Chapter 6

Stationary localized states in fully nonlinear systems

6.1 Introduction

So far we have studied the formation of stationary localized states due to the presence of nonlinear impurities in linear lattices. But for further understanding of this subject, one needs to know the effect on the formation of SL states if nonlinear impurities are present in a perfect nonlinear lattice. This system can be visualized from the fact that the exciton (quasiparticle) does not interact with all the lattice oscillators equally strongly even though the interaction strengths are appreciable. As far as linear lattices are concerned, we know that all states are extended if the system is perfect or periodic. One, therefore, needs to see if nonlinearity plays any role on the formation of SL states in a perfect or a periodic nonlinear lattice. We therefore consider various nonlinear systems and look for the possibility of the formation of SL states.

We write a general Hamiltonian given by

\[
H = 2 \sum_{m=-\infty}^{0} \frac{X_m}{\sigma + 2} V^{|m|\sigma} |C_m|^{\sigma + 2} + 2 \sum_{m=1}^{\infty} \frac{X_m}{\sigma + 2} V^{(m-1)\sigma} |C_m|^{\sigma + 2} \\
+ \sum_{m=-\infty}^{\infty} (C_m C_{m+1}^* + C_{m+1}^* C_m) + V_0 (C_0 C_1^* + C_1 C_0^*)
\]  

(6.1)
where $C_n$ is the probability amplitude of the quasiparticle at the $n^{th}$ site of a lattice, $\sigma$ is arbitrary, $\chi_n$ is the strength of the interaction between the the quasiparticle and the $n^{th}$ oscillator in the lattice. The nearest neighbor hopping is taken to be unity without any loss of generality. $V_0 = V - 1$. $V$ is arbitrary and it determines the underlying lattice. The systems we consider in this chapter for the study of the formation of SL states are as follows.

(1) The system described by the Hamiltonian (eq. (6.1) ) with $V=1$ and $\chi_m = \chi_0 + \chi (1 - \delta_{m,0})$, i.e., a fully nonlinear one dimensional chain with an impurity at the zeroth site, is discussed in section 6.2.1.

(2) The system described by the Hamiltonian (eq. (6.1) ) with $V=1$, and $\chi_m = \chi_0 + \chi (1 - \delta_{m,0}) + \chi (1 - \delta_{m,1})$, i.e., a fully nonlinear one dimensional chain with a dimeric impurity occupying sites 0 and 1, is studied in sec. 6.2.2.

(3) The system described by the Hamiltonian (eq. (6.1) ) with $V=1$ and $\chi_m = \chi$, i.e., a perfect nonlinear chain, is discussed in sec. 6.2.3.

(4) The system described by the Hamiltonian (eq. (6.1) ) with $V=1$, $\sigma = 2$ and $\chi_{2m} = \chi_1$, $\chi_{2m+1} = \chi_2$, i.e., a periodic one dimensional nonlinear (quadratic nonlinear) lattice where alternative sites are of same nonlinear strength, is discussed in the sec. 6.2.4.

(5) The Hamiltonian given by eq. (6.1) with $V < 1$ and $\chi_m = \chi$ describes a perfect nonlinear Cayley tree. This system is discussed in sec. 6.3. The system described by the Hamiltonian ( given by eq. (6.1) ) with $V > 1$ and $\chi_m=\chi$ is also discussed in sec. 6.3.

The stability of the SL states are discussed graphically in sec. 6.4 and finally the findings in this chapter are summarized in sec. 6.5.

### 6.2 One dimensional nonlinear chain

#### 6.2.1 Single nonlinear impurity

Here we consider the one dimensional nonlinear (power-law nonlinear) chain with a single nonlinear impurity at the zeroth site. The system is described by the Hamiltonian given in eq. (6.1) with $V = 1$ and $\chi_m = \chi_0 + \chi(1 - \delta_{m,0})$ i.e., the nonlinear strength at all sites is $\chi$ except at the zeroth site where it is $\chi_0$. Total probability of the quasiparticle in the
system is conserved and therefore the normalization condition is given by \( \sum_m |C_m|^2 = 1 \).
Since we are interested in the SL states, the form of \( C_m \) is given by
\[
C_m(t) = \phi_m e^{iBt}
\]
and assume that
\[
\phi_m = \phi_0 \eta^{|m|}
\]
Furthermore, in the asymptotic limit, i.e., \( |m| \to \infty \), we get
\[
\eta = \frac{|E| - \sqrt{E^2 - 4}}{2}
\]
The parameter \( \eta \in [0,1] \). This ansatz (eq. (6.3)) corresponds to the solution peaked at the impurity site and it is supported by the symmetry of the system under consideration.
We also note that for a single nonlinear impurity in a totally perfect chain this ansatz gives the correct form of localized states. Other possible ansatz is presented in chapter 3.

After introducing this ansatz in the appropriate Hamiltonian as well as the normalization condition, we obtain the effective Hamiltonian, \( H_{\text{eff}} \) for the reduced dynamical system. \( H_{\text{eff}} \) for this case is
\[
H_{\text{eff}} = \frac{\chi}{\sigma + 2} \left[ (1 - \eta^2)^{\sigma/2+1}(1 + \eta^{\sigma+2}) \right] + \frac{(\chi_0 - \chi)(1 - \eta^2)^{\sigma/2+1}}{(\sigma + 2)(1 + \eta^2)^{\sigma/2+1}} + \frac{2\eta}{1 + \eta^2}
\]
The relevant equation governing the formation of SL states is obtained from the fixed points of the reduced dynamical Hamiltonian, i.e., by setting \( \frac{\partial H_{\text{eff}}}{\partial \eta} = 0 \). After some algebra we get,
\[
\frac{1}{\chi} = \frac{\eta^{\sigma+1}(1 - \eta^2)^{\sigma/2-1}(2\eta^{\sigma+4} - \eta^4 - 2\eta^2 + 1)}{(1 - \eta^{\sigma+2})^2[\chi_0 \eta(1 - \eta^2)^{\sigma/2-1} - (1 + \eta^2)^{\sigma/2}]} = F(\sigma, \eta)
\]
Note that for \( \chi_0 = \chi \), from eq. (6.6) we get
\[
\frac{1}{\chi} = \frac{\eta(1 - \eta^2)^{\sigma/2-1}(1 - \eta^\sigma)(1 + \eta^{\sigma+4})}{(1 + \eta^2)^{\sigma/2}(1 - \eta^{\sigma+2})}
\]
Equation (6.7) can be used to get the phase diagram for SL states in a perfectly nonlinear chain [24, 85].
We now make a rudimentary analysis of eq. (6.6). By setting the denominator of $F(a, \eta)$ equal to zero, we obtain,

$$\frac{1}{\chi_0} = \frac{\eta(1-\eta^2)\sigma/2-1}{(1+\eta^2)\sigma/2} \quad (6.8)$$

We note that eq. (6.8) gives SL states for a single nonlinear impurity in a perfect 1-d chain. We further know that for $\sigma < 2$, we have one solution. For $\sigma \geq 2$ if $\chi_0 > \chi_0^c$ (critical value of $\chi_0$) we have two solutions. Otherwise we have no solution. So, for $\sigma < 2$ there will be one $\eta$ say $\eta_0$ at which $F(\sigma, \eta)$ will diverge. For $\chi_0 > \chi_0^c$ and $\sigma > 2$, there will be two values of $\eta$ for which $F(\sigma, \eta)$ will diverge. For $\chi_0 < \chi_0^c$ there will be no $\eta$ between 0 and 1 where $F(\sigma, \eta)$ will diverge. We further note that $F(\sigma, 1-\varepsilon) = \frac{2\sigma}{\chi_0(\sigma+2)^2} \varepsilon^{-1}$ for $\sigma < 2$ and $F(\sigma, 1-\varepsilon) = \sigma(\sigma+2)^2 \varepsilon^{\sigma/2-2}$ for $\sigma > 2$. Therefore, as $\eta \to 1$, $F(\sigma, \eta) \to -\infty$ for $\sigma < 2$. On the other hand, for $4 > \sigma > 2$, $F(\sigma, \eta) \to \infty$ as $\eta \to 1$. Furthermore, for $\sigma > 4$, $F(\sigma, \eta) \to 0$ as $\eta \to 1$. We now consider the following cases.

**Fig. 6.1** The phase diagram of SL states in the $\chi - \chi_0$ plane for a fully nonlinear system with one nonlinear impurity in the middle of the system. The nonlinear strength of the impurity site is $\chi_0$ and that at other sites is $\chi$. The critical value of $\chi_0$, $\chi_0^c$, is shown in the figure. Numbers in different regions indicate the number of possible SL states. $\sigma = 3$ in this case.

Case 1: $\sigma < 2$. Since $F(\sigma, \eta)$ diverges only at one value of $\eta$ between 0 and 1, we shall have only one SL state for small value of $\chi$. But for general values of $\chi$ and $\chi_0$,
the number of SL states depends on the exact nature of the function \( F(\sigma, \eta) \) i.e., local maximum, minimum, etc. But for this case we have numerically found that for any combination of \( \chi \) and \( \chi_0 \) there exists only one SL state.

Case 2: \( 2 < \sigma < 4 \). In this case, as long as \( \chi_0 < \chi_0^c \), there will be no divergence of \( F(\sigma, \eta) \) for \( 0 < \eta < 1 \). But \( F(\sigma, \eta) \) diverges as \( \eta \to 1 \). We therefore expect only one SL state for small value of \( \chi \). We have numerically seen that for this case as long as \( \chi_0 < \chi_0^c \), there will be only one SL states irrespective of the combination of \( \chi \) and \( \chi_0 \). Since for \( \chi_0 > \chi_0^c \), there are altogether three divergence of the of \( F(\sigma, \eta) \), we will have three SL states for small value of \( \chi \). Analytically it is not possible to find the number of SL states for general combination of \( \chi \) and \( \chi_0 \). We have therefore seen numerically that there exists one, two and three SL states depending on the combination of \( \chi \) and \( \chi_0 \). In Fig. 6.1 we have shown the phase diagram of SL states in the \((\chi, \chi_0)\) plane. We have taken \( \sigma = 3 \in [2, 4] \). It is clear from the Fig. 6.1 that there are regions containing one and three SL states. The lines separating one and three SL states regain contain two SL states.

Case 3: \( \sigma > 4 \). In this case \( F(\sigma, \eta = 1) = 0 \) and there is no divergence of \( F(\sigma, \eta) \). Therefore, we will have no SL state for smaller value of \( \chi \). We have seen numerically that for \( \chi_0 < \chi_0^c \), there is a critical value of \( \chi \) (say \( \chi^c \)) below which there will be no SL state and above it there will be two SL states. There is only one SL state at \( \chi^c \). For \( \chi_0 > \chi_0^c \), \( F(\sigma, \eta) \) diverges at two values of \( \eta \). Thus we will have two SL states for smaller values of \( \chi \). But for general values of \( \chi \) and \( \chi_0 \) we numerically find that there exists one to four SL states. Therefore, for \( \sigma > 4 \) there exists 0 to 4 SL states depending the combination of \( \chi \) and \( \chi_0 \). However we have not presented the phase diagram for this case.

6.2.2 Two nonlinear impurities

Here we consider the system of one dimensional nonlinear chain. Two consecutive sites in the middle of the chain have nonlinearity strength, \( \chi_0 \). Nonlinearity strength at other sites is \( \chi \). Therefore the Hamiltonian for this system is given by eq. (6.1) with \( \chi_m=\chi_0 \), \( (\delta_{m,0}+\delta_{m,1})+\chi(1-\delta_{m,0}-\delta_{m,1}) \). Considering the symmetry of the system we introduce the dimeric ansatz (presented in chapter 3 as well as chapter 4) for the solution of the
SL states. Using the dimeric ansatz and the procedure used in chapter 4, we obtain the required equation for SL states. The equation is given by

\[
\frac{1}{\chi} = \frac{\eta^{\sigma+1}(1 - \eta^2)^{\sigma/2}[1 - 2\eta^2 + \eta^{\sigma+4}]}{(1 - \eta^{\sigma+2})^2[\chi_0\eta(1 - \eta^2)^{\sigma/2} - 2^{\sigma/2}(1 - \eta^2)]}. 
\] (6.9)

**Fig. 6.2** The phase diagram of SL states in the \((\chi, \chi_0)\) plane for a fully nonlinear system with a dimeric nonlinear impurity in the middle of the system. The nonlinear strength of the impurity sites is \(\chi_0\) and that at other sites is \(\chi\). The critical value of \(\chi_0\), \(\chi_0^c\), is shown in the figure. Numbers in different regions indicate the number of possible SL states. \(\sigma = 3\) in this case.

Using the same analysis as in the case of single impurity, we can show to have three distinct regions of \(\sigma\). These regions are (i) \(\sigma \leq 2\), (ii) \(2 > \sigma \geq 4\), and \(\sigma > 4\). The nature of phase diagram of SL states in \((\chi, \chi_0)\) plane will be independent of \(\sigma\) in a given region. For \(\sigma < 2\), we will have only one SL state for general \(\chi\) and \(\chi_0\). In the region \(2 < \sigma \leq 4\), there is again a critical value of \(\chi_0\). We define it as \(\chi_{0c}\). Below \(\chi_{0c}\), one, two and three SL states may appear depending on \(\chi_0\) and \(\chi\) whereas for one impurity case only one SL state appear below \(\chi_0^c\). Here two and three states appear because of local maximum and minimum in \(F(\sigma, \eta)\). Above \(\chi_{0c}\) there will be one, two and three SL states depending on the combination of \(\chi\) and \(\chi_0\). Here two and three SL states appear because of divergence in \(F(\sigma, \eta)\). The phase diagram in the \((\chi, \chi_0)\) plane is shown in Fig. 6.2 for \(\sigma = 3 \in [2, 4]\).
\( \chi_{0c} \) is shown in the figure. The curves separating one and three SL states contain two SL states. For any \( \sigma > 4 \), as long as \( \chi_0 < \chi_{0c} \), we will have a critical value of \( \chi \) below and above of which there will be no and two SL states respectively. For \( \chi_0 > \chi_{0c} \), no, one, two, three and four states will appear depending on the values of \( \chi \) and \( \chi_0 \). These things are clear from Fig. 6.3. \( \chi_{0c} \) is shown in the figure.

![Graph showing the separation of SL states](image)

**Fig. 6.3** Same as Fig. 6.2 except \( \sigma = 5 \) in this case.

### 6.2.3 Perfect nonlinear chain

The system we consider here is 1-d chain where all the sites have nonlinearity strength \( \chi \). The Hamiltonian for this system is same as given in eq. (6.1) with \( \chi_m = \chi \). Here we consider the existence of central peaked soliton as well as inter site peaked and dipped soliton solution. Again it is well known that in one dimension or in pseudo one dimension, states appearing outside the band are exponentially localized [82]. If nonlinearity can induce self-localization, states at the band edges of the system will have the propensity to undergo localization. Then low energy localized modes will have no node or all nodes and one node or \((N - 1)\) nodes, depending on the sign of \( \chi \). \( N \) is the number of sites in the system. In case of states having no node, states can peak either at lattice site or at the middle of two lattice points. For the on-site peaked localized states, the monomeric
ansatz is then suitable choice [30, 85]. On the other hand, for the intersite peaked or dipped states dimeric ansatz is the rational choice [33]. For central site peaked solution we can analyze the eq. (6.7). Doing the same analysis as mentioned above we draw the phase diagram in the $(\chi, \sigma)$ plane for the central peaked solution in this system. This is shown in Fig. 6.4. We see that there are two critical values of $\sigma$. We define them as $\sigma_1$ and $\sigma_2$ and shown in the figure. For $\sigma \leq \sigma_1$, only one SL state exists for arbitrary $\chi$. For $\sigma_1 < \sigma \leq \sigma_2$, there are two critical curves for $\chi$ and they are shown by dotted and solid curves. In this range of $\sigma$, there exists one state below the solid line, three states between the solid and dotted line, two states on the lines and again one state above the dotted line. For $\sigma > 4$, only one critical line exists and that is shown by the solid curve. In this region of $\sigma$, there exists no state below the curve, one state on the curve and two states above the curve.

Fig. 6.4 The phase diagram of SL states in the $\chi - \sigma$ plane for a perfectly nonlinear system. The nonlinear strength at all sites is $\chi$. Only the central site peaked solution is shown. $\sigma_1$ and $\sigma_2$ are the critical values of $\sigma$. Numbers in different regions indicate the number of possible SL states.

Next we consider the intersite peaked and dipped soliton solution for this system. For that we consider same ansatz and same procedure as used in the two impurity case. After
a little bit algebra we obtain the required equation and given by

\[
\frac{1}{\chi} = \frac{\eta(\eta^\sigma - 1)(1 - \eta^2)^{\sigma/2}}{2^{\sigma/2}(1 - \eta^{\sigma+2})^2[\text{sgn}(\beta)\eta - \text{sgn}(E)]}
\]  

(6.10)

where $\beta = \pm 1$. This $\beta$ has been introduced as well as its interpretation has been given in chapter 4. Positive sign of $\beta$ refers to the intersite peaked soliton and the negative sign refers to the intersite dipped soliton. For both the cases we get same kind of phase diagram as in the case of central site peaked solution except that the critical values are different.

![Phase diagram](image)

**Fig. 6.5** The phase diagram of SL states in the $\chi - \sigma$ plane for a perfectly nonlinear system. The nonlinear strength at all sites is $\chi$. Critical lines for central site peaked solution (dotted curves), intersite peaked solutions (solid curves) and intersite dipped solutions (dashed curves) are shown. Numbers in different regions indicate the number of possible SL states. The shaded region contains six SL states.

We have drawn the phase diagram for all three cases in Fig. 6.5. We see that there might exist all kind of solutions. The dashed curves are for intersite dipped solution, the solid curves are for intersite peaked solution and the dotted curves are for the central site peaked solution respectively. We have marked different regions by numbers. Those numbers indicate the possible number of SL states. We also note that maximum number
of SL states is six. The shaded region also contains six SL states. There is also continuity throughout the phase space as we go from one region to other.

We further notice that for the single impurity case maximum number of possible SL states is four. But form Fig. 6.4 we see that for central peaked soliton solution, maximum number of states is three. For two impurity case also maximum number of SL states is four. Thus, it is clear that the inhomogeneity in nonlinearity increases the number of SL states. As far as stability is concerned the presence of inhomogeneity does not do anything better. Inhomogeneity increases number of unstable SL states for some combination of $\chi$ and $\chi_0$. Even in the presence of inhomogeneity in nonlinearity the maximum number of stable SL states is two. We also note that for lower value of $\sigma$ and $\chi$, intersite peaked, dipped and central site peaked stationary localized solutions have equal probability to appear and all of them are stable.

6.2.4 Nonlinear chain with alternative nonlinear strengths

Here we consider a fully nonlinear (quadratic nonlinear) 1-dimensional system where the neighboring sites are of different nonlinear strengths but alternative sites are of same nonlinear strength. The Hamiltonian of the system is given by eq. (6.1) with $V = 1$, $\sigma = 2$ and

$$\chi_{2m} = \chi_1, \quad -\infty \leq m \leq \infty$$
$$\chi_{2m+1} = \chi_2, \quad -\infty \leq m \leq \infty.$$  \hspace{1cm} (6.11)

Hopping matrix elements between neighboring sites are taken to be unity without any loss of generality. To obtain SL states we consider $C_n = \phi_n e^{-iEt}$ and $\phi_n = \phi_0 |n|$. The expression for $\eta$ is given earlier. Here we have considered monomeric ansatz because the system is symmetric about the 0-th site. After going through the same procedure as earlier and using the normalization condition, $\sum_m |C_m|^2 = 1$, which in turn gives $\phi_0^2 = (1 - \eta^2)/2$, we obtain the reduced Hamiltonian of the dynamical system given by

$$H_{\text{eff}} = \frac{1}{\chi(\sigma + 2)} \left[ 1 - \eta^2 \right]^{\sigma+2} \left[ \frac{1 + \eta^{\sigma+2}}{1 - \eta^{\sigma+2}} + \delta \frac{1 - \eta^{\sigma+2} + \eta^{2\sigma+2}}{1 + \eta^{\sigma+2}} + \frac{2\eta}{1 + \eta^2} \right].$$  \hspace{1cm} (6.12)
where $\chi$ and $\delta$ are defined as $\chi = (\chi_1 + \chi_2)/2$ and $\delta = (\chi_1 - \chi_2)/2$. Setting $\partial H_{eff}/\partial \eta = 0$ we get

$$
\frac{1}{\chi} = \frac{(\eta^\sigma - 1)(\eta^{\sigma+4} + 1)}{\left[\frac{\eta^2 - 1}{\eta} \left(\frac{1+\eta^2}{1-\eta^2}\right)^{\frac{\sigma}{2}} - \delta \frac{(\eta^{\sigma+1})(\eta^{\sigma+4}+1)}{(1+\eta^{\sigma+2})^2}\right] (1 - \eta^{\sigma+2})^2}.
$$

(6.13)

Eq. (6.13) can be used to analyze number of possible states for different values of $\chi$, $\delta$ and $\sigma$. For $\sigma = 0$ and $\delta = 0$ from eq. (6.13) we see that to get a SL state $\chi$ needs to be infinite. It, therefore, means that no SL states can be obtained. This is true for $\sigma = 0$ and $\delta = 0$ because the system reduces to a perfect linear system. This is also true for $\sigma = 0$ and $\delta \neq 0$. Analysis of SL states can be done for arbitrary $\sigma$ but we consider the case of $\sigma = 2$ for the fact that this case is more physical. For $\sigma = 2$, eq. (6.13) reduces to

$$
\frac{1}{\chi} = \frac{\eta(1+\eta^\delta)(1+\eta^4)}{(1+\eta^2)^3[(1-\eta^2)(1+\eta^4)^2 - \delta \eta(1-\eta^2)(1-\eta^6)]}
$$

(6.14)

\[\text{Fig. 6.6 Phase diagram of SL states in } (\chi, \delta) \text{ plane for a fully nonlinear one dimensional system where alternative sites are of same nonlinear strengths. The number in a region indicates the number of possible SL states in the region.}\]

Eq. (6.14) directly tells that for $\delta = 0$ there will be always one SL state and this is consistent with our earlier result shown in section 6.2.3. Using eq. (6.14) the phase diagram of SL states in $(\chi, \delta)$ plane is obtained and shown in Fig. 6.6. As usual numbers
in different regions indicate number of SL states in those regions. Here also maximum number of SL states increases compared to a perfect nonlinear chain. Furthermore the phase diagram in this case is quite rich (see Fig. 6.6).

### 6.3 Fully nonlinear Cayley tree

We consider here the formation of SL states in a Cayley tree with each site having a power law nonlinearity and the same coupling constant, \( \chi \). We note that the system under consideration has the required translational invariance. So, the transformation proposed in section 5.2 will be applicable here. The problem then reduces to the study of formation of SL states in a one dimensional system with a bond defect between the zeroth and the first site. The tunneling matrix between the sites, \( V \) is reduced by a factor of \( 1/\sqrt{K} \) (\( K \) is the connectivity of the Cayley tree) and site energies are

\[
\epsilon_n = V^{n-1}\chi|C_n| \quad \text{if } n \geq 1
\]

and

\[
\epsilon_{-|n|} = V^{|n|}\chi|C_{-|n|}^\sigma| \quad \text{if } n \geq 0.
\] (6.15)

Furthermore, the Hamiltonian is given by eq.(6.1) with \( \chi_n = \chi \), and \( V = \frac{1}{\sqrt{K}} < 1 \). We first note that for \( \chi = 0 \) and \( V \leq 1 \), we have a band of states. For \( V > 1 \), we can have localized states. Since for \( V = 1 \) and \( V = 1/\sqrt{K} \) we have translationally invariant one dimensional system and a Cayley tree with connectivity, \( K \), the localized solutions will not show any space dependence. For \( V > 1 \), we, however, do not have the required translational invariance in the system. Here localized states peaked in the vicinity of the bond defect can show the space dependence. So, for this case (\( V > 1 \)) we restrict our study to the formation of self-localized modes pinned at the bond defect.

### 6.3.1 Inter-site peaked and dipped solutions

We first consider inter-site peaked and dipped solutions. It is clear from the previous discussion that the use of dimeric ansatz is justified for this purpose. The corresponding
effective Hamiltonian with $|\beta|=1$ is given by,

$$H_{\text{eff}} = \frac{2\chi}{2\sigma^2(\sigma + 2)} (1 - \eta^2)^{\sigma/2+1} + \frac{2\eta V}{2\sigma(\sigma^2 + 2)} + \frac{\eta V^2}{2\sigma} + \frac{\eta^2}{(\sigma + 2)}$$

(6.16)

By setting $\partial H_{\text{eff}}/\partial \eta = 0$, we obtain the equation governing the formation of SL states in this system. It is given by

$$\frac{2\sigma/2}{\chi} \frac{\eta (1 - \eta^2)^{\sigma/2} (1 - (V\eta)^\sigma)}{\eta^2 (V\eta)^\sigma} = G(\eta, \sigma)$$

(6.17)

From the asymptotic analysis of the equation of motion for $|n| \to \infty$, we obtain $\eta = (|E| - \sqrt{E^2 - 4})/2$ [30,85]. We further note that the effective nonlinear coupling constant in the equation of motion decreases exponentially with $|n|$ for $K > 1$ and $\sigma > 0$ (see eq. (6.15)). We are also assuming that $\chi > 0$. Since $[1 - (V\eta)^\sigma] \to -\sigma \ln(V\eta)$ as $\sigma \to 0$, $G(\eta, \sigma) \to 0$ as $\sigma \to 0$. Consequently, $\chi \to \infty$. This implies that no SL state will be formed in this limit. We now consider various cases.

![Phase diagram](image_url)

**Fig. 6.7** Phase diagram for SL states in a fully nonlinear Cayley tree. The solid line is the critical line for the on site (zeroth site) peaked solution. The lower dotted line is the critical line for the inter site peaked (symmetric) solution and the uppermost line defines the critical line for the inter site dipped (antisymmetric) solution. Here $K = 4(V = 0.5)$. For inter site solutions region I, II and III contains no, two and three SL states respectively. For on site solutions there is no state below the solid curve and two states above the solid curve.
(I) Symmetric Case

Here \( \text{sgn}(E) = \text{sgn}(\beta) = +1 \). We note that \( G(\eta, \sigma) \) have a removable singularity and a divergence at \( \eta_0 = 1/V \) and \( \eta_1 = 1/V^{2\pi^2} \) respectively. But for \( V < 1 \), \( \eta_1 > 1 \). So, the singularity of \( G(\eta, \sigma) \) at \( \eta_1 \) will not play any role in the formation of SL states. Again \( G(0, \sigma) = 0 = G(1, \sigma) \). So, \( G(\eta, \sigma) \) will have at least one maximum at \( \eta_m \in [0, 1] \). It can be seen numerically that \( G(\eta, \sigma) \) has only one maximum for \( \eta \in [0, 1] \). Consequently, in the \((\chi, \sigma)\) plane there will be a critical line separating two states region from the no state region. Since one of the states in the two states region \( \eta \to 1 \) as \( \chi \to \infty \), it is an unstable state. On the other hand for \( V > 1 \), \( G(\eta, \sigma) \) has a divergence at \( \eta_1 < 1 \). So, in this case the system will always produce a SL state even if \( \chi \) is infinitesimally small. Furthermore, \( \lim_{\epsilon \to 0} G(\eta_1 - \epsilon, \sigma) \to \infty \) from the positive side only. So, for \( \chi > 0 \), \( \eta_{\text{max}} = \eta_1 \) and \( \eta \to 0 \) as \( \chi \to \infty \). Hence this is a stable SL state.

(II) Antisymmetric Case

In this limit \( \text{sgn}(\beta) = -1 \) but \( \text{sgn}(E) \) can be either \(+1\) or \(-1\). We note that for \( \text{sgn}(E)=+1 \), \( G(\eta_0, \sigma)=0 \) here. But for \( V < 1 \) both \( \eta_0 \) and \( \eta_1 \) lie beyond unity. Since \( G(\eta, \sigma)=0 \) both at \( \eta=0 \) and \( \eta=1 \), it has a maximum at \( \eta_m \in [0, 1] \). This implies that in the \((\chi, \sigma)\) plane there will be a critical line. This line again separates the no state region and the two states region. Furthermore, in the two state region one state will be unstable for the same argument given earlier. For \( V > 1 \), we note that \( G(\eta, \sigma) \) goes to zero and infinity at \( \eta_0 \) and \( \eta_1 \) respectively. Furthermore, if \( \sigma \) is finite, \( \eta_0 < \eta_1 \). So, for \( \eta \in (\eta_0, \eta_1) \), \( G(\eta, \sigma) \) is negative. This in turn implies that \( \lim_{\epsilon \to 0} G(\eta_1 - \epsilon, \sigma) \to -\infty \) and \( \lim_{\epsilon \to 0} G(\eta_1 + \epsilon, \sigma) \to \infty \). Consequently, \( \lim_{\epsilon \to 0} G(1-\epsilon, \sigma) \to 0 \) from the positive direction for \( \eta \in [\eta_1, 1] \). Therefore, we shall always obtain a SL state even if \( \chi \) is infinitesimally small. In this SL state, however, \( \eta_{\text{min}} = \eta_1 \) and as \( \eta \to 1 \), \( \chi \to \infty \). So, this is an unstable state. We further note that \( G(0, \sigma)=0=G(\eta_0, \sigma) \). Then there will be a maximum of \( G(\eta, \sigma) \) at \( \eta_m \in [0, \eta_0] \). So, there will be a critical line in the \((\chi, \sigma)\) plane also. This line will separate one state region and three states region. It is further seen that two of the states in the later region are unstable. We can also have \( \text{sgn}(E)=-1 \). However, for \( V < 1 \) no SL states will be obtained in this limit. On the other hand for \( V > 1 \), \( \lim_{\epsilon \to 0} G(\eta_1 \pm \epsilon, \sigma) \to \pm \infty \). So, we shall always get a SL state below the band even if \( \chi \) is infinitesimally small. However, this SL state is unstable.
We now combine our results to obtain the phase diagram. For $V < 1$, we have three regions, namely I, II and III containing no SL state, two SL states and four SL states respectively.

\[\text{Fig. 6.8} \text{ Stability diagram for SL states in a fully nonlinear Cayley tree. Here } K = 4(V = 0.5). \text{ Stability of the states in various regions are marked in the figure.}\]

\[\text{Fig. 6.9} \text{ Total phase diagram for SL states of a fully nonlinear (nonlinear strength reduces as a function of distance measured from zeroth as well as first site of the chain) one dimensional chain with a bond defect between sites 0 and 1. Here } V = \sqrt{2}. \text{ The region I contains three SL states and the region II contains five SL states.}\]
This is shown in Fig. 6.7 for \( K = 4 \) or \( V = 0.5 \). The stability diagram \([124]\) of SL states for this case is shown in Fig. 6.8. However, for \( V > 1 \) we do not have any no SL state region. Instead we have a three state region separated from a five state region by a critical line. One of these states appears below the band. In the three state region we have two stable states while in the other region we have three stable states. For \( V = \sqrt{2} \), the phase diagram is shown in Fig. 6.9.

### 6.3.2 On site peaked soliton

We discuss here the formation of on-site soliton in the fully power-law nonlinear Cayley tree. For this purpose we first consider a power law nonlinear impurity with strength, \( \chi \) embedded at the zeroth site of the one dimensional system transformed from a Cayley tree. After introducing the dimeric ansatz in the appropriate form of the Hamiltonian we obtain

\[
H_{\text{eff}} = \frac{2\chi}{(\sigma + 2)} \left( \frac{1 - \eta^2}{1 + \beta^2} \right)^{\sigma/2+1} + 2\text{sgn}(E) \eta + 2V\beta \left( \frac{1 - \eta^2}{1 + \beta^2} \right)
\]

(6.18)

Again relevant equations are obtained by \( \partial H_{\text{eff}}/\partial X_i = 0 \) where \( X_1 = \beta \) and \( X_2 = \eta \). After a trite algebra we then obtain \( \beta = \text{sgn}(E)V\eta \) and

\[
\frac{\text{sgn}(E)}{\chi} = \frac{\eta(1 - \eta)^{\sigma/2}}{(1 + V^2\eta^2)^{\sigma/2}(1 - V^2\eta^2)} = f(\eta, \sigma)
\]

(6.19)

For \( V = 1/\sqrt{K} \), eq. (6.19) describes the formation of SL states due to a nonlinear impurity in a Cayley tree. This has been discussed in detail in chapter 3. So, we see that the dimeric ansatz reduces to the appropriate monomeric ansatz. When \( V > 1 \), \( f(\eta, \sigma) \) has a divergence at \( \eta_u = 1/V \). Furthermore we have, \( f(0, \sigma) = 0 = f(1, \sigma) \) and \( \lim_{\epsilon \to 0} f(\eta_u - \epsilon, \sigma) \to \infty \). So, we shall obtain two SL states even if \( \chi \) is infinitesimally small and \( \sigma > 0 \). However, one state will appear below the band. In this state \( \eta_{\text{min}} = \eta_u \) and \( \eta \to 1 \) as \( \chi \to \infty \). So, this is an unstable state. For, \( \sigma = 0 \), this state will appear if \( 0 < \chi < (V^2 - 1) \).

To study the formation of on-site peaked SL states in the fully nonlinear chain we put the dimeric ansatz with \( \beta = \text{sgn}(E)V\eta \) in the Hamiltonian, \( H \) given by eq. (6.1) with

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\( \chi_m = \chi \). We then obtain the effective Hamiltonian, \( H_{\text{eff}} \), given by

\[
H_{\text{eff}} = \frac{2\chi}{(\sigma + 2)(1 + V^2 \eta^2)^{\sigma/2+1}(1 - V^\sigma \eta^{\sigma+2})} + 2\text{sgn}(E)\eta + \text{sgn}(E) \frac{2V^2 \eta(1 - \eta^2)}{1 + V^2 \eta^2}.
\]

We then set \( \partial H_{\text{eff}} / \partial \eta = 0 \). After a trite algebra we finally obtain

\[
\frac{\text{sgn}(E)}{\chi} = \frac{\eta(1 - \eta^2)^{\sigma/2}(1 + V^\sigma + 2 \eta^{\sigma+4}) (1 - V^\sigma \eta^{\sigma})}{(1 + V^2 \eta^2)^{\sigma/2}(1 - V^\sigma \eta^{\sigma+2})^2 (1 - V^2 \eta^2)} = f_1(\eta, \sigma)
\]

(6.20)

When \( V = 1 \), eq. (6.20) reduces to the relevant equation in ref. [24]. We note that \( f_1(\eta, \sigma) \) has a removable singularity at \( \eta_m \) and a divergence at \( \eta_1 = V^{\sigma+\frac{\sigma}{2}} \). However, for \( V < 1 \) the divergence at \( \eta_1 \) does not play any role in the formation of SL states. Since \( f_1(0, \sigma) = 0 = f_1(1, \sigma) \) we expect at least one maximum of \( f_1(\eta, \sigma) \) at \( \eta_m \in [0, 1] \). It is seen numerically that for \( \eta \in [0, 1] \) \( f_1(\eta, \sigma) \) has only one maximum. So in the \( (\chi, \sigma) \) plane there is a critical line separating the no state region from the two states region. Of course, one of the states is unstable. In Fig. 6.7 the critical line is shown by solid curve for \( K = 4 \).

On the other hand, for \( V > 1 \), \( f_1(\eta, \sigma) \) diverges at \( \eta_1 \). Since, \( \lim_{\epsilon \to 0} f_1(\eta_1 \mp \epsilon, \sigma) \to \infty \) we have also two states and these states are formed even if \( \chi \) is infinitesimally small. One of the states is unstable. However, numerical calculation shows that there exists a critical value of \( \sigma \) say \( \sigma_{cr} \) such that for \( \sigma > \sigma_{cr} \) there will be a four state region bounded by two critical values of \( \chi \). For example, for \( V = \sqrt{2} \) \( \sigma_{cr} \sim 3.85 \). But the four states region occurs at larger value of \( \chi \). Two of the states are again unstable.

We now end this section with a brief discussion on the exactness of the calculation. The method adopted here is similar to the well known effective medium theory for the linear system. This is quite clear from the form of \( H_{\text{eff}} \) given in Eqs.(16) and (20) In the first case we have an effective nonlinear dimer in which \( \chi_{\text{eff}} \) is a function of \( \eta, \chi \) and \( \sigma \). In the second case we have a effective nonlinear monomer. But the use of the dimeric as well as the monomeric ansatzs are quite justified for the study of low energy self-localized modes. This has been discussed. Therefore, basic features obtained here will also be reproduced by rigorous calculations. But quantitative agreement may not be obtained. More work is therefore necessary.
6.4 Stability

The stability of the SL states can be understood from a simpler graphical analysis. We choose the system discussed in sec. 6.2.4 and do the stability analysis for the SL states graphically. For this purpose we look at the effective Hamiltonian for the system given in eq.(6.12). The Hamiltonian is function of one dynamical variable, namely, $\eta$ where $\chi$ and $\delta$ are parameters. For fixed values of $\chi$ and $\delta$, $\eta$ and $\partial H_{\text{eff}} f / \partial \eta = f(\eta)$ can be treated as generalized coordinate and generalized momentum respectively for the reduced dynamical system. Solutions of the equation $f(\eta) = 0$ will give the fixed points for fixed value of $\chi$ and $\delta$ and number of fixed points with $\eta \in [0,1]$ will be the number of SL states.

![Diagram](image.png)

**Fig. 6.10** $f(\eta)$ is plotted as a function of $\eta$. Points indicated by A, B and C are fixed points. Here $\chi$ and $\delta$ are taken to be 0.35 and 2.5 respectively.

We now plot $f(\eta)$ as a function of $\eta$ for $\chi = 0.35$ and $\delta = 2.5$ as shown in Fig. 6.10. If $f(\eta) > 0$, then the flow of the dynamical variable will be along the positive direction and on the other hand if $f(\eta) < 0$ the flow will be along the negative direction. These flows are indicated by arrows in the figure and the fixed points are denoted by A, B and C respectively. In the neighborhood of A the flow is always towards A and same for C. Therefore, A and C are stable fixed points. On the other hand in the neighborhood of B the flow is away from B. Hence, this is an unstable fixed point. Same thing happens for all $(\chi, \delta)$ in the three state region in Fig. 6.6. We therefore notice that among three SL
states (fixed points) two are stable and the other one is unstable. To get a connection of the stability of the states with the energy and \( \chi \), we plot the energy of the states as a function of \( \chi \) in the neighborhood of \( \chi = 0.35 \) for fixed value of \( \delta = 2.5 \) in Fig. 6.11. Energies of the SL states arising from the fixed points A, B and C are denoted by \( A_1, B_1 \) and \( C_1 \) respectively in Fig. 6.11.

![Energy of SL states](image)

**Fig. 6.11** Energy of SL states are plotted as a function of \( \chi \). Here \( \delta = 2.5 \). \( A_1, B_1 \) and \( C_1 \) are the points corresponding to the points A, B and C (in Fig. 6.10) respectively. The dotted line is \( \chi = 0.35 \).

The figure clearly shows that in the neighborhood of \( \chi = 0.35 \), the energy of two stable SL states increases with \( \chi \) and that of the unstable state decreases. We, therefore, conclude that the SL state is stable if the energy of the state increases with \( \chi \) otherwise unstable.

### 6.5 Summary

The formation of SL states in fully nonlinear one dimensional chain as well as in the perfectly nonlinear Cayley tree is studied. The full phase diagram for SL states is presented for all cases. It is interesting to note that even if the translational symmetry remains preserved, the nonlinearity causes the formation of SL states in the system. The introduction of nonlinear impurity caused the formation of new localized states. For a
perfectly nonlinear one dimensional chain, the central site peaked, the intersite peaked and the intersite dipped solutions have equal probability to appear for certain range of the parameters. This is also true for the perfect nonlinear Cayley tree. The stability of SL states are verified graphically for a particular system. The connection of the stability of a SL state with the variation of its energy as a function of the nonlinear strength is discussed.