CHAPTER V

ON CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS
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WITH NEGATIVE COEFFICIENTS

5.1. Let $T$ denote the class of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0 \text{ for } k \geq 2,$$

which are analytic in the unit disc $E$. We denote by $T^*(\alpha)$ and $C^*(\alpha)$, the subclasses of $T$ which are respectively, starlike of order $\alpha$ and convex of order $\alpha$ ($0 < \alpha < 1$).

Let the function $g(z)$ be defined in the unit disc $E$ by

$$g(z) = z - \sum_{k=2}^{\infty} b_k z^k, \quad b_k \geq 0 \text{ for } k \geq 2.$$

Let $n$ be any non-negative integer. Let $A, B, \alpha, \beta$ and $\gamma$ be arbitrary fixed real numbers such that $-1 \leq B < A \leq 1$, $0 < \alpha < 1$, $0 < \beta \leq 1$ and $0 < \gamma \leq 1$. Then a function $f(z) \in T$ is said to be in the class $U_n(\alpha, \beta, \gamma, A, B)$ if it satisfies the condition

$$\left| \frac{\mathcal{D}^n f(z)}{g(z)} - 1 \right| < \gamma \quad (z \in E)$$

for some $g(z) \in T^*(\alpha)$ and where $\mathcal{D}^n f(z)$ is the Ruscheweyh's derivative of $f(z)$ given by
\( \frac{z}{(1-z)^{n+1}} f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!} = z - \sum_{k=2}^{\infty} \delta(n,k) a_k z^k, \)

Here \( \delta(n,k) = (n+k-1)!/(n!(k-1)!). \)

We say that a function \( f(z) \in T \) is said to be in the class \( V_n(\alpha, \beta, \gamma, A, B) \) if \( f(z) \) satisfies the condition (5.1.3) for some \( g(z) \in C^*(\alpha) \).

It is easily seen that for \( n = 1 \) and \( (A, B) = (1, -1) \), \( U_n(\alpha, \beta, \gamma, A, B) \) and \( V_n(\alpha, \beta, \gamma, A, B) \) coincide with the classes \( T^*(\alpha, \beta, \gamma) \) and \( C^*(\alpha, \beta, \gamma) \) introduced and studied by Srivastava and Owa [110, 111]. The class \( T^*(\alpha, \beta, \gamma) \), further, reduces to the class \( P^*(\beta, \gamma) \) studied by Gupta and Jain [34] for \( \alpha = 1 \) (for \( \alpha = 1, g(z) = z \)). We denote \( U_n(1, \beta, \gamma, A, B) \) by \( U_n(\beta, \gamma, A, B) \).

We, further, observe that by specializing the parameters \( A, B, \alpha \) and \( n \), our classes give rise to the following new subclasses of analytic functions with negative coefficients. For instance,

\[ U_n(1, \beta, \gamma, A, B) \equiv U_n(\beta, \gamma, A, B) \]

\[ = \left\{ f(z) \in T : \frac{D^n f(z)}{z} - 1 < \gamma, z \in E \right\}, \]

\[ V_1(1, \beta, \gamma, 1, -\mu) \equiv P^*(\mu, \beta, \gamma) \]

\[ = \left\{ f(z) \in T : \frac{f'(z)}{\mu f'(z) + 1 - (1+\mu)\beta} - 1 < \gamma, 0 < \mu \leq 1, z \in E \right\}. \]
\[ U_0(\alpha, \beta, \gamma, A, B) = \{ f(z) \in T : \left| \frac{f(z)}{g(z)} - 1 \right| < \gamma, \quad (A-B)(1-B) - B \frac{|\gamma|}{g(z) - 1} \} \]

and

\[ U_0(1, \beta, \gamma, A, B) \equiv U_0(\beta, \gamma, A, B) \]

\[ = \{ f(z) \in T : \left| \frac{f(z)}{z} - 1 \right| < \gamma, \quad z \in E \} \].

As noticed above, the class \( U_{n}(\alpha, \beta, \gamma, A, B) \) (respectively \( V_{n}(\alpha, \beta, \gamma, A, B) \)) includes various subclasses of analytic functions with negative coefficients, a study of its properties will lead to a unified study of these classes.

In section 5.2 of this chapter we prove certain coefficient inequalities of functions belonging to the classes \( U_{n}(\alpha, \beta, \gamma, A, B) \) and \( V_{n}(\alpha, \beta, \gamma, A, B) \). We use some of these coefficient inequalities to study some other aspects such as convolution properties, integral transforms etc. For functions in the class \( U_{n}(\alpha, \beta, \gamma, A, B) \) (resp. \( V_{n}(\alpha, \beta, \gamma, A, B) \)) in section 5.3. In section 5.4, we derive general distortion theorems involving fractional derivatives and fractional integrals of functions belonging to these classes. Finally, section 5.5 deals with the determination of radii of convexity for functions in the classes \( U_{n}(\alpha, \beta, \gamma, A, B) \) and \( V_{n}(\alpha, \beta, \gamma, A, B) \) respectively. The results found in this
chapter besides generalizing the work of Gupta and Jain [34], Srivastava and Owa [110], Sarangi and Uralegaddi [99] give a number of new results.

5.2. In this section, we find a necessary condition for a $f(z)$ to be in the class $U_n(\alpha, \beta, \gamma, A, B)$ or $V_n(\alpha, \beta, \gamma, A, B)$. We also obtain the sharp coefficient estimates and some coefficient inequalities for functions belonging to the classes $U_n(\alpha, \beta, \gamma, A, B)$ and $V_n(\alpha, \beta, \gamma, A, B)$.

Unless otherwise mentioned, we assume throughout this chapter that $-1 < B < A < 1, 0 < \alpha < 1, 0 < \beta < 1, 0 < \gamma < 1$ and $n \in N_o = \{0, 1, 2, \ldots\}$.

To establish our main results, we need the following Lemmas due to Silverman [102],

**Lemma 5.2.1.** Let the function $g(z)$ be defined by (5.1.2). Then $g(z) \in T^*(\alpha)$, if and only if

\[
\sum_{k=2}^{\infty} \frac{(k-\alpha)}{(1-\alpha)} b_k \leq 1.
\]

**Lemma 5.2.2.** Let the function $g(z)$ be defined by (5.1.2). Then $g(z) \in C^*(\alpha)$, if and only if

\[
\sum_{k=2}^{\infty} \frac{k(k-\alpha)}{(1-\alpha)} b_k \leq 1.
\]
Applying Lemma 5.2.1, we first prove:

**Theorem 5.2.1.** Let the function $f(z)$ defined by (5.1.1) be in the class $U_n(\alpha, \beta, \gamma, A, B)$. Then

\[
\sum_{k=2}^{\infty} \left[ (1-\gamma B)\delta(n,k)a_k - \frac{(1-\alpha)}{k-\alpha} \right] \left\{ (1-\gamma A)+(A-B)\gamma B \right\} a_k \leq (A-B)\gamma (1-\beta)
\]

where $\delta(n,k) = (n+k-1)!/(n!(k-1)!)$.

**Proof.** Since $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in U_n(\alpha, \beta, \gamma, A, B)$, there exists a function $g(z) = z - \sum_{k=2}^{\infty} b_k z^k \in T^*(\alpha)$ such that

\[
\left| \frac{D^nf(z)}{g(z)} - 1 \right| < \gamma, \; z \in E.
\]

On replacing the power series expansions of $D^nf(z)$ and $g(z)$ in (5.2.4) followed by using the fact that $\text{Re}(z) < |z|$ for all $z \in \mathbb{C}$, we get

\[
\sum_{k=2}^{\infty} \left\{ \delta(n,k)a_k - b_k \right\} z^{k-1}
\]

\[
\text{Re} \left[ \frac{\sum_{k=2}^{\infty} \left\{ \delta(n,k)a_k - b_k \right\} z^{k-1}}{(A-B)(1-\beta) + \sum_{k=2}^{\infty} \left\{ B \delta(n,k)a_k - (A-B)(1-\beta) + B \right\} b_k z^{k-1}} \right] \leq \gamma, \; z \in E.
\]

We choose values of $z$ on the real axis so that $D^nf(z)/g(z)$ is real. Upon clearing the denominator in (5.2.5) and letting $z \to 1^-$ through real values, we deduce that

\[
\sum_{k=2}^{\infty} \left\{ \delta(n,k)a_k - b_k \right\} \leq \gamma \left[ (A-B)(1-\beta) + \sum_{k=2}^{\infty} \left\{ B \delta(n,k)a_k - (A-B)(A-B) \right\} b_k \right].
\]
which after simplification yields

\[(5.2.6) \sum_{k=2}^{\infty} [(1-\gamma B)\delta(n,k)a_k - \{(1-\gamma A)+(A-B)\gamma \} b_k] \leq (A-B)\gamma (1-\beta).\]

Since \(g(z) \in T^{*}(\alpha)\), Lemma 5.2.1 implies

\[(5.2.7) \quad b_k \leq \frac{1-\alpha}{k-\alpha}, \quad k \geq 2.\]

Making use of (5.2.7) in (5.2.6), we complete the proof of Theorem 5.2.1.

**Corollary 5.2.1.** Let \(f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in U_n(\alpha,\beta,\gamma,A,B)\).

Then

\[(5.2.8) \quad a_k \leq \frac{(1-\alpha)(1-\gamma B)+(A-B)\gamma (k-1)(1-\beta)}{(1-\gamma B)\delta(n,k)(k-\alpha)}, \quad k \geq 2.\]

The estimate (5.2.8) is sharp for the function \(f(z)\) defined by

\[(5.2.9) \quad f(z) = z - \frac{(1-\alpha)(1-\gamma B)+(A-B)(k-1)(1-\beta)}{(1-\gamma B)\delta(n,k)(k-\alpha)} z^k, \quad k \geq 2\]

with respect to

\[g(z) = z - \left(\frac{1-\alpha}{k-\alpha}\right) z^k, \quad (k \geq 2).\]

Putting \(\alpha = 1\) in Theorem 5.2.1 and Corollary 5.2.1, we obtain the following necessary condition and coefficient estimate for functions belonging to the class \(U_n(\beta,\gamma,A,B)\).
Corollary 5.2.2. Suppose \( f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in U_n(\beta, \gamma, A, B) \).

Then

\[
\sum_{k=2}^{\infty} (1-\gamma B) \delta(n,k) a_k \leq (A-B) \gamma (1-\beta)
\]

and

\[
a_k \leq \frac{\gamma(A-B)(1-\beta)}{(1-\gamma B) \delta(n,k)}, \quad k \geq 2.
\]

The estimate (5.2.11) is sharp for the function

\[
f(z) = z - \frac{(A-B) \gamma (1-\beta)}{(1-\gamma B) \delta(n,k)} \cdot z^k, \quad k \geq 2,
\]

where \( \delta(n,k) \) is given as in Theorem 5.2.1.

**Remark.** In the special case when \( n = 1 \) and \( (A,B) = (1,-1) \), Theorem 5.2.1 corresponds to an earlier result due to Srivastava and Owa [110].

The next result shows that if \(-1 < B < 0\), the condition (5.2.10) is sufficient for functions belonging to the class \( U_n(\beta, \gamma, A, B) \).

**Theorem 5.2.2.** Let \( f(z) = z - \sum_{k=2}^{\infty} a_k z^k \), \( a_k \geq 0 \) be analytic in \( E \). If for \(-1 < B < 0\),

\[
\sum_{k=2}^{\infty} (1-\gamma B) \delta(n,k) a_k \leq (A-B) \gamma (1-\beta),
\]
where \( \delta(n,k) = \frac{(n+k-1)!}{(n!(k-1)!)} \), then \( f(z) \in U_n(\beta, \gamma, A, B) \).

The result is sharp.

**Proof.** Suppose that (5.2.13) holds. Then, for \( z \in E \), we have

\[
\left| \frac{D^n f(z)}{z} - 1 \right| - \gamma \left| (A-B)(1-\beta) - B \left( \frac{D^n f(z)}{z} - 1 \right) \right|
\]

\[
= \left| - \sum_{k=2}^{\infty} \delta(n,k)a_k z^{k-1} \right| - \gamma \left| (A-B)(1-\beta) + B \sum_{k=2}^{\infty} \delta(n,k)a_k z^{k-1} \right|
\]

\[
\leq \sum_{k=2}^{\infty} \delta(n,k)a_k |z|^{k-1} - \gamma \left| (A-B)(1-\beta) + B \sum_{k=2}^{\infty} \delta(n,k)a_k z^{k-1} \right|, \text{ (since } -1 \leq B < 0 \)
\]

\[
< \sum_{k=2}^{\infty} \delta(n,k)a_k - \gamma \left| (A-B)(1-\beta) + B \sum_{k=2}^{\infty} \delta(n,k)a_k \right|
\]

\[
= \sum_{k=2}^{\infty} (1-\gamma B)\delta(n,k)a_k - (A-B)\gamma (1-\beta) \leq 0, \text{ by (5.2.13)}.
\]

Hence

\[
\left| \frac{D^n f(z)}{z} - 1 \right| < \gamma, \ z \in E,
\]

which implies that \( f(z) \in U_n(\beta, \gamma, A, B) \).

We note that the function \( f(z) \) defined by (5.2.12) is an extremal function with respect to the above theorem, since for this function
and the inequality is attained in (5.2.13).

Substituting \( n = 1 \) and \((A, B) = (1, -\mu) \) \((0 < \mu \leq 1)\) in Theorem 5.2.2 and Corollary 5.2.2, we have the following result which in turn gives the corresponding result obtained by Gupta and Jain [34] for \( \mu = 1 \).

**Corollary 5.2.3.** Let \( f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \ a_k \geq 0, \) be analytic in \( E \). Then \( f(z) \in P(\mu, \beta, \gamma) \), if and only if

\[
\sum_{k=2}^{\infty} k(1 + \mu \gamma)a_k \leq \gamma(1+\mu)(1-\beta)
\]

and

\[
a_k \leq \frac{\gamma(1+\mu)(1-\beta)}{k(1+ \mu \gamma)}, \ k \geq 2.
\]

The estimates (5.2.14) and (5.2.15) are sharp.

Following the lines of proof of Theorem 5.2.1 and using Lemma 5.2.2 instead of Lemma 5.2.1, we can prove

**Theorem 5.2.3.** Let the function \( f(z) \) defined by (5.1.1) be in the class \( V_n(\alpha, \beta, \gamma, A, B) \). Then

\[
\sum_{k=2}^{\infty} [(1-\gamma B)\delta(n,k)a_k \frac{(1-\alpha)}{k(k-\alpha)}](1-\gamma A)+(A-B)\gamma B] \leq (A-B)\gamma(1-\beta)
\]

and

\[
a_k \leq \frac{(1-\gamma B)(1-\alpha)+(A-B)(1-\beta)(k-1)(k+1-\alpha)}{(1-\gamma B)\delta(n,k)k(k-\alpha)}, \ k \geq 2.
\]
The estimate (5.2.17) is sharp for the function

\[ f(z) = z - \frac{(1-\gamma B)(1-\alpha)+\gamma(A-B)(1-\beta)(k-1)(k+1-\alpha)}{(1-\gamma B)\delta(n,k)k(k-\alpha)} z^k, \quad k \geq 2, \]

with respect to

\[ g(z) = z - \frac{(1-\alpha)}{k(k-\alpha)} z^k, \quad k \geq 2. \]

5.3. Using the coefficient inequalities obtained in section 5.2, we prove certain results on convolutions and integral transforms for functions belonging to the class \( U_n(\beta, \gamma, A, B) \).

We first prove a inclusion relation.

**Theorem 5.3.1.** For \(-1 < B < A \leq 1 \) \((-1 < B < 0)\) and \( n \in \mathbb{N}_0 \), we have

\[ U_{n+1}(\beta, \gamma, A, B) \subset U_n(\lambda, \gamma, A, B) \]

where \( \lambda = \lambda(n, \beta) = \left\{ 1 + (n+1)\beta \right\} / (n+2) \). The result is best possible.

**Proof.** Suppose \( f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in U_{n+1}(\beta, \gamma, A, B) \).

Then by (5.2.13)

\[ \sum_{k=2}^{\infty} \frac{(1-\gamma B)\delta(n+1,k)a_k}{(A-B)\gamma(1-\beta)} \leq 1. \]
To show that \( f(z) \in U_n(\lambda, \gamma, A, B) \), we have to prove that

\[
\sum_{k=2}^{\infty} \frac{(1-\gamma B)(n,k)a_k}{(A-B)\gamma(1-\lambda)} \leq 1.
\]

In view of (5.3.1) the above inequality is true if

\[
\frac{(1-\gamma B)(n,k)}{\gamma(A-B)(1-\lambda)} a_k \leq \frac{(1-\gamma B)(n+1,k)}{\gamma(A-B)(1-\beta)} a_k , \quad k \geq 2.
\]

that is, if

\[
(5.3.2) \quad \lambda \leq \frac{(k+1) + (n+1)\beta}{n + k}, \quad k \geq 2.
\]

Since the right hand side of (5.3.2) is an increasing function \( k \geq 2 \), putting \( k = 2 \) in (5.3.2) we get

\[
\lambda \leq \frac{1 + (n+1)\beta}{n + 2}.
\]

This completes the proof.

The estimate is sharp for the function

\[
f(z) = z - \frac{(1-\gamma B)(n+2)}{\gamma(A-B)(1-\beta)} z^2.
\]

Hence the theorem.

Remarks. Letting \( \gamma = 1 \) and \( (A, B) = (1, -1) \) in Theorem 5.3.1, we get a result of Sarangi and Uralegaddi [99].
Theorem 5.3.2. For $-1 < B < A < 1$ ($-1 < B < 0$) and $n \in \mathbb{N}_0$, let
\[ f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in U_n(\beta_1, \gamma, A, B) \]
and
\[ g(z) = z - \sum_{k=2}^{\infty} b_k z^k \in U_n(\beta_2, \gamma, A, B). \]
Then
\[ (f \ast g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k \text{ belongs to } U_n(\mathcal{U}, \gamma, A, B) \]
where
\[ \mathcal{U} = \frac{(n+1)(1-\gamma B) - \gamma (A-B)(1-\beta_1)(1-\beta_2)}{(n+1)(1- \gamma B)} \]
The result is best possible.

Proof. Since $f(z) \in U_n(\beta_1, \gamma, A, B)$ and $g(z) \in U_n(\beta_2, \gamma, A, B)$, we have by Corollary 5.2.2
\[ \sum_{k=2}^{\infty} \frac{(1-\gamma B) \delta(n,k)}{\gamma (A-B)(1-\beta_1)} \cdot a_k \leq 1 \]
and
\[ \sum_{k=2}^{\infty} \frac{(1-\gamma B) \delta(n,k)}{\gamma (A-B)(1-\beta_2)} \cdot b_k \leq 1. \]
Therefore

\[
(5.3.6) \quad \sum_{k=2}^{\infty} \frac{(1-\gamma B)^2 \delta(n,k)^2}{\gamma^2(A-B)^2(1-\beta_1)(1-\beta_2)} a_k b_k
\]

\[
\leq \left[ \sum_{k=2}^{\infty} \frac{(1-\gamma B)\delta(n,k)}{\gamma(A-B)(1-\gamma)} a_k \right] \left[ \sum_{k=2}^{\infty} \frac{(1-\gamma B)\delta(n,k)}{\gamma(A-B)(1-\beta_2)} b_k \right]
\]

\[
\leq 1.
\]

To show that \((f \ast g)(z) \in U_n(\mathcal{U}, \tau, A, B)\), we have to prove that

\[
\sum_{k=2}^{\infty} \frac{(1-\gamma B)\delta(n,k)}{\gamma(A-B)(1-\gamma)} a_k b_k \leq 1.
\]

In view of (5.3.6), the above inequality is true if

\[
\frac{(1-\gamma B)\delta(n,k)}{\gamma(A-B)(1-\gamma)} \leq \frac{(1-\gamma B)^2 \delta(n,k)^2}{\gamma^2(A-B)^2(1-\beta_1)(1-\beta_2)}, \quad k \geq 2,
\]

that is, if

\[
(5.3.7) \quad \mathcal{U} \leq 1 - \frac{\gamma(A-B)(1-\beta_2)(1-\gamma)}{(1-\gamma B)\delta(n,k)}, \quad k \geq 2.
\]

Since \(\frac{1}{\delta(n,k)}\) is a decreasing function of \(k\), the right hand side of (5.3.1) is an increasing function of \(k \geq 2\). Taking \(k = 2\) in (5.3.1), we get the required result.

The result is best possible for the functions of the form

\[
f(z) = z - \frac{(n+1)(1-\gamma B)}{\gamma(A-B)(1-\beta_1)} z^2
\]

and

\[
g(z) = z - \frac{(n+1)(1-\gamma B)/(\gamma(A-B)(1-\beta_2))}{(n+1)(1-\gamma B)/(\gamma(A-B)(1-\beta_1))} z^2.
\]
Putting $\beta_1 = \beta_2 = \beta$ in Theorem 5.3.2 we get the following result.

**Corollary 5.3.1.** If the functions $f(z)$ and $g(z)$ are in $U_n(\beta, \gamma, A, B)$, then $(f \ast g)(z)$ belongs to $U_n(\gamma, A, B)$, where

$$\gamma = \frac{(n+1)(1-\gamma B) - \gamma (A-B)(1-\beta)^2}{(1-\gamma B)(n+1)}$$

The result is best possible for the functions

$$f(z) = g(z) = z - \frac{(n+1)(1 - \gamma B)}{\gamma (A-B)(1 - \beta)^2} z^2.$$

**Theorem 5.3.3.** Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ be in the class $U_n(\beta, \gamma, A, B)$. If

$$(n+1)(1-\gamma B) - 2\gamma (A-B)(1-\beta) > 0 \text{ for } n \in \mathbb{N}_0,$$

then

$$h(z) = z - \sum_{k=2}^{\infty} (a_k^2 + b_k^2) z^k \in U_n(\beta, \gamma, A, B).$$

**Proof.** Since $f(z) \in U_n(\beta, \gamma, A, B)$, we have by Corollary 5.3.2

$$\sum_{k=2}^{\infty} \frac{\gamma (1-\gamma B) \delta(n,k)}{\gamma (A-B)(1-\beta)} a_k \leq 1.$$ 

Therefore

$$\sum_{k=2}^{\infty} \left\{ \frac{(1-\gamma B) \delta(n,k)}{\gamma (A-B)(1-\beta)} \right\}^2 a_k^2 \leq 1.$$
Similarly, for \( g(z) \in \mathcal{U}_n(\beta, \gamma, A, B) \)

\[
\sum_{k=2}^{\infty} \left\{ \frac{(1 - \gamma B) \cdot \delta(n,k)}{\gamma(A-B)(1-\beta)} \right\}^2 \cdot b_k^2 < 1.
\]

Thus,

\[ (5.3.8) \quad \frac{1}{2} \cdot \sum_{k=2}^{\infty} \left\{ \frac{(1 - \gamma B) \cdot \delta(n,k)}{\gamma(A-B)(1-\beta)} \right\} \cdot \left( a_k^2 + b_k^2 \right) \leq 1. \]

To show that \( h(z) \in \mathcal{U}_n(\beta, \gamma, A, B) \), we have to prove that

\[
\sum_{k=2}^{\infty} \frac{(1 - \gamma B) \cdot \delta(n,k)}{\gamma(A-B)(1-\beta)} \cdot (a_k^2 + b_k^2) \leq 1.
\]

In view of (5.3.8), the above inequality is true if

\[
\frac{(1 - \gamma B) \cdot \delta(n,k)}{\gamma(A-B)(1-\beta)} \leq \frac{1}{2} \cdot \left\{ \frac{(1 - \gamma B) \cdot \delta(n,k)}{\gamma(A-B)(1-\beta)} \right\}^2, \quad k \geq 2,
\]

that is, if

\[ (5.3.9) \quad (1 - \gamma B) \cdot \delta(n,k) \geq 2\gamma(A-B)(1-\beta), \quad k \geq 2. \]

Since the left hand side of (5.3.9) is an increasing function of \( k \geq 2 \), the inequality (5.3.9) is valid for each \( k \geq 2 \) and \( n \in \mathbb{N}_0 \) if

\[
(n+1)(1 - \gamma B) - 2\gamma(A-B)(1-\beta) \geq 0.
\]

This completes the proof of Theorem 5.3.3.
Theorem 5.3.4. Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in U_n(\beta, \gamma, A, B)$ and $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$, $|b_k| \leq 1$, $k = 2, 3, \ldots$, then

$$(f \ast g)(z) \in U_n(\beta, \gamma, A, B).$$

Proof. Since

$$\sum_{k=2}^{\infty} \frac{(1-\gamma B) \delta(n,k)}{\gamma(A-B)(1-\beta)} |a_k \cdot b_k| \leq \sum_{k=2}^{\infty} \frac{(1-\gamma B) \delta(n,k)}{\gamma(A-B)(1-\beta)} a_k \leq 1,$$

by Corollary 5.2.2, it follows that $(f \ast g)(z) \in U_n(\beta, \gamma, A, B)$.

Remark. The function $g(z)$ in the above theorem need not be univalent in $E$. For instance, if $g(z) = z - \frac{a}{a+b} z^2 (0 < b < a)$, then $|\frac{a}{a+b}| < 1$, but $g'(z) = 1 - \frac{2a}{a+b} z = 0$ which lies inside $E$. Hence $g(z)$ is not univalent in $E$.

Theorem 5.3.5. Let $-1 \leq B < A \leq 1 (-1 \leq B < 0)$ and $c > -1$. If $f(z) \in U_n(\beta, \gamma, A, B)$, then the integral transforms

$$F_c(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt$$

belong to $U_n(\lambda, \gamma, A, B)$, where $\lambda = \lambda(\beta, c) = \frac{1+\beta(c+1)}{c+2}$. This result is best possible.
Proof. Suppose \( f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in U_n(\beta, \gamma, A, B) \). Then

\[
F_c(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) \, dt = z - \sum_{k=2}^{\infty} \frac{(c+1)}{(c+k)} a_k z^k.
\]

To prove that \( F_c(z) \) are in \( U_n(\lambda, \gamma, A, B) \), we have to show that

\[
\sum_{k=2}^{\infty} \frac{(1-\gamma B) \delta(n,k)(c+1)}{\gamma(A-B)(1-\lambda)(c+k)} a_k \leq 1
\]

(5.3.10)

Since \( f(z) \in U_n(\beta, \gamma, A, B) \), we get

\[
\sum_{k=2}^{\infty} \frac{(1-\gamma B) \delta(n,k)}{\gamma(A-B)(1-\beta)} a_k \leq 1.
\]

In view of the above inequality, (5.3.10) will be satisfied if

\[
\frac{c + 1}{(c+k)(1-\lambda)} \leq \frac{1}{1-\beta} \quad (k \geq 2)
\]

that is, if

\[
\lambda \leq \frac{(k-1) + \beta(c+1)}{c + k} \quad (k \geq 2).
\]

(5.3.11)

Since the right hand side of (5.3.11) is an increasing function of \( k \), putting \( k = 2 \) in (5.3.11), we get

\[
\lambda \leq \frac{1 + \beta(c+1)}{c + 2}
\]

This proves Theorem 5.3.5.
It is easy to check that the result is best possible for the function

\[ f(z) = z - \frac{\gamma(A-B)(1-\beta)}{(n+1)(1-\gamma)} z^2. \]

Remark. It is interesting to note that for \( n = 0, c = 1, \gamma = 1 \) and \( (A,B) = (1, -1) \), Theorem 5.3.5 gives that if \( \text{Re} \left( \frac{f(z)}{z} \right) > \beta \ (0 < \beta < 1) \), then the Libera transform \( F_1(z) = \frac{1}{2} \int_0^z f(t) dt \) satisfies \( \text{Re} \left( \frac{F_1(z)}{z} \right) > \frac{1+2\beta}{3} \) for \( z \in E \).

5.4. In this section, we apply some of the coefficient inequalities obtained in section 5.2 in order to deduce substantially more general distortion theorems involving fractional derivatives and fractional integrals for functions belonging to the classes \( U_n(\alpha,\beta,\gamma,A,B) \) and \( V_n(\alpha,\beta,\gamma,A,B) \).

Theorem 5.4.1. Let the function \( f(z) \) defined by (5.1.1) be in the class \( U_n(\alpha,\beta,\gamma,A,B) \). Then for \( |z| = r < 1 \) and \( \lambda > 0 \),

\[ |D_z^{-\lambda} f(z)| \geq \frac{r^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \frac{2A_n(\alpha,\beta,\gamma,A,B)}{2 + \lambda} r \right\} \]

and

\[ |D_z^{-\lambda} f(z)| \leq \frac{r^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 + \frac{2A_n(\alpha,\beta,\gamma,A,B)}{2 + \lambda} r \right\}. \]
where

\[(5.4.3) \quad A_n(\alpha, \beta, \gamma, A, B) = \{(1-\gamma B)(1-\alpha)+\gamma(A-B)(1-\beta)\} \cup \{(n+1)(1-\gamma B)(2-\alpha)\}.\]

The results (5.4.1) and (5.4.2) are sharp.

**Proof.** Since \(f(z) \in U_n(\alpha, \beta, \gamma, A, B)\), we get from (5.4.6)

\[(5.4.4) \quad (1-\gamma B)(n, 2) \cdot \sum_{k=2}^{\infty} a_k - \{(1-\gamma A)+(A-B)\gamma\} \cdot \sum_{k=2}^{\infty} b_k \leq \gamma(A-B)(1-\beta).\]

Again for \(g(z) \in T(\alpha)\), Lemma 5.2.1 yields

\[\sum_{k=2}^{\infty} b_k \leq \frac{1-\alpha}{2-\alpha}.\]

Using the above estimate in (5.4.4), we deduce that

\[(5.4.5) \quad \sum_{k=2}^{\infty} a_k \leq \frac{(1-\gamma B)(1-\alpha)+\gamma(A-B)(1-\beta)}{(n+1)(1-\gamma B)(2-\alpha)} = A_n(\alpha, \beta, \gamma, A, B) \text{ (say)}.\]

Now, consider the function \(F(z)\) defined by

\[(5.4.6) \quad F(z) = \Gamma(2+\lambda)z^\lambda D_z f(z) = z - \sum_{k=2}^{\infty} \frac{\Gamma(k+1) \cdot \Gamma(2+\lambda)}{\Gamma(k+1+\lambda)} a_k z^k,\]

for \(\lambda > 0\) and \(z \in E\). Since the function \(\Psi(k)\) given by

\[\Psi(k) = \frac{\Gamma(k+1) \cdot \Gamma(2+\lambda)}{\Gamma(k+1+\lambda)}\]

is a decreasing function of \(k \quad (k \geq 2)\), we have

\[(5.4.7) \quad 0 < \Psi(k) \leq \Psi(2) = \frac{2}{\lambda+2} \quad (k \geq 2).\]
With the aid of (5.4.5) and (5.4.7), we now have for $|z| = r < 1$,

$$|F(z)| \geq r - \sum_{k=2}^{\infty} \psi(k) a_k r^k$$

$$\geq r - \left(\frac{2}{2+\lambda}\right)(\sum_{k=2}^{\infty} a_k)r^2$$

$$\geq r - \frac{2\lambda_n(\alpha, \beta, \gamma, \lambda, A, B)}{\lambda + 2}r^2$$

which gives (5.4.1). Similarly,

$$|F(z)| \leq r + \sum_{k=2}^{\infty} \psi(k) a_k r^k$$

$$\leq r + \left(\frac{2}{\lambda+2}\right)(\sum_{k=2}^{\infty} a_k)r^2$$

$$\leq r + \frac{2\lambda_n(\alpha, \beta, \gamma, A, B)}{\lambda + 2}r^2$$

which implies (5.4.2).

Finally, it is easy to verify that the results (5.4.1) and (5.4.2) are sharp for the function $f(z)$ defined by

$$f(z) = \frac{z^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \left(\frac{2}{\lambda+2}\right)\lambda_n(\alpha, \beta, \gamma, A, B)z \right\}_{\lambda}^{z}$$

where $\lambda_n(\alpha, \beta, \gamma, A, B)$ is defined as in (5.4.3).

**Corollary 5.4.1.** Let the function $f(z)$ defined by (5.1.1) be in the class $U_n(\alpha, \beta, \gamma, A, B)$. Then the disc $E$
is mapped onto a domain that contains the disc $|w| < r$, where

$$ r = \frac{1}{\Gamma(2+\lambda)} \left[ 1 - \frac{2 \left\{ (1-\gamma \beta)(1-\alpha) + \gamma (A-B)(1-\beta) \right\}}{(n+1)(1-\gamma \beta)(2-\alpha)(2+\lambda)} \right]. $$

The result is sharp with the extremal function being given by (5.4.8).

Letting $\lambda \to 0$ in Theorem 5.4.1, we get

**Corollary 5.4.2.** Let the function $f(z)$ defined by (5.1.1) be in the class $U_n(\alpha, \beta, \gamma, A, B)$. Then for $|z| = r < 1$,

$$ |f(z)| \geq r \left\{ 1 - A_n(\alpha, \beta, \gamma, A, B) r \right\} $$

and

$$ |f(z)| \leq r \left\{ 1 + A_n(\alpha, \beta, \gamma, A, B) r \right\}. $$

The estimates are sharp for the function $f(z)$ given by

$$ f(z) = z - A_n(\alpha, \beta, \gamma, A, B) z^2. $$

Using Theorem 5.4.1, we next prove

**Theorem 5.4.2.** Let the function $f(z)$ defined by (5.1.1) be in the class $U_n(\alpha, \beta, \gamma, A, B)$ for $n \geq 1$. Then for $|z| = r < 1$ and $0 \leq \lambda < 1$,

$$ (5.4.9) \quad |D_z^{1-\lambda} f(z)| \leq \frac{r^\lambda}{\Gamma(2+\lambda)} \left\{ (1+\lambda) + 2A_n(\alpha, \beta, \gamma, A, B) r \right\}. $$
and

\[(5.4.10) \quad |D_z^{1-\lambda} f(z)| \geq \frac{r^\lambda}{\Gamma(2+\lambda)} \left\{ (1-\lambda) - 2A_n(\alpha, \beta, \gamma, A, B)r \right\}^2 \]

where $A_n(\alpha, \beta, \gamma, A, B)$ is defined as in Theorem 5.4.1. The result (5.4.9) is sharp.

Proof. From (5.2.3) we deduce that

\[(5.4.11) \quad \sum_{k=2}^{\infty} k \alpha_k \leq \frac{2}{(n+1)} \frac{(1-\gamma B)(1-\alpha)+\gamma(A-B)(1-\beta)}{(1-\gamma B)(2-\alpha)} \]

\[\leq 2A_n(\alpha, \beta, \gamma, A, B) \quad (n \geq 1).\]

By using (5.4.7) and (5.4.11), we have for $|z| = r < 1$

\[(5.4.12) \quad |F'(z)| \leq 1 + \sum_{k=2}^{\infty} \frac{\Gamma(k+1) \cdot \Gamma(2+\lambda)}{\Gamma(k+1+\lambda)} k \alpha_k \cdot r^{k-1} \]

\[\leq 1 + \frac{2}{\lambda + 2} \left( \sum_{k=2}^{\infty} k \alpha_k \right) r \]

\[\leq 1 + \frac{4A_n(\alpha, \beta, \gamma, A, B)}{\lambda + 2} \cdot r,\]

where $F(z)$ is given by (5.4.6). Since

\[(5.4.13) \quad F'(z) = \Gamma(2+\lambda) \left\{ z^{-\lambda} D_z^{1-\lambda} f(z) - \lambda z^{-\lambda-1} D_z^{-\lambda} f(z) \right\}, \]

the assertion (5.4.9) follows by using (5.4.1) and (5.4.12) in (5.4.13).
Further, by using (5.4.7) and (5.4.11), we get for 
\[ |z| = r < 1 \]

(5.4.14) \[ |F'(z)| \geq 1 - \sum_{k=2}^{\infty} \frac{\Gamma(k+1) \cdot \Gamma(\lambda+2)}{\Gamma(\lambda+k+1)} a_k r^{k-1} \]

\[ \geq 1 - (\lambda + 2)^{-4} \lambda_n(\alpha, \beta, \gamma, A, B). \]

Now, from (5.4.13)

\[ \Gamma(\lambda+2) |D_z^{1-\lambda} f(z)| \geq (r^\lambda |F'(z)| - \Gamma(\lambda+2) r^{-\lambda} |D_z f(z)|) \]

\[ \geq r^\lambda \left( 1 - \frac{4^\lambda A_n(\alpha, \beta, \gamma, A, B)}{\lambda + 2} - \alpha \frac{A_n(\alpha, \beta, \gamma, A, B)}{\lambda + 2} r \right) \]

\[ = r^\lambda \left\{ (1 - \alpha) - 2\lambda_n(\alpha, \beta, \gamma, A, B) r \right\}. \]

The last inequality follows by using (5.4.14) and (5.4.2). Thus

\[ |D_z^{1-\lambda} f(z)| \geq \frac{r^\lambda}{\Gamma(\lambda+2)} \left\{ (1 - \alpha) - 2\lambda_n(\alpha, \beta, \gamma, A, B) r \right\}. \]

This proves (5.4.10). The estimate (5.4.9) is seen to be sharp for the function \((5.4.8). This completes the proof of the theorem.

**Theorem 5.4.3.** Let the function \(f(z)\) defined by (5.1.1) be in the class \(U_n(\alpha, \beta, \gamma, A, B)\) for \(n \geq 1\). Then for \(|z| = r < 1\) and \(0 \leq \lambda < 1\),
The results (5.4.15) and (5.4.16) are sharp.

Proof. Define the function \( G(z) \) by

\[
G(z) = \sum_{k=2}^{\infty} \frac{\Gamma(k+1) \cdot \Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} a_k z^k
\]

for \( 0 \leq \lambda < 1 \) and \( z \in E \). Since

\[
1 \leq \frac{\Gamma(k+1) \cdot \Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} < k \quad (k \geq 2)
\]

for \( 0 \leq \lambda < 1 \), it follows from (5.4.11) and (5.4.18) that

\[
|G(z)| \geq r - \sum_{k=2}^{\infty} \frac{\Gamma(k+1) \cdot \Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} a_k r^k
\]

\[\geq r - (\sum_{k=2}^{\infty} ka_k)r^2 \geq r - 2A_n(a,\beta,\gamma,A,B)r^2,\]

which implies (5.4.15) and that

\[
|G(z)| \leq r + r^2(\sum_{k=2}^{\infty} ka_k) \leq r + 2A_n(a,\beta,\gamma,A,B)r^2,
\]

where \( A_n(a,\beta,\gamma,A,B) \) is defined as in Theorem 5.4.1. The above inequality yields (5.4.16).
Finally, by taking the function $f(z)$ defined by

\begin{equation}
D^\lambda f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left[ 1 - \frac{2 \{ (1-\gamma B)(1-\alpha)+\gamma(A-B)(1-\beta) \}}{(n+1)(1-\gamma B)(2-\alpha)} z \right]^{1-\lambda}
\end{equation}

we see that the estimates in (5.4.15) and (5.4.16) are sharp.

The results of this section can also be established for the class $V_n(\alpha, \beta, \gamma, A, B)$. For instance, Theorem 5.4.1 in this case would read as follows:

**Theorem 5.4.4.** Let the function $f(z)$ be defined by (5.1.1) be in the class $V_n(\alpha, \beta, \gamma, A, B)$. Then for $|z| = r < 1$ and $\lambda > 0$,

\begin{equation}
|D_z^{-\lambda} f(z)| \geq \frac{r^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \frac{2B_n(\alpha, \beta, \gamma, A, B)}{2 + \lambda} r \right\}
\end{equation}

and

\begin{equation}
|D_z^{-\lambda} f(z)| \leq \frac{r^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 + \frac{2B_n(\alpha, \beta, \gamma, A, B)}{2 + \lambda} r \right\},
\end{equation}

where

\[ B_n(\alpha, \beta, \gamma, A, B) = \frac{(1-\gamma B)(1-\alpha)+\gamma(A-B)(1-\beta)(3-\alpha)}{2(n+1)(1-\gamma B)(2-\alpha)}. \]

The results (5.4.21) and (5.4.22) are sharp for the function $f(z)$ defined by
\[ \text{D}_z f(z) = \frac{1^\lambda}{\Gamma(2+\lambda)} \left\{ 1 - \frac{2B_n(\alpha, \beta, \gamma, A, B)}{2 + \lambda} z \right\}. \]

**Remarks.** 1. Setting \( n = 1 \) and \( (A, B) = (1, -1) \) in our theorems of this section, we obtain the corresponding results of Srivastava and Owa [111].

2. Letting \( \lambda \to 0 \) in Theorems 5.4.1 and 5.4.3, we get distortion theorems, that is, bounds of \(|f(z)|\) and \(|f'(z)|\) for function \( f(z) \) belonging to the class \( U_n(\alpha, \beta, \gamma, A, B) \).

3. For \( \alpha = 1, n = 1 \) and \( (A, B) = (1, -\mu) \) \((0 < \mu \leq 1)\) and \( \lambda \to 0 \), Theorems 5.4.1 and 5.4.2 give distortion theorems for the class \( F(\mu, \beta, \gamma) \) which in turn yield the corresponding distortion theorems obtained by Gupta and Jain [34] for \( \mu = 1 \).

5.5. The radii of convexity for functions belonging to the classes \( U_n(\alpha, \beta, \gamma, A, B) \) and \( V_n(\alpha, \beta, \gamma, A, B) \), \( n \geq 1 \), are determined in this section.

From (5.2.1), we note that a function \( f(z) \) defined by (5.1.1) is starlike in \( E \) if and only if

\[ \sum_{k=2}^{\infty} k a_k \leq 1. \]

For \( f(z) \in U_n(\alpha, \beta, \gamma, A, B) \), \( n \geq 1 \), we have from (5.4.11)

\[ \sum_{k=2}^{\infty} k a_k \leq 2A_n(\alpha, \beta, \gamma, A, B) \leq 1 \]

Similarly, for \( f(z) \in V_n(\alpha, \beta, \gamma, A, B) \) \((n \geq 1)\) we have
where $A_n(\alpha, \beta, \gamma, A, B)$ and $B_n(\alpha, \beta, \gamma, A, B)$ are defined as in Theorem 5.4.1 and Theorem 5.4.4.

Thus, we observe that for $n \geq 1$, $U_n(\alpha, \beta, \gamma, A, B)$ and $V_n(\alpha, \beta, \gamma, A, B)$ are the subclasses of starlike functions.

Therefore, it is natural to find the radii of starlikeness for the class $U_0(\alpha, \beta, \gamma, A, B)$ (respectively $V_0(\alpha, \beta, \gamma, A, B)$) and the radii of convexity for the class $U_n(\alpha, \beta, \gamma, A, B)$ (respectively $V_n(\alpha, \beta, \gamma, A, B)$) for $n \geq 1$.

**Theorem 5.5.1.** If the function $f(z)$ defined by

$$(5.5.1) \quad \sum_{k=2}^{\infty} ka^k \leq 2B_n(\alpha, \beta, \gamma, A, B) \leq 1,$$

where $A_n(\alpha, \beta, \gamma, A, B)$ and $B_n(\alpha, \beta, \gamma, A, B)$ are defined as in Theorem 5.4.1 and Theorem 5.4.4.

Thus, we observe that for $n \geq 1$, $U_n(\alpha, \beta, \gamma, A, B)$ and $V_n(\alpha, \beta, \gamma, A, B)$ are the subclasses of starlike functions.

Therefore, it is natural to find the radii of starlikeness for the class $U_0(\alpha, \beta, \gamma, A, B)$ (respectively $V_0(\alpha, \beta, \gamma, A, B)$) and the radii of convexity for the class $U_n(\alpha, \beta, \gamma, A, B)$ (respectively $V_n(\alpha, \beta, \gamma, A, B)$) for $n \geq 1$.

**Theorem 5.5.1.** If the function $f(z)$ defined by

$$(5.5.1) \quad \sum_{k=2}^{\infty} ka^k \leq 2B_n(\alpha, \beta, \gamma, A, B) \leq 1,$$

where $A_n(\alpha, \beta, \gamma, A, B)$ and $B_n(\alpha, \beta, \gamma, A, B)$ are defined as in Theorem 5.4.1 and Theorem 5.4.4.

Thus, we observe that for $n \geq 1$, $U_n(\alpha, \beta, \gamma, A, B)$ and $V_n(\alpha, \beta, \gamma, A, B)$ are the subclasses of starlike functions.

Therefore, it is natural to find the radii of starlikeness for the class $U_0(\alpha, \beta, \gamma, A, B)$ (respectively $V_0(\alpha, \beta, \gamma, A, B)$) and the radii of convexity for the class $U_n(\alpha, \beta, \gamma, A, B)$ (respectively $V_n(\alpha, \beta, \gamma, A, B)$) for $n \geq 1$.

**Theorem 5.5.1.** If the function $f(z)$ defined by

$$(5.5.1) \quad \sum_{k=2}^{\infty} ka^k \leq 2B_n(\alpha, \beta, \gamma, A, B) \leq 1,$$

where $A_n(\alpha, \beta, \gamma, A, B)$ and $B_n(\alpha, \beta, \gamma, A, B)$ are defined as in Theorem 5.4.1 and Theorem 5.4.4.

Thus, we observe that for $n \geq 1$, $U_n(\alpha, \beta, \gamma, A, B)$ and $V_n(\alpha, \beta, \gamma, A, B)$ are the subclasses of starlike functions.

Therefore, it is natural to find the radii of starlikeness for the class $U_0(\alpha, \beta, \gamma, A, B)$ (respectively $V_0(\alpha, \beta, \gamma, A, B)$) and the radii of convexity for the class $U_n(\alpha, \beta, \gamma, A, B)$ (respectively $V_n(\alpha, \beta, \gamma, A, B)$) for $n \geq 1$.

**Theorem 5.5.1.** If the function $f(z)$ defined by

$$(5.5.1) \quad \sum_{k=2}^{\infty} ka^k \leq 2B_n(\alpha, \beta, \gamma, A, B) \leq 1,$$

where $A_n(\alpha, \beta, \gamma, A, B)$ and $B_n(\alpha, \beta, \gamma, A, B)$ are defined as in Theorem 5.4.1 and Theorem 5.4.4.

Thus, we observe that for $n \geq 1$, $U_n(\alpha, \beta, \gamma, A, B)$ and $V_n(\alpha, \beta, \gamma, A, B)$ are the subclasses of starlike functions.

Therefore, it is natural to find the radii of starlikeness for the class $U_0(\alpha, \beta, \gamma, A, B)$ (respectively $V_0(\alpha, \beta, \gamma, A, B)$) and the radii of convexity for the class $U_n(\alpha, \beta, \gamma, A, B)$ (respectively $V_n(\alpha, \beta, \gamma, A, B)$) for $n \geq 1$.

**Theorem 5.5.1.** If the function $f(z)$ defined by

$$(5.5.1) \quad \sum_{k=2}^{\infty} ka^k \leq 2B_n(\alpha, \beta, \gamma, A, B) \leq 1,$$

where $A_n(\alpha, \beta, \gamma, A, B)$ and $B_n(\alpha, \beta, \gamma, A, B)$ are defined as in Theorem 5.4.1 and Theorem 5.4.4.

Thus, we observe that for $n \geq 1$, $U_n(\alpha, \beta, \gamma, A, B)$ and $V_n(\alpha, \beta, \gamma, A, B)$ are the subclasses of starlike functions.
(5.5.2) \[ \left| \frac{zf'(z) - f(z)}{f(z)} \right| \leq 1 - \lambda_0 \quad \text{for} \quad |z| < R_s. \]

We have

\[
\left| \frac{zf(z)}{f(z)} - 1 \right| = \left| \sum_{k=2}^{\infty} \frac{(k-1)a_k z^k}{1 - \sum_{k=2}^{\infty} a_k z^k} \right| \leq \sum_{k=2}^{\infty} |a_k| |z|^{k-1}
\]

The above inequality will be bounded by \(1 - \lambda_0\) if

(5.5.3) \[ \sum_{k=2}^{\infty} \frac{(k-\lambda_0)a_k |z|^{k-1}}{1 - \lambda_0} \leq 1. \]

In view of (5.3.1), it follows that (5.5.3) is true if

\[
\frac{(k-\lambda_0)|z|^{k-1}}{1 - \lambda_0} \leq \frac{(1 - \gamma B)(2 - \alpha)}{(1 - \gamma B)(1 - \alpha) + \gamma (A-B)(1-\beta)}
\]

or

\[
|z| \leq \left[ \frac{(1 - \gamma B)(2 - \alpha)(1 - \lambda_0)}{(k-\lambda_0) \{(1-\gamma B)(1-\alpha) + \gamma (A-B)(1-\beta)\}} \right]^{1/k-1}
\]

This proves the Theorem 5.5.1. The result is sharp for the function \(f(z)\) given by

\[ f(z) = z - \frac{(1 - \gamma B)(1 - \alpha) + \gamma (A-B)(1-\beta)}{k(1 - \gamma B)(2 - \alpha)} z^k. \]

**Theorem 5.5.2.** If the function \(f(z)\) defined by (5.1.1) belongs to the class \(U_0(\alpha, \beta, \gamma, A, B)\) then \(f(z)\) is convex of order \(\lambda_0\) in the disc \(|z| < R_0\), where
\begin{align*}
(5.5.4) \quad R_c &= R_c(\lambda_0, \beta, \gamma, A, B) = \inf_{k \geq 2} \left[ \frac{1 - \lambda_0}{2(k-\lambda_0)A_n(\alpha, \beta, \gamma, A, B)} \right]^{(k-1)}
\end{align*}

where \( A_n(\alpha, \beta, \gamma, A, B) \) is defined as in Theorem 5.4.1. The result is sharp.

\textbf{Proof.} It is sufficient to show that

\begin{align*}
(5.5.5) \quad \left| \frac{zf''(z)}{f(z)} \right| &\leq 1 - \lambda_0 \quad \text{for } |z| < R_c = R_c(\lambda_0, \beta, \gamma, A, B).
\end{align*}

We have

\begin{align*}
\left| \frac{zf''(z)}{f(z)} \right| &= \left| \frac{\sum_{k=2}^{\infty} k(k-1)a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} ka_k z^{k-1}} \right| \\
&\leq \frac{\sum_{k=2}^{\infty} k(k-1)a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} ka_k |z|^{k-1}}
\end{align*}

This will be bounded by \((1 - \lambda_0)\) if

\begin{align*}
(5.5.6) \quad \sum_{k=2}^{\infty} \frac{k(k-\lambda_0)}{1 - \lambda_0} a_k |z|^{k-1} &\leq 1.
\end{align*}

In view of (5.4.11), it follows that (5.5.6) is true if

\begin{align*}
\frac{k - \lambda_0}{1 - \lambda_0} |z|^{k-1} &\leq \frac{1}{2A_n(\alpha, \beta, \gamma, A, B)} \\
&= \frac{(1-\gamma B)(2-\alpha)}{2 \{(1-\gamma B)(1-\alpha)+\gamma(A-B)(1-\beta)\}}
\end{align*}
or

\[ |z| \leq \left[ \frac{(1 - \gamma B)(2 - \alpha)(1 - \lambda_o)}{2(k-\lambda_o) \left\{ (1 - \gamma B)(1 - \alpha) + \gamma (A - B)(1 - \beta) \right\}} \right]^{\frac{1}{(k-1)}} \]

\[ = \left[ \frac{1 - \lambda_o}{2(k-\lambda_o)A_n(\alpha, \beta, \gamma, A, B)} \right]^{\frac{1}{(k-1)}} \quad (k \geq 2). \]

This completes the proof of (5.5.4).

The result (5.5.4) is sharp with extremal function being \( f(z) \) given by

\[
f(z) = z - \frac{2 \left\{ (1 - \gamma B)(1 - \alpha) + \gamma (A - B)(1 - \beta) \right\}^3}{k(1 - \gamma B)(2 - \alpha)} z^k
\]

for some integer \( k \geq 2 \).

Similarly, we can prove

**Theorem 5.5.3.** If the function \( f(z) \) defined by (5.1.1) belongs to the class \( U_n(\alpha, \beta, \gamma, A, B) \) for \( n \geq 1 \), then \( f(z) \) is convex in the disc \( |z| < R_c \), where

\[
(5.5.7) \quad R_c \equiv R_c(\alpha, \beta, \gamma, A, B) = \inf_{k \geq 2} \left[ \frac{1}{2k A_n(\alpha, \beta, \gamma, A, B)} \right]^{(k-1)}
\]

where \( A_n(\alpha, \beta, \gamma, A, B) \) is defined as in Theorem 5.4.1. The result is sharp.
Proof. It is sufficient to show that

\[(5.5.8) \quad \left| \frac{zf''(z)}{f(z)} \right| \leq 1 \text{ for } |z| \leq R_0 = R_0(\alpha, \beta, \gamma, \Lambda, B).\]

We have

\[
\left| \frac{zf''(z)}{f(z)} \right| = \left| \sum_{k=2}^{\infty} \frac{k(k-1)a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} ka_k z^{k-1}} \right| \leq \sum_{k=2}^{\infty} \frac{k(k-1)a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} ka_k |z|^k} \leq \sum_{k=2}^{\infty} \frac{k(k-1)a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} ka_k |z|^k}.
\]

This will be bounded by 1 if

\[(5.5.9) \quad \sum_{k=2}^{\infty} \frac{k^2 a_k |z|^{k-1}}{k} \leq 1
\]

In view of \((5.4.11)\), \((5.5.9)\) is true if

\[k |z|^{k-1} \leq \frac{1}{2A_n(\alpha, \beta, \gamma, \Lambda, B)} \quad (k \geq 2, n \geq 1)
\]

which on simplification gives

\[|z| \leq \left[ \frac{1}{2A_n(\alpha, \beta, \gamma, \Lambda, B)} \right]^{\frac{1}{(k-1)}} \quad (k \geq 2).
\]

This proves \((5.5.7)\).

The result \((5.5.7)\) is sharp with the extremal function being \(f(z)\) given by

\[f(z) = z - \frac{2A_n(\alpha, \beta, \gamma, \Lambda, B)}{k} z^k
\]

for some integer \(k \geq 2\) and \(n \geq 1\).
Following the lines of proof of Theorems 5.5.1, 5.5.2 and 5.5.3, we can prove the following results.

**Theorem 5.5.4.** If the function \( f(z) \) defined by (5.1.1) belongs to the class \( V_\alpha(\alpha, \beta, \gamma, A, B) \), then \( f(z) \) is convex of order \( \lambda_0 \) in the disc \( |z| < R_c \), where

\[
R_c \equiv R_c(\lambda_0, \alpha, \beta, \gamma, A, B) = \inf_{k \geq 2} \left[ \frac{1 - \lambda_0}{2 (k - \lambda_0) B_n(\alpha, \beta, \gamma, A, B)} \right]^{\frac{1}{(k-1)}}
\]

and \( B_n(\alpha, \beta, \gamma, A, B) \) is given as in Theorem 5.4.4.

The result is sharp.

**Theorem 5.5.5.** If the function \( f(z) \) defined by (5.1.1) belongs to the class \( V_n(\alpha, \beta, \gamma, A, B) \) for \( n \geq 1 \), then \( f(z) \) is convex in the disc \( |z| < R_c \), where

\[
R_c \equiv R_c(\alpha, \beta, \gamma, A, B) = \inf_{k \geq 2} \left[ \frac{1}{2k B_n(\alpha, \beta, \gamma, A, B)} \right]^{\frac{1}{(k-1)}}
\]

and \( B_n(\alpha, \beta, \gamma, A, B) \) is given as in Theorem 5.4.4. The result is sharp for the function

\[
f(z) = z - \frac{2B_n(\alpha, \beta, \gamma, A, B)}{k} z^k \quad (k \geq 2, \ n \geq 1).
\]

**Theorem 5.5.6.** If the function \( f(z) \) defined by (5.1.1) belongs to the class \( V_\alpha(\alpha, \beta, \gamma, A, B) \) then \( f(z) \) is starlike of order \( \lambda_0 \) in the disc \( |z| < R_s \), where
The result is sharp.

Proof. Let \( f(z) \in V_0(\alpha, \beta, \gamma, A, B) \). Then, from (5.4.11) for \( n = 0 \)

\[
\sum_{k=2}^{\infty} \frac{(1 - \gamma B)(2 - \alpha)}{(1 - \gamma B)(1 - \alpha) + \gamma (A - B)(1 - \beta)(3 - \alpha)} a_k \leq 1
\]

In view of (1.2.17), it is sufficient to show that

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq (1 - \lambda_0) \text{ for } |z| < R_s = R_s(\lambda_0, \alpha, \beta, \gamma, A, B)
\]

or, equivalently,

\[
(5.5.11) \quad \left| \frac{zf'(z) - f(z)}{f(z)} \right| \leq (1 - \lambda_0) \text{ for } |z| < R_s.
\]

We have

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \sum_{k=2}^{\infty} \frac{(k-1)a_k z^k}{z - \sum_{k=2}^{\infty} a_k z^k} \right| \leq \sum_{k=2}^{\infty} \frac{(k-1)a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k |z|}
\]

This will be bounded by \( (1 - \lambda_0) \)

\[
(5.5.12) \quad \sum_{k=2}^{\infty} \frac{(k-1)a_k |z|^{k-1}}{(1 - \lambda_0)} \leq 1.
\]
In view of (5.4.11), it follows that (5.5.12) is true if

\[ \frac{(k-\lambda_o)}{1-\lambda_o} \left| \frac{1}{2B_n(\alpha, \beta, \gamma, A, B)} \right| \leq \left| \frac{(1-\gamma B)(2-\alpha)}{(1-\gamma B)(1-\alpha)+\gamma(A-B)(1-\beta)(3-\alpha)} \right| \]

or

\[ |z| \leq \left[ \frac{(1-\gamma B)(2-\alpha)(1-\lambda_o)}{(k-\lambda_o)\{(1-\gamma B)(1-\alpha)+\gamma(A-B)(1-\beta)(3-\alpha)\}} \right]^{(k-1)} \]

This proves the Theorem 5.5.6. The result is sharp for the function \( f(z) \) given by

\[ f(z) = z - \frac{(1-\gamma B)(1-\alpha)+\gamma(A-B)(1-\beta)(3-\alpha)}{2k(1-\gamma B)(2-\alpha)} z^k \]

for some integer \( k \geq 2 \).

Remarks. 1. The radius of convexity for the classes \( T^*(\alpha, \beta, \gamma) \) and \( C^*(\alpha, \beta, \gamma) \) considered by Srivastava and Owa [110] can be obtained from our Theorem 5.5.3 and Theorem 5.5.5 by setting \( n = 1 \) and \( (A,B) = (1,-1) \).

2. The radius of convexity for the class \( P^*(\mu, \beta, \gamma) \) can be deduced from Theorem 5.5.3 by putting \( n = 1 \) and \( (A,B) = (1, -\mu) \) \( (0 < \mu \leq 1) \) which also yields the radius of convexity for the class \( P(\beta, \gamma) \) by letting \( \mu = 1 \) [34].