cases}
\]

The estimates are sharp, being attained for the function

\[
f(z) = \begin{cases} 
\frac{A}{B} z + \frac{1}{B} (1 - \frac{A}{B}) \ln(1+Bz), & B \neq 0 \\
z + \frac{A}{2} z^2, & B = 0.
\end{cases}
\]

If we put \( A = 1-2\alpha, 0 \leq \alpha < 1 \) and \( B = -1 \) in (3.2.20) of Corollary 3.2.2, we obtain
Corollary 3.2.3. Let \( f(z) \in H \) and \( \text{Re}\{f'(z)\} > \alpha, \) 
\( 0 \leq \alpha < 1, \ z \in E. \) Then

\[
\text{Re}\left(\frac{f(z)}{z}\right) \geq (2\alpha - 1) + 2(1 - \alpha)\ln 2,
\]

and this result is sharp.

This improves an earlier result of Owa and Obradovic [78].

Corollary 3.2.4. Let \( f(z) \in H \) and

\[
f'(z) + \frac{1}{2}zf''(z) < \frac{1+Az}{1+Bz}, \ z \in E
\]

then

\[
(3.2.21) \quad f'(z) < q(z) = \begin{cases} \frac{A}{B} - \frac{2}{B^2} & \text{if } B \neq 0 \\ 1 + \frac{2A}{B}z & \text{if } B = 0 \end{cases}
\]

and the right hand side of (3.2.21) is the best dominant. Furthermore,

\[
(3.2.22) \quad \text{Re}\{f'(z)\} \geq \begin{cases} \frac{A}{B} - \frac{2}{B^2}(1 - \frac{A}{B})[\ln(1+B) + B] & \text{if } B \neq 0 \\ 1 - \frac{2}{3}A & \text{if } B = 0 \end{cases}
\]

The function \( q(z) \) in (3.2.21) shows that the estimate (3.2.22) is sharp.

The proof of the Corollary 3.2.4 is obtained by setting \( n = 1 \) and \( \lambda = 1 \) in Corollary 3.2.1 and by using the identities (3.2.9) and (3.2.10) in the resulting equation.
Corollary 3.2.5. Let \( f(z) \in H \) and
\[
\frac{f'(z)}{z} + \frac{1}{3} z f''(z) < \frac{1 + Az}{1 + Bz}, \quad z \in E
\]
then
\[
f'(z) < q(z) = \begin{cases}
\frac{A}{B} + \frac{3}{(Bz)^3} \left( 1 - \frac{A}{B} \right) \ln(1+Bz) - Bz + \frac{(Bz)^2}{2}, & B \neq 0 \\
1 + \frac{3}{4} Az, & B = 0
\end{cases}
\]
and the right hand side of (3.2.23) is the best dominant. Furthermore,
\[
\text{Re} \left\{ f'(z) \right\} > \begin{cases}
\frac{A}{B} - \frac{3}{B^3} \left( 1 - \frac{A}{B} \right) \ln(1-B) + B - \frac{B^2}{2}, & B \neq 0 \\
1 - \frac{3}{4} A, & B = 0
\end{cases}
\]
The result (3.2.24) is sharp.

The proof of Corollary 3.2.5 is obtained by taking \( n = 1 \) and \( \lambda = \frac{2}{3} \) is Corollary 3.2.1 followed by applying the identities in (3.2.10) and (3.2.11) in the resulting equation.

Remarks. 1. In view of Corollary 3.2.2, we note that if \( f(z) \in P'(A,B) \), where \( A' = \left\{ B \ln(1-B) \right\} / \left\{ B + \ln(1-B) \right\} \), \( B \neq 0 \) then \( \text{Re} \left\{ f(z)/z \right\} > 0 \) in \( E \).

From this it follows that if \( \text{Re} \left\{ f'(z) \right\} > \frac{\left( \log 4-1 \right)}{\left( \log 4-2 \right)} = -0.62944 \), then \( \text{Re} \left\{ f(z)/z \right\} > 0 \) in \( E \).

2. For \( A = (1-2 \alpha) \), \( 0 \leq \alpha < 1 \) and \( B = -1 \), Corollary 3.2.4 gives the corresponding result obtained by Owa, Obradovic and Nunokawa [80].
3. We observe from Corollary 3.2.4 that if $B \neq 0$ and

\[ f'(z) + \frac{1}{2} zf''(z) < \frac{1+A''}{1+Bz}, \quad z \in E \]

where $A'' = \frac{2B[B+\ln(1-B)]}{2[B+\ln(1-B)]+B^2}$, then

\[ \text{Re} \{f'(z)\} > 0 \quad \text{in} \quad E \]

and hence $f(z)$ is univalent in $E$. This gives a new criteria for univalency.

In particular, if $B = -1$ then

\[ \text{Re} \left\{ f'(z) + \frac{1}{2} zf''(z) \right\} \geq \frac{4\ln 2 - 3}{4\ln 2 - 2} = -0.2943 \]

implies that

\[ \text{Re} \{f'(z)\} > 0 \quad \text{in} \quad E. \]

4. It is shown by Saitoh [98] that for $\lambda > 0$ and $0 < \alpha < 1$

\[ \text{Re} \left\{ f'(z) + \lambda zf''(z) \right\} > \alpha (z \in E) \quad \text{implies that} \]

\[ \text{Re} \{f'(z)\} > \frac{(2\alpha + \lambda)/(2 + \lambda)}{z \in E}. \]

However, (3.2.22) and (3.2.24) shows that (for $A = 1 - 2\alpha$ and $B = -1$) if

\[ \text{Re} \left\{ f'(z) + \frac{1}{2} zf''(z) \right\} > \alpha (z \in E) \quad \text{then} \]

\[ \text{Re} \{f'(z)\} > 3 - 2\alpha - 4(1 - \alpha)\ln 2 \]

and

\[ \text{Re} \left\{ f'(z) + \frac{1}{2} zf''(z) \right\} > \alpha (z \in E) \quad \text{implies that} \]

\[ \text{Re} \{f'(z)\} > (2\alpha - 1) + 3(1 - \alpha)(2 \ln 2 - 1). \]
Comparing these results with the result of Saitoh [98]
(c.f. equn. 3.2.26), we easily conclude that the result
(3.2.26) is not best possible. In that sense, our results
contained in Corollary 3.2.4 and Corollary 3.2.5 are an
improvement of the result (3.2.26).

We next prove

**Theorem 3.2.2.** Let $f(z) \in H_p$ for $p \geq 2$. If for
\[-1 < B < 1, B \neq 0,\]
\[
\frac{f(z)}{z^{p-1}} < \frac{1+A^t}{1+ Bz}, \quad z \in E
\]
where $A^t = \{B \cdot \ln(1-B)\} / \{B + \ln(1-B)\}$, then $f(z)$ is
$p$-valent in $E$.

**Proof.** Let $p(z) = f^{(p-1)}(z)/p! z$. Then $p(z)$ is
analytic in $E$ with $p(0) = 1$. An easy calculation yields
\[
p(z) + zp'(z) = \frac{f^{(p)}(z)}{p!} < \frac{1+A^t}{1+ Bz} (z \in E).
\]
Thus an application of Lemma 3.2.1 for $\gamma = 1$ gives

\[
(3.2.27) \quad p(z) < q(z) = z^{-1} \int_0^1 \left( \frac{1+AT}{1+BT} \right) dt = \frac{A^t}{B} + (1-\frac{A^t}{B}) \frac{\ln(1+Bz)}{Bz}
\]
by using (3.2.8) and (3.2.9). Since the right hand side
of (3.2.27) has real coefficients and its image is convex
with respect to the real axis, it follows from (3.2.27) that
Re \{ p(z) \} = \frac{1}{p!} \Re \left\{ \frac{f^{(p-1)}(z)}{z} \right\} > \frac{A}{B} - \frac{1}{B} (1 - \frac{A}{B}) \ln(1-B) = 0.

This shows that \( \Re \left\{ \frac{f^{(p-1)}(z)}{z} \right\} > 0 \) in \( E \) which is equivalent to

\[ \frac{f^{(p-1)}(z)}{z^2} > 0 \quad \text{in } E. \]

Since \( g(z) = z^2 \) is 2-valently starlike in \( E \), in view of Lemma 3.2.2, it follows that \( f(z) \) is \( p \)-valent in \( E \).

Remark. In the case \( B = -1 \), we get \( A' = \ln 2/(1-\ln 2) \) so that Theorem 3.2.2 gives the corresponding result obtained by Nunokawa [74].

By a similar method to that used in Theorem 3.2.2, we may obtain the following result.

**Theorem 3.2.3.** Let \( f(z) \in H_p \) for \( p \geq 1 \). If for 

\[-1 < A \leq 1 \]

\[ \frac{f^{(p)}(z)}{p!} < \frac{1+Az}{1+Bz}, \quad z \in E \]

then

\[ \frac{f^{(p-1)}(z)}{p! z} < q(z) = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B}) \frac{\ln(1+Bz)}{Bz}, & B \neq 0 \\ 1 + \frac{A}{2} z, & B = 0, \end{cases} \]

and the function \( q(z) \) is the best dominant. Furthermore,
For $A \geq 1$ and $B \leq -1$, the above theorem shows that

$$\text{Re}\left\{ f^{(p-1)}(z) \right\} \geq \begin{cases} p \left[ \frac{A}{B} - \frac{1}{B(1-A)} \ln(1-B) \right], & B \neq 0 \\ p \left( 1 - \frac{A}{2} \right), & B = 0. \end{cases}$$

This implies that $f(z) \in H_p$ and $\text{Re}\left\{ f(z) \right\} > 0$ in $E$, then $\text{Re}\left\{ f^{(p-1)}(z)/z \right\} > p! \ln 2 - 1$. This improves a result due to Saitoh [98] who proved that if $f(z) \in H_p$ and $\text{Re}\left\{ f(z) \right\} > 0$ in $E$ then $\text{Re}\left\{ f^{(p-1)}(z)/z \right\} > p \ln 2 - 1$.

3.3. In this section we use the Briot-Bouquet differential subordination to the investigation of Libera and Bernadi transformations of the class $T_{n,p}(A,B,\lambda)$. Here we consider the following integral transform

$$(3.3.1) \quad F_C(z) = \frac{c+p}{z^c} \int_0^z t^c f(t) dt,$$

where $f(z) \in H_p$ and $c + p > 0$. We prove the following theorem.

**Theorem 3.3.1.** Let $n > -p$ and $c$ be a real number such that $c + p > 0$. If $f(z) \in H_p$ satisfies

$$D_{z^p}^{n+p-1} f(z) = 1, \quad z \in E, -1 \leq B < A \leq 1$$
then

\[(3.3.2) \quad \frac{n+p-1}{D} F_c(z) \quad \frac{z^p}{q(z)} < q(z) < \frac{1+Az}{1+Bz}\]

where \( F_c(z) \) is defined by (3.3.1) and \( q(z) \) is given by

\[
q(z) = (1+Bz)^{-1} \left[ F(1,1; c+p+1; \frac{Bz}{Bz+1}) + \right.
\]

\[
+ \frac{(c+p)Az}{c+p+1} F(1,1; c+p+2; \frac{Bz}{Bz+1})].
\]

Furthermore,

\[(3.3.3) \quad \text{Re} \left\{ \frac{n+p-1}{D} F_c(z) \right\} \geq (1-B)^{-1} \left[ F(1,1; c+p+1; \frac{B}{B-1}) - \right.
\]

\[
- \frac{(c+p)A}{c+p+1} F(1,1; c+p+2; \frac{B}{B-1}) \right].
\]

**Proof.** Since \( F_c(z) = \sum_{k=p}^{\infty} \frac{(c+p)z^p}{(c+k)z^k} \neq f(z) \)

and

\[
\frac{n+p-1}{D} f(z) = \frac{z^p}{(1-z)^{n+p}} f(z) = z^p + \sum_{k=1}^{\infty} \frac{\Gamma(n+p+k)}{\Gamma(n+p)k!} a_{k+p} z^p
\]

a simple calculation gives

\[(3.3.4) \quad z(D^n F_c(z))^\prime = (c+p)D^{n+p-1} f(z) - cD^{n+p-1} F_c(z). \]

Let \( p(z) = (D^{n+p-1} F_c(z))^\prime z^p. \) Then \( p(z) \) is analytic in \( E \)

with \( p(0) = 1. \) In view of (3.3.4), we have
which with the aid of Lemma 3.2.1 for \( \gamma = c+p \) yields

\[
(3.3.5) \quad D f(z) \prec q(z) = (c+p)z \quad \frac{-(c+p)^z}{\frac{1}{1+Az}} \int_0^1 \frac{(1+At)}{1+Bt} dt,
\]

\[\prec \frac{1+Az}{1+Bz}, \quad z \in E.\]

Applying the identities (3.2.4) and (3.2.5) to the right hand side of (3.3.5), we get

\[
q(z) = (1+Bz)^{-1} \left[ F(1,1; c+p+1; \frac{Bz}{Bz+1}) + \frac{(c+p)Az}{c+p+1} F(1,1; c+p+2; \frac{Bz}{Bz+1}) \right].
\]

This proves (3.3.2). The estimate (3.3.3) can be proved on the same line as that of (3.2.13). Hence the theorem.

Taking \( n = 0, p = 1, A = (1-2\alpha), 0 < \alpha < 1 \) and \( B = -1 \) in Theorem 3.3.1, we get the following result.

**Corollary 3.3.1.** Let \( f(z) \in H \). If \( \text{Re} \{ f(z)/z \} > \alpha (0 < \alpha < 1) \) in \( E \), then

\[
\text{Re} \left\{ \frac{c+1}{z^{c+1}} \int_0^z t^{c-1} f(t) dt \right\} > \mathfrak{g}, \quad (z \in E)
\]

where

\[
\mathfrak{g} = [F(1,1; c+2; \frac{1}{2}) + \frac{(c+1)(2\alpha-1)}{c + 2} F(1,1; c+3; \frac{1}{2})] / 2.
\]
The above Corollary shows that if \( f(z) \in H \) satisfies \( \text{Re} \left\{ \frac{f(z)}{z} \right\} > \alpha, \ 0 < \alpha < 1 \), then the Libera transform satisfies

\[
\text{Re}\left\{ \frac{F_1(z)}{z} \right\} > (4\ln 2 \cdot 2)\alpha + (3 - \ln 2), \quad z \in E.
\]

This shows that the result (1.5.4) obtained by Obradovic [76] is not best possible. Thus, the result of Theorem 3.3.1 is an improvement of the result (1.5.3).

3.4. In this section for functions \( f(z) \in B_{n,p}(\alpha, \beta, \mu, \lambda) \), we find a sufficient condition on the function \( g(z) \in H_p \) which can guarantee that \( \text{Re} \left\{ D^{n+p-1} f(z) / D^{n+p} g(z) \right\} > \alpha \) implies

\[
\text{Re}\left\{ D^{n+p-1} f(z) / D^{n+p} g(z) \right\} > \alpha > \alpha' > 0 \text{ in } E.
\]

Some applications of this result is also obtained.

We now prove

**Theorem 3.4.1.** Let \( f(z) \in B_{n,p}(\alpha, \beta, \mu, \lambda) \) and \( \lambda > 0 \).

Then

\[
(3.4.1) \quad \text{Re}\left\{ \frac{D^{n+p-1} f(z)}{D^{n+p} g(z)} \right\} \mu > \frac{2(n+p)\alpha \mu + \beta \lambda}{2(n+p)\mu + \beta \lambda}, \quad z \in E
\]

where the function \( g(z) \in H_p \) satisfies the condition

\[
\text{Re}\left\{ \frac{D^{n+p-1} g(z)}{D^{n+p} g(z)} \right\} > \beta, \quad z \in E.
\]
Proof. Let \( \gamma = \frac{2(n+p)\alpha \mu + \beta \lambda}{2(n+p)\mu + \beta \lambda} \) and consider the function

\[ p(z) = (1-\gamma)^{-1} \left[ \left( \frac{f(z)}{g(z)} \right)^\mu - \gamma \right]. \]

The function \( p(z) \) is analytic in \( E \) and \( p(0) = 1 \). If we set

\[ \Psi_0(z) = \frac{n+p-1}{D} g(z) / \frac{n+p}{D} g(z), \]

then by the hypothesis

\[ \text{Re} \{ \Psi_0(z) \} > \beta \text{ in } E. \]

Differentiating \( p(z) \) and using the identity (1.4.5) in the resulting equation we deduce that

\[ (1-\gamma) \left\{ \frac{f(z)}{g(z)} \right\}^\mu + \lambda \frac{f(z)}{g(z)} \left\{ \frac{f(z)}{g(z)} \right\}^{\mu-1} \]

\[ = \gamma + (1-\gamma)p(z) + \frac{(1-\gamma)\lambda}{\mu(n+p)} \Psi_0(z) \cdot zp'(z) \]

\[ = \Psi(p(z), zp'(z)) \]

where

\[ \Psi(r,s) = \gamma + (1-\gamma)r + \frac{\lambda(1-\gamma)}{\mu(n+p)} \Psi_0(z) \cdot s. \]

Using (3.4.2) and the fact that \( f(z) \in B_{n,p}(\alpha,\beta,\mu,\lambda) \), we obtain for each \( z \in E \),

\[ \{ \Psi(p(z), zp'(z)) : z \in E \} \subset \bigcap \{ w \in \mathbb{C} : \text{Re} w > a \} \].
Now for all real $r_2$, $s_1 \leq -(1 + r_2^2)/2$ we have for each $z \in E$

$$\text{Re} \left\{ \Psi(ir_2, s_1) \right\} = \gamma + \frac{\lambda(1-\gamma)s_1}{\mu(n+p)} \text{Re} \left\{ \Psi_0(z) \right\}$$

$$\leq \gamma - \frac{\lambda(1-\gamma)\beta}{2\mu(n+p)} (1 + r_2^2)$$

$$\leq \gamma - \frac{\lambda(1-\gamma)\beta}{2\mu(n+p)} = \alpha.$$

Hence for each $z \in E$, $\Psi(ir_2, s_1) \notin \bigcap$. Thus by Theorem 1.5.1,

$$\text{Re} \{ p(z) \} > 0 \text{ in } E. \text{ This proves (3.4.1).}$$

**Corollary 3.4.1.** Let $f(z)$ and $g(z)$ are in $H_p$ and $g(z)$ satisfies the condition

$$\text{Re} \left\{ \frac{D^{n+p-1} g(z)}{D^{n+p} g(z)} \right\} > \beta \quad (0 \leq \beta < 1, \ z \in E).$$

If $\lambda \geq 1$ and

$$\text{Re} \left\{ (1-\lambda) \frac{D^{n+p-1} f(z)}{D^{n+p} g(z)} + \lambda \frac{D^{n+p} f(z)}{D^{n+p} g(z)} \right\} > \alpha, \ z \in E$$

for some $\alpha (0 \leq \alpha < 1)$, then

$$\text{Re} \left\{ \frac{D^{n+p} f(z)}{D^{n+p} g(z)} \right\} > \gamma = \frac{\alpha(2n+2p+\beta) + \beta(\lambda-1)}{2(n+p) + \lambda\beta}. $$
Proof. We have for each \( z \in E \),

\[
\frac{D^{n+p} f(z)}{D^{n+p} g(z)} = [(1-\lambda) \frac{D^{n+p-1} f(z)}{D^{n+p-1} g(z)} + \lambda \frac{D^{n+p} f(z)}{D^{n+p} g(z)}] \\
+ (\lambda - 1) \frac{D^{n+p-1} f(z)}{D^{n+p-1} g(z)}.
\]

Since \( \lambda \geq 1 \), making use of (3.4.1) and (3.4.3) (for \( \mu = 1 \)), we deduce that for \( z \in E \)

\[
\lambda \Re\left\{ \frac{D^{n+p} f(z)}{D^{n+p} g(z)} \right\} > \frac{2\alpha(n+p) + \lambda\beta^2(\lambda - 1)}{2(n+p) + \lambda\beta} + \alpha
\]

so that

\[
\Re\left\{ \frac{D^{n+p} f(z)}{D^{n+p} g(z)} \right\} > \gamma = \frac{\alpha(2(n+p) + \beta^2 + \beta(\lambda - 1))}{2(n+p) + \lambda\beta}
\]

This proves (3.4.4).

Corollary 3.4.2. Let \( \lambda \) be a complex number with \( \Re(\lambda) > 0 \) (\( \lambda \neq 0 \)). If \( f(z) \in H_p \) satisfies

\[
\Re\left\{ (1-\lambda)(\frac{D^{n+p-1} f(z)}{z^p}) + \lambda \frac{D^{n+p} f(z)}{z^p} (\frac{D^{n+p-1} f(z)}{z^p}) \right\} > \alpha, z \in E
\]

for \( 0 < \alpha < 1 \) and \( \mu > 0 \), then
Further, if \( \lambda \geq 1 \) and \( f(z) \in H_p \) satisfies

\[
\text{Re} \left\{ \frac{D^{n+p-1} f(z)}{z^p} \right\} > \frac{2(n+p)\mu \alpha + \text{Re} \lambda}{2(n+p)\mu + \text{Re} \lambda}, \quad z \in E.
\]

then

\[
\text{Re} \left\{ (1-\lambda) \frac{D^{n+p-1} f(z)}{z^p} + \lambda \frac{D^n f(z)}{z^p} \right\} > \alpha, \quad z \in E.
\]

**Proof.** The result (3.4.5) follows by putting 
\( g(z) = z^p \) and by following the lines of proof of Theorem 3.4.1. The estimate (3.4.6) is obtained by setting 
\( g(z) = z^p \) in Corollary 3.4.1.

**Remarks.** 1. For \( n = 0, p = 1, \mu = 1 \) and \( \alpha = 0 \), Theorem 3.4.1 yields the corresponding result due to Karunakaran and Ponnusamy [47].

2. Choosing \( n = 0, p = 1, \lambda = 1 \) in Corollary 3.4.2, we obtain (1.4.9), a result due to Ponnusamy and Karunakaran [91].

3. Choosing \( n, \mu, p \) and \( \lambda \) appropriately in Corollary 3.4.2, we obtain the following results.

\[
(3.4.7) \quad \text{Re} \left\{ f'(z) + \lambda zf''(z) \right\} > \alpha (0 \leq \alpha < 1, z \in E)
\]

implies 
\[ \text{Re} \left\{ f'(z) \right\} > \frac{2\alpha + \text{Re} \lambda}{2 + \text{Re} \lambda}, (z \in E). \]
For $\lambda$ complex and $\text{Re}(\lambda) > 0$ ($\lambda \neq 0$),

\begin{equation}
(3.4.8) \quad \text{Re} \left\{ (1-\lambda) \frac{f(z)}{z} + \lambda f'(z) \right\} > \alpha \quad (0 \leq \alpha < 1, \ z \in E)
\end{equation}

implies

\begin{equation*}
\text{Re} \left\{ \frac{f(z)}{z} \right\} > \frac{2\alpha + \text{Re}\lambda}{2 + \text{Re}\lambda}, \quad (z \in E),
\end{equation*}

while for $\lambda$ real with $\lambda \geq 1$,

\begin{equation}
(3.4.9) \quad \text{Re} \left\{ (1-\lambda) \frac{f(z)}{z} + \lambda f'(z) \right\} > \alpha \quad (0 \leq \alpha < 1, \ z \in E)
\end{equation}

implies

\begin{equation*}
\text{Re} \left\{ f'(z) \right\} > \frac{3\alpha + \lambda - 1}{2 + \lambda}, \quad (z \in E).
\end{equation*}

We note that the results (3.4.7), (3.4.8) and (3.4.9) were obtained by Juneja and Ponnusamy [45].

**Theorem 3.4.2.** Let $\lambda$ be a complex number satisfying $\text{Re}(\lambda) > 0$. Let $f(z) \in H_d$ satisfies the condition

\begin{equation}
(3.4.10) \quad \text{Re} \left\{ (1-\lambda) \left( \frac{D^{n+p-1} f(z)}{z^p} \right)^\mu + \right.
\end{equation}

\begin{equation*}
\left. + \lambda \left( \frac{D^{n+p} f(z)}{z^p} \right) \left( \frac{n+p-1 f(z)}{z^p} \right)^{\mu-1} \right\} > \alpha, \quad z \in E
\end{equation*}

for some $\alpha$ ($0 \leq \alpha < 1$) and $\mu > 0$. Then

\begin{equation}
(3.4.11) \quad \text{Re} \left\{ \left( \frac{D f(z)}{z^p} \right)^\mu \right\} > \alpha + (1-\alpha)(2\varepsilon - 1), \quad z \in E
\end{equation}
where \( \mathcal{F} = F(1,1;1 + \frac{\mu(n+p)}{\text{Re}(\lambda)} ; \frac{1}{2}) / 2 \). The estimate is sharp in the sense that the bound in (3.4.11) cannot be improved.

**Proof.** Let \( p(z) = \left\{ \frac{n+p-1}{D f(z)/z^p} \right\}^\mu \). Then \( p(z) \) is analytic in \( E \) with \( p(0) = 1 \). Differentiating \( p(z) \) and using the identity (1.4.5), we get

\[
(1-\lambda) \left\{ \frac{D f(z)}{z^p} \right\}^\mu + \lambda \frac{D f(z)}{z^p} \left\{ \frac{D f(z)}{z^p} \right\}^{\mu-1}
\]

\[
= p(z) + \lambda \frac{zp'(z)}{\mu(n+p)},
\]

so that by the hypothesis (3.4.10),

\[
\text{Re} \left\{ p(z) + \frac{\lambda}{\mu(n+p)} zp'(z) \right\} > \alpha, \ z \in E.
\]

In view of Theorem 1.5.2, this implies that

\[
(3.4.12) \quad \text{Re} \{ p(z) \} > \alpha + (1-\alpha)(2 \mathcal{F} - 1)
\]

where

\[
\mathcal{F} = \int_0^{\frac{\text{Re}(\lambda)}{\mu(n+p)}} \left\{ 1 + t^\frac{\text{Re}(\lambda)}{\mu(n+p)} \right\}^{-1} \text{dt}.
\]

Setting \( \text{Re}(\lambda) = \lambda_1 > 0 \), we have
\[ \Phi = \int_0^1 \left\{ 1 + \frac{\lambda_1}{\mu(n+p)} \right\}^{-1} dt \]

\[ = \frac{\mu(n+p)}{\lambda_1} \int_0^1 \frac{\mu(n+p)}{\lambda_1} (1+u)^{-1} du \]

\[ = F(1, \frac{\mu(n+p)}{\lambda_1}; 1 + \frac{\mu(n+p)}{\lambda_1}, -1), \text{ by (3.2.4)} \]

\[ = F(1,1; 1 + \frac{\mu(n+p)}{\lambda_1}; \frac{1}{2}) / 2, \text{ by (3.2.5).} \]

Substituting the value of \( \Phi \) in (3.4.12), we get the required result.

Since the estimate (1.5.7) is sharp, our estimate (3.4.11) is sharp in the sense that the bound can not be improved.

This completes the proof of Theorem 3.4.2.

**Corollary 3.4.3.** Let \( \lambda \) be a real number satisfying \( \lambda \geq 1 \). If \( f(z) \in H_p \) satisfies

\[ (3.4.13) \quad \text{Re} \left\{ (1-\lambda) \frac{D^{n+p-1} f(z)}{z^p} + \lambda \frac{D^n f(z)}{z^p} \right\} > \alpha, \; z \in E \]

for \( \alpha \) \((0 \leq \alpha < 1)\), then

\[ \text{Re}(\frac{D^{n+p} f(z)}{z^p}) > \alpha + (1-\alpha)(2 \Phi' -1)(1- \lambda^{-1}), \; z \in E \]

where \( \Phi' = F(1,1; 1 + \frac{n+p}{\lambda}; \frac{1}{2}) / 2 \). The result is best possible.
Proof. The result follows by using the hypothesis (3.4.13) and the estimate (3.4.11) for \( \lambda \) real and \( \mu = 1 \)
in the identity

\[
\lambda \left\{ \frac{D^{n+p} f(z)}{z^p} \right\} = \left[ (1-\lambda) \frac{D^{n+p-1} f(z)}{z^p} + \lambda \frac{D^{n+p} f(z)}{z^p} \right] + (\lambda-1) \frac{D^{n+p-1} f(z)}{z^p}.
\]

Remark. We note that if \( n = 0, p = 1 \) and \( \mu = \lambda > 0 \), Corollary 3.4.2 show that if

\[
(3.4.14) \quad \text{Re}\left[ (1-\lambda) \left\{ \frac{f(z)}{z} \right\}^\lambda + \lambda f'(z) \left\{ \frac{f(z)}{z} \right\}^{\lambda-1} \right] > \alpha
\]

for \( 0 < \alpha < 1 \), \( z \in E \), then

\[
(3.4.15) \quad \text{Re} \left\{ \frac{f(z)}{z} \right\}^\lambda > \frac{2\alpha+1}{3}, \quad z \in E.
\]

Whereas, if \( f(z) \in H \) satisfies the condition (3.4.14) then by (3.4.11)

\[
\text{Re} \left\{ \frac{f(z)}{z} \right\}^\lambda > 2(1-\ln 2)\alpha + (2 \ln 2 - 1), \quad z \in E.
\]

which is certainly better than (3.4.15).

Similarly, if \( n = p = 1, \lambda = 2 \), Corollary 3.4.2 yields the following:
If

\[
\text{Re} \left\{ 2 \frac{D^2 f(z)}{z} - \frac{D^1 f(z)}{z} \right\} > \alpha, \quad z \in E
\]

then \( \text{Re} \left\{ D^2 f(z)/z \right\} > (5\alpha+1)/6 \) whereas by using (3.2.6), Corollary 3.4.3 implies that
\[ \text{Re}\left\{ \frac{D^2 f(z)}{z} \right\} > \left[ \{(3-2 \ln 2)\alpha + (2 \ln 2-1)\} / 2 \right] > (5\alpha + 1)/6, \quad z \in E. \]

With these observations we conclude that Theorem 3.4.2 and Corollary 3.4.3 are improvement over Corollary 3.4.2. Thus, suitably choosing the parameters \( \mu, \lambda, n, \ p \) and \( \alpha \) in Theorem 3.4.2 and Corollary 3.4.3, we can improve the results contained in the remarks following Corollary 3.4.2.

We note that if \( X \) is real with \( X > 0 \) and

\[
 h(z) = \frac{D^{n+p}}{D^{n+p}} \frac{f(z)}{g(z)} + (\frac{1}{\lambda} - 1) \frac{D^{n+p-1}}{D} \frac{f(z)}{f(z)}
\]

then Theorem 3.4.1 (for \( \mu = 1 \)) reduces to

\[
 (3.4.16) \quad \text{Re}\left\{ h(z) \right\} > \frac{\alpha}{\lambda} \quad (0 \leq \alpha < 1, \ z \in E) \quad \text{implies}
\]

\[
 \text{Re}\left\{ \frac{D^{n+p-1}}{D} \frac{f(z)}{g(z)} \right\} > \frac{2(n+p)\alpha + \lambda \beta}{2(n+p) + \lambda \beta} \quad (0 < \beta \leq 1, \ z \in E).
\]

whenever

\[
 \text{Re}\left\{ \frac{D^{n+p}}{D} \frac{g(z)}{g(z)} \right\} > \beta \quad (0 < \beta \leq 1, \ z \in E).
\]

In the following theorem we shall extend the above result as follows:
Theorem 3.4.3. Suppose $f(z)$ and $g(z)$ are in $H_p$ and $g(z)$ satisfies the condition

$$\text{Re} \left\{ \frac{D^{n+p-1}g(z)}{D^{n+p}g(z)} \right\} > \beta \quad (0 \leq \beta < 1), \quad z \in \mathcal{E}.$$ 

If

$$\text{Re} \left\{ \frac{D^{n+p}f(z)}{D^{n+p+1}g(z)} - \frac{D^{n+p-1}f(z)}{D^{n+p}g(z)} \right\} > \frac{(1-\alpha)\beta}{2(n+p)}, \quad z \in \mathcal{E}$$

for some $\alpha \ (0 \leq \alpha < 1)$, then

$$\text{Re} \left\{ \frac{D^{n+p-1}f(z)}{D^{n+p}g(z)} \right\} > \alpha, \quad z \in \mathcal{E}$$

and

$$\text{Re} \left\{ \frac{D^{n+p}f(z)}{D^{n+p+1}g(z)} \right\} > \frac{(2n+2p+\beta)\alpha - \beta}{2(n+p)}, \quad z \in \mathcal{E}.$$

**Proof.** Let $p(z) = (1-\alpha)^{-1} \left[ \frac{D^{n+p-1}f(z)}{D^{n+p}g(z)} - \alpha \right]$. Then $p(z)$ is analytic in $\mathcal{E}$ with $p(0) = 1$. Setting $
\Psi_o(z) = \left\{ D^{n+p}g(z) / D^{n+p}f(z) \right\}, \quad z \in \mathcal{E},$ we observe that by the hypothesis, $\text{Re} \ \Psi_o(z) > \beta$ in $\mathcal{E}$. A simple computation shows that

$$\frac{(1-\alpha)zp'(z)\Psi_o(z)}{n+p} = \frac{D^{n+p}f(z)}{D^{n+p+1}g(z)} - \frac{D^{n+p-1}f(z)}{D^{n+p}g(z)} = \Psi(p(z),zp'(z)),$$

where $\Psi(r,s) = \{(1-\lambda)\Psi_o(z)s \}/(n+p)$. Using the hypothesis
(3.4.17), we get for each \( z \in E \)

\[
\mathcal{C}(p(z), zp'(z)); z \in E \subseteq \bigcap = \{ w \in C : \text{Re}(w) > -\frac{(1-\alpha)\beta}{2(n+p)} \}.
\]

Now, for all real \( r_2, s_1 \leq -\frac{(1+ r_2^2)}{2} \), we have for each \( z \in E \)

\[
\text{Re}\{\mathcal{C}(ir_2, s_1)\} = \frac{s_1(1-\alpha)\text{Re}\{\mathcal{C}_0(z)\}}{n + p} \leq -\frac{(1-\alpha)\beta(1+ r_2^2)}{2(n+p)}
\]

\[
\leq -\frac{(1-\alpha)\beta}{2(n+p)}.
\]

This shows that \( \mathcal{C}(ir_2, s_1) \notin \bigcap \) for each \( z \in E \). Hence by Theorem 1.5.1, \( \text{Re}\{p(z)\} > 0 \) in \( E \); that is,

\[
\text{Re}\{D^{n+1} f(z) / D^{n+1} g(z)\} > \alpha, z \in E. \] \hfill (3.4.18)

This proves (3.4.18).

The proof of (3.4.19) follows from the identity

\[
\text{Re}\{D^{n+p} f(z) / D^{n+p} g(z)\} = \text{Re}\{D^{n+p} f(z) / D^{n+p} g(z)\} + \text{Re}\{D^{n+p} f(z) / D^{n+p} g(z)\} + \text{Re}\{D^{n+p} f(z) / D^{n+p} g(z)\}
\]

by using the hypothesis (3.4.17) and the estimate in (3.4.18). This completes the proof of Theorem 3.4.3.

Remark. For \( n = 0, p = 1 \) and \( g(z) = z \), Theorem 3.4.3 gives
(a) \( \text{Re} \left\{ f'(z) - \frac{f(z)}{z} \right\} > -\left(\frac{1-a}{2}\right)(0 \leq a < 1, z \in E) \) implies

\[ \text{Re} \left\{ \frac{f(z)}{z} \right\} > a \quad (z \in E) \]

and

\[ \text{Re} \left\{ f'(z) \right\} > \frac{3a-1}{2} \quad (z \in E). \]

For \( n = p = 1 \) and \( g(z) = z \) in Theorem 3.4.3, we have

(b) \( \text{Re} \left\{ (1-a) + 2zf''(z) \right\} > 0(z \in E) \) implies

\[ \text{Re} \left\{ f'(z) \right\} > a \quad (z \in E). \]

We next prove

**Theorem 3.4.4.** Let \( f(z) = z^p + a_{p+k} z^{p+k} + \ldots \) be

analytic in \( E \) (\( k \) is integer \( \geq 1 \)). If \( F_\lambda(z) = (1-\lambda)f(z) + \lambda zf'(z), \lambda \geq 0, \) and

\[ \text{Re}\left\{ \frac{F_\lambda(j)(z)}{z^{p-j}} \right\} > a \quad (0 \leq a < \frac{p!(1-\lambda + \lambda p)}{(p-j)!}; \ z \in E) \]

then

\[ \text{Re}\left\{ \frac{f(j)(z)}{z^{p-j}} \right\} > \frac{(p-j)! 2a + p! \lambda k}{(p-j)! 2+(k-2)\lambda + 2\lambda p}, \ z \in E \]

where \( 0 \leq j < p. \)

**Proof.** On differentiating \( F_\lambda(z) \), we get for each \( j \ (0 \leq j < p) \)

\[ F_\lambda^{(j)}(z) = (1-\lambda + \lambda j)f^{(j)}(z) + \lambda zf^{(j+1)}(z). \]
Thus, the hypothesis is equivalent to

\[(3.4.20) \quad \Re \left\{ \frac{(p-j)!(1-\lambda + \lambda j)}{p!(1-\lambda + \lambda p)} \cdot \frac{f^{(j)}(z)}{z^{p-j}} + \right. \]

\[+ \frac{(p-j)! \lambda}{p!(1-\lambda + \lambda p)} \cdot \frac{f^{(j+1)}(z)}{z^{p-j-1}} \right\} > \eta \quad (0 \leq \eta < 1, \quad z \in \mathbb{E})\]

where \( \eta = \{ (p-j)! \alpha \} / \{ p!(1-\lambda + \lambda p) \} \). Now, taking

\[M(z) = f^{(j)}(z)(p-j)! / p! \quad \text{and} \quad N(z) = z^{p-j}, \quad (3.4.20) \text{can be written as}\]

\[\Re \left\{ (1-\lambda) \frac{M(z)}{N(z)} + \lambda \frac{M'(z)}{N(z)} \right\} > \eta \quad (0 \leq \eta < 1)\]

where \( \lambda = \{ \lambda(p-j) \} / (1 - \lambda + \lambda p) > 0 \). Thus, the function \( M(z) \) and \( N(z) \) satisfies the hypothesis of Theorem 1.5.3 for \( m = p - j \). Hence by (1.5.8), we obtain

\[\Re\left\{ \frac{f^{(j)}(z)}{z^{p-j}} \right\} > \frac{(p-j)! \lambda k}{(p-j)! \left\{ 2+(k-2)\lambda + 2\lambda p \right\}}, \quad z \in \mathbb{E}.\]

This proves Theorem 3.4.4.

**Remark.** For \( k = 1 \), the above result was obtained by Saitoh [98] by another method.

Using Theorem 1.5.3 (for \( \lambda = 1 \)), we obtain the following theorem which can be proved in a manner similar to that of Theorem 3.4.4.
Theorem 3.4.5. Let \( f(z) = z^p + a_{p+k} z^{p+k} + \ldots \) be analytic in \( E \) (\( k \) is an integer \( \geq 1 \)).

If

\[
\text{Re} \left\{ \frac{f^{(j)}(z)}{z^{p-j}} \right\} > \alpha \left( 0 \leq \alpha < \frac{p!}{(p-j)!} \right) ; \, z \in E
\]

then

\[
\text{Re} \left\{ \frac{f^{(j-1)}(z)}{z^{p-j-1}} \right\} > \frac{(p-j+1)! \, 2\alpha p! \, k}{(p-j+1)! \, 2(p-j)+(k+2)} , \, z \in E
\]

where \( 1 \leq j < p \).

Remark. For \( k = 1 \), the above theorem was proved by Saitoh [98] by using a different technique.