STATISTICAL PROPERTIES OF STRUCTURAL SAFETY

6.1. Introduction

The safety of a structure against failure having a capacity (resistance) \( R \), subjected to loading effects \( L \), is given as:

\[
\text{Safety factor} = S_F = \frac{R}{L} \\
\text{Safety zone} = S_Z = (R-L)
\]

As \( R \) & \( L \) are random variables so are \( S_F \) and \( S_Z \) with density functions, say, \( f_{S_F} \) & \( f_{S_Z} \). For a given limit state, say, \( i^{th} \) state, with \( R = R_i \) and \( L = L_i \), we have for safety factor and safety zone as:

\[
S_{F_i} = \frac{R_i}{L_i} \quad \text{&} \quad S_{Z_i} = R_i - L_i \quad \ldots \quad (6.1)
\]

The capacity (Resistance) \( R \) depends on a set of variables \( X_1, X_2, \ldots X_p \), representing material properties and geometrical parameters of the structure and can be given by the following function \( h \):

\[
R = h (X_1, X_2, \ldots X_p) \quad \ldots \quad (6.2)
\]

Similarly, \( L \) is a function of variables \( Y_1, Y_2, \ldots Y_q \) representing loads.

\[
L = g (Y_1, Y_2, \ldots Y_q) \quad \ldots \quad (6.3)
\]

The main object of the investigation is the evaluation of \( S_F \) and/or \( S_Z \) and hence finding probability of failure of
structures, given by \( \text{Prob} (Sg \leq 1) \) or \( \text{Prob} (Sg \leq 0) \) as shown in Fig. 6.1. The shaded portion in Fig. 6.1 shows the "interference area" which is indicative of the probability of failure.

Fig. 6.1 - Interference of \( R \& L \).

Since \( R \& L \) are known functions of variables representing material properties, geometry of the sections and various loads supported by the structure, the distributions of \( R \) and \( L \) will depend on the joint distribution of these variables. A general procedure is presented to evaluate the distribution of a function of several independent variables with known distributions and then applying it for obtaining the distributions of \( S_F \) and \( S_g \).

6.2. Exact sampling distribution of a function of independent variables: 2131

Let the distribution of \( P \) independent variables, \( x_1, x_2, \ldots, x_P \) be \( f_1(x_1), \ldots, f_p(x_P) \) and \( z = z(x_1, x_2, \ldots, x_P) \) be a function
The distribution of $z$ is given by:

$$P(z_0) = \int_D \int dF_1(x_1) \, dF_2(x_2) \cdots dF_p(x_p) \quad (6.4)$$

where $D$ is the domain of $x_1, x_2, \ldots, x_p$ such that

$$B(x_1, x_2, \ldots, x_p) \subseteq z_0$$

The above integral may be evaluated by convenient change of variables involved. This procedure consists of simplification of the domain of integration as defined by the limits of the new (changed) variables. In general we take the statistic whose sampling distribution is being sought as one of the new variables and choose $(n-1)$ others ($\theta_1, \theta_2, \ldots \theta_{p-1}$) conveniently. Thus,

$$x_1 = H_1(z_1 \theta_1, \ldots \theta_{p-1})$$
$$x_2 = H_2(z_1 \theta_1, \ldots \theta_{p-1})$$
$$\vdots$$
$$x_p = H_p(z_1 \theta_1, \ldots \theta_{p-1})$$

where $H_1, H_2, \ldots, H_p$ are chosen functions of $z_1, \theta_1, \ldots \theta_{p-1}$ to represent $x_1, x_2, \ldots, x_p$ respectively. The Jacobian $J$ of the transformation $(x_1, x_2, \ldots, x_p) \rightarrow (z_1, \theta_1, \theta_2, \ldots, \theta_{p-1})$ is given by the determinant: 

$$J = \left| \begin{array}{ccc}
\frac{\partial x_1}{\partial z_1} & \cdots & \frac{\partial x_1}{\partial \theta_{p-1}} \\
\vdots & \ddots & \vdots \\
\frac{\partial x_p}{\partial z_1} & \cdots & \frac{\partial x_p}{\partial \theta_{p-1}}
\end{array} \right|$$
say \[
\begin{bmatrix}
\frac{\partial x_1}{\partial z_1}, & \frac{\partial x_1}{\partial z_2}, & \ldots & \frac{\partial x_{p-1}}{\partial z_1} \\
\frac{\partial x_2}{\partial z_1}, & \frac{\partial x_2}{\partial z_2}, & \ldots & \frac{\partial x_{p-1}}{\partial z_2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial x_p}{\partial z_1}, & \frac{\partial x_p}{\partial z_2}, & \ldots & \frac{\partial x_p}{\partial z_{p-1}}
\end{bmatrix}
\]  

Thus integral \( F(z) \) becomes,

\[
F(z) = \int f_1(x_1) f_2(x_1) \ldots f_p(x_p) J \, dz_1 \ldots dz_{p-1} \quad (6.5)
\]

Where \( f_1(x_1) \) is the probability density function of \( x_1 \) and is expressed as function of \( z \) and \( e \)'s. The sign of \( J \) must be determined to yield positive transformed integral.

In the present study, we apply the above procedure to see whether the distribution of \( S_f \) and \( S_g \) can be simplified. It is easy to notice that:

\[
S_f = h_F(f_y, f_c, D, L) = C_1(f_y - C_2 \frac{f_y}{f_c}) \frac{1}{C_3(D + L)} \quad (6.7)
\]

\[
S_g = h_g(f_y, f_c, D, L) = C_1 f_y (1 - C_2 f_y / f_c) - C_3 (D + L) \quad (6.8)
\]

Where \( f_y, f_c, D, L \) have density functions denoted, say by:

\( g_y(f_y), g_c(f_c), g_D(D), g_L(L) \).
Considering the transformation:

\[ f_y = e_1 \]
\[ D = e_2 \]
\[ L = e_3 \]
\[ f_c = \frac{C_2 \cdot e_1}{e_1 - C_3 \cdot \frac{S_F (e_2 + e_3)}{C_1}} \]

.: Jacobian of the transformation is

\[
J = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
2 \left( e_1 \frac{C_3}{C_1} S_F (e_2 + e_3) \right)^2 - \left( e_1 \frac{C_3 S_F}{C_1} (e_2 + e_3) \right)^2
\]

\[
\frac{C_2 C_3 S_F e_1}{\left( e_1 \frac{C_3 S_F}{C_1} (e_2 + e_3) \right)^2}
\]

\[
\frac{C_2 C_3 e_1 (e_2 + e_3)}{\left( e_1 \frac{C_3 S_F}{C_1} (e_2 + e_3) \right)^2}
\]
\[ F(s_p) = \int \int \int e_y(e_1) \cdot e_p(e_2) \cdot e_L(e_3) \cdot e_c \left( \frac{c_2 c_1 e_1}{c_{12} - c_3 s_p(e_2 + e_3)} \right)^2 \cdot d e_1 \cdot d e_2 \cdot d e_3 \] .... (6.10)

The functions \( e_y(.) \), \( e_p(.) \), \( e_L(.) \) are density functions of normal distributions while that of \( e_c(.) \) is lognormal. It is easy to see the complexity involved in evaluating the above triple integral explicitly. A similar complicated integral will be obtained for the distribution function of \( s_p \) also. Therefore we need to evaluate their distributions using other alternative procedures. Two such approaches have been employed based respectively on simulations and approximation using moments.

6.3. Evaluation of Probability of Failure:

The probability density functions of most of the variables are not known precisely because of general scarcity of data, let alone the joint density functions (as required for \( R, L, R/L \) & \( R-L \)).

With regard to the distribution functions, it should be emphasised that it is the distribution of safety factor (\( R/L \)) or (\( R-L \)) that is important. The expression for the probability...
failure $P_f$ is multi-dimensional integral (as evident from Equation No.6.10) and a direct approach for the solution is exceedingly complex. At very low risk, say $P_f \leq 10^{-6}$, the calculated failure probability is sensitive to the assumed distribution, since it is normally difficult to ascertain the correct distribution. For high risk condition, say $P_f > 10^{-3}$, the failure probability is not significantly affected by the shape of the distribution. Therefore $P_f$ has been estimated in the following two ways.

1. Simulation approach.

2. Approximation approach based on higher order moments.

6.3.1. Simulation approach: If a sufficiently large data can be generated, it should, in principle, be possible to make a direct count of the failure events and thereby obtain the probability of failure. As the direct analytical approach for the solution of $P_f$ is not practically possible, the Monte Carlo simulation technique provides a versatile and powerful tool.

6.3.1.1. Monte Carlo Technique: The principle utility of Monte Carlo simulation is in the analysis of complex probability problems where analytical models are mathematically intractable. It is a method of approximately solving mathematical and physical problems by the simulation of random variables, from prescribed probability distributions. A sample from a Monte Carlo simulation is similar to that of a laboratory or field experiment, except that it is a synthetic or computer generated experiment. It is evident that, irrespective of the complexity of the simulation
model, it remains an idealization of reality and as such the results are only as good as the reliability of the assumptions underlying the model and the quality of the input information.

6.3.1.2. Generation of pseudo random values of R & L : The procedure consists of generating a set of four independent uniform variables \( u_1, u_2, u_3, u_4 \).

1) **Uniform variable:** A random variable, \( u \), is uniform in \((0,1)\) if its probability density function is given by:

\[
f(u) = 1, \quad 0 \leq u \leq 1
\]

the algorithm for generating uniform random numbers in \((0,1)\) is given in Appendix. Moreover, uniform random variables are available as "library routine".

From these uniform variables, 4 independent standard normal variables \( x_1, x_2, x_3, x_4 \) are computed.

ii) **Normal random variable:** If \( u_1 \) & \( u_2 \) are two independent standard uniform variates, then the following functions:

\[
\begin{align*}
x_1 &= \sqrt{-2 \log u_1} \cdot \sin (2 \pi u_2) \\
x_2 &= \sqrt{-2 \log u_1} \cdot \cos (2 \pi u_2)
\end{align*}
\]

constitute a pair of statistically independent standard normal variates. (Box & Muller 1958).

In order to generate a normal variable with non zero mean \( \mu \) and standard deviation \( \sigma \), we use the following expression:

\[
y_1 = \mu + \sigma x_1 \sim N(\mu, \sigma^2)
\]
iii). Lognormal random variable: If \( x \) is a normal random variable with mean \( u \) and standard deviation \( \sigma \), then

\[ y = e^x, \text{ has lognormal distribution with} \]

\[ \text{mean } = \log_e \left( \frac{\mu}{\sqrt{\mu^2 + \sigma^2}} \right) \]

\[ \text{standard deviation } = \left( \log_e \left( \frac{\sigma^2 + \mu^2}{\mu^2} \right) \right)^{1/2} \]

Then random values of \( f_y, f_c, D \) and \( L \) are computed as given below:

\[ f_y = \mu_y + \sigma_y x_1 \quad \ldots \quad (6.11) \]

\[ f_c = \mu_c + \sigma_c x_2 \quad \ldots \quad (6.12) \]

\[ \text{Dead load } = d = \mu_d + \sigma_d x_3 \quad \ldots \quad (6.13) \]

\[ \text{Live load } = l = \exp \left( \sigma_y + s_y x_y \right) \quad \ldots \quad (6.14) \]

Hence random values of \( R \) and \( L \) are computed from their governing equations:

\[ R = c_1 \left( f_y - c_2 \frac{f_y^2}{f_c} \right) \text{ for beams.} \quad (6.15) \]

\[ R = c_1 f_c^2 + c_2 f_y \quad \text{for columns.} \quad (6.16) \]

\[ \& \quad L = c_3 (d + l) \quad \ldots \quad (6.17) \]

These values of \( R \) and \( L \) form one set of simulated (generated) values. This process was repeated \( N \) times independently to generate \( N \) values \((R_i, L_i) i = 1 \ldots N\) of \( R \) and \( L \). \( N \) is termed the number of simulation runs.

6.3.2. Approximations based on Moments: More often concise descriptors summarizing only the dominant features of the behaviour of
a random variable are sufficient. One or more simple parameters are used in place of whole probability density function. The numbers usually take the form of weighted averages of certain functions of a random variable. The weights used are the Probability Mass Function (PMF) or Probability Density Function (PDF) of the variable and the average is called the expectation of the function. When compared with entire probability laws, these expectations are easier to work within the analysis of uncertainty, as well as much easier to obtain estimates of from the available data. The evaluation of first four moments has been considered to study the distributional behaviour of the random variables considered here.

6.3.2.1. Moments of $R$

**Case 1 (Beans):** $R$ is defined by

$$R = c_1 \left( f_y - c_2 \frac{f_y^2}{f_c} \right) \quad \ldots \quad (6.18)$$

Let

$$f_y = \mu_y (1 + e_1)$$
$$f_c = \mu_c (1 + e_2)$$

**Hence**

$$R = c_1 (\mu_y (1 + e_1) - c_2 \frac{\mu_y^2}{\mu_c} (1+e_1)^2 (1-e_2 e_1 - e_2)^{-1}) \quad (6.19)$$

Expanding the terms up to 4th power:

$$R = c_1 \left( \mu_y (1+e_1) - c_2 \frac{\mu_y^2}{\mu_c} (1+2 \epsilon_1 + \epsilon_1^2)^2 (1- \epsilon_2 + \epsilon_2 - \epsilon_2 + \epsilon_2) \right)$$

$$= c_1 \left( \mu_y (1+e_1) - c_2 \frac{\mu_y^2}{\mu_c} (1-3 \epsilon_2 + \epsilon_2 + 3 \epsilon_2 + \epsilon_2 - 2 \epsilon_1) \right)$$

$$2 \epsilon_1 \epsilon_2 \quad 2 \epsilon_1 \epsilon_2$$

$$-2 \epsilon_1 \epsilon_2 + 2 \epsilon_1 \epsilon_2 - 2 \epsilon_1 \epsilon_2 + \epsilon_1 \epsilon_2 \epsilon_2$$
\[
\begin{align*}
\alpha_1 &= c_1 \left( \frac{\mu_y^2}{\mu_c} \right) + \mu_y \xi_1 - c_2 \frac{\mu_y^2}{\mu_c} (\xi_1^2 + \xi_2^2 - 2 \xi_1 \xi_2) \\
&\quad - 2 \xi_1 \xi_2 - 2 \xi_2^2 - 3 \xi_1^2 - 2 \xi_1 \xi_2 - 2 \xi_1^2 + \xi_1 \xi_2 + \xi_2^2 ) \\
&\quad \ldots (6.20)
\end{align*}
\]

Let \( R = \mu_R + \xi_R \), where
\[
\begin{align*}
\mu_R &= c_1 \left( \frac{\mu_y - c_2}{\mu_c} \right) \\
\xi_R &= c_1 \left( \frac{\mu_y(1-c_2)}{\mu_c} \right) \xi_1 + c_2 \frac{\mu_y}{\mu_c} (\xi_2 - \xi_1) + \xi_2 \\
&\quad + \xi_2 - \xi_1 \xi_2 + 2 \xi_1 \xi_2 + \xi_1 \xi_2 - 2 \xi_1^2 - \xi_1 \xi_2 + 2 \xi_1 \xi_2 ) \\
&\quad \ldots (6.22)
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}(R) &= \mu_R + \mathbb{E}(\xi_R) \\
&\quad \ldots (6.23)
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}(\xi_R) &= -c_1 \left( \frac{\mu_y^2}{\mu_c} \right) \frac{\sigma_y^2}{\mu_y^2} + \frac{\sigma_c^2}{\mu_c^2} + \frac{\mu_4c}{\mu_c^4} - \frac{\sigma_c^2 \sigma_y^2}{\mu_c^4} - \frac{\mu_3c}{\mu_c^3} \\
&\quad \ldots (6.24)
\end{align*}
\]

Where \( \mu_{1s} = \mathbb{E} (S - \mathbb{E}(S))^i \), \( i \) th central moment of \( S \).

\[
S = y, c; i = 3, 4 \ldots
\]

\( \mathbb{E}(\xi_R) \) gives the general expression for the bias \( B(R) \) in estimating the quantity \( \mu_R \) based on approximation up to fourth order of moments.

Since \( \mu_y \) and \( \mu_c \) are normal random variables,
\[
\mu_4c = 3 \sigma_c^4
\]

\[
\mathbb{E}(R) = \mathbb{E}(\xi_R) = \frac{\mu_y^2}{\mu_c^2} \left( \frac{\sigma_y^2}{\mu_y^2} + \frac{\sigma_c^2}{\mu_c^2} + \frac{3 \sigma_c^4}{\mu_c^4} - \frac{\sigma_c^2 \sigma_y^2}{\mu_c^4} - \frac{\mu_3c}{\mu_c^3} \right) \\
&\quad \ldots (6.25)
\]
Let \( \lambda_1 = \mu_y(1 - 2c_2 \frac{\mu_y}{\mu_c}) \) & \( \lambda_2 = c_2 \frac{\mu_y}{\mu_c} \).

\[ (6.26) \]

Bias in \( R \): \( B(R) = -c_1 \left( \frac{\sigma_y^2}{\mu_y^2} + \frac{\sigma_c^2}{\mu_c^2} + \frac{3 \sigma_c^4}{\mu_c^4} + \frac{\sigma_c \sigma_y}{\mu_c \mu_y} \right) \)

Mean square error of \( R = MSE(R) = E(R - \mu_R)^2 \) \[ (6.27) \]

\[ \text{Var} \ (R) = MSE(R) - (B(R))^2 \] \[ (6.28) \]

We can write \( \mathbb{E}_R = (\lambda \mathbb{E}_1 + \lambda \mathbb{E}_2) - \lambda_1 (\mathbb{E}_1 + \mathbb{E}_2 - 2 \mathbb{E}_1 \mathbb{E}_2) \)
\[ \text{from Eqn. 6.22} \]
\[ \mathbb{E}_1 \]
\[ (6.29) \]

retaining terms up to 4th power only:

\[ \sigma^2_{R/c_1} = (\frac{\lambda \mathbb{E}_1^2}{2} + 2 \lambda \mathbb{E}_1 \mathbb{E}_2 + \lambda \mathbb{E}_2^2) - 2 \lambda_1 (\mathbb{E}_1^2 + \mathbb{E}_2^2) \]
\[ \mathbb{E}_1 \]
\[ (6.30) \]

\[ \sigma^3_{R/c_1} = \frac{\lambda \mathbb{E}_1^3}{2} + \frac{\lambda \mathbb{E}_1^3}{2} + 3 \lambda \mathbb{E}_1 (\mathbb{E}_1^2 \mathbb{E}_2 + \mathbb{E}_1 \mathbb{E}_2^2) - 3 \lambda_1 (\mathbb{E}_1^2 \mathbb{E}_2 + \frac{\lambda \mathbb{E}_1^2}{2} \mathbb{E}_2^2) \]
\[ \mathbb{E}_1 \]
\[ (6.31) \]

\[ \sigma^4_{R/c_1} = \frac{\lambda \mathbb{E}_1^4}{2} + 4 \lambda \mathbb{E}_1 \mathbb{E}_2^3 + 4 \mathbb{E}_1^2 \mathbb{E}_2^2 + \mathbb{E}_1 \mathbb{E}_2^3 + 2 \mathbb{E}_1^2 \mathbb{E}_2^2 + \mathbb{E}_1 \mathbb{E}_2^3 + 4 \mathbb{E}_1 \mathbb{E}_2^3 \]
\[ \mathbb{E}_1 \]
\[ (6.32) \]

The expectation simplify to:

\[ E(\frac{\sigma_R}{c_1})^2 = \lambda \frac{\sigma_y^2}{\mu_y} + \lambda \left( \frac{\sigma_c^2}{\mu_c^2} - \frac{2 \mu_3c}{\mu_c^3} + \frac{\mu_4y}{\mu_y^4} + \frac{3 \mu_4c}{\mu_c^4} \right) \]
\[ \frac{\sigma_c \sigma_y}{\mu_c \mu_y} \]
\[ (6.33) \]
The third and fourth central moments of $R$ can be obtained from the above expressions of $E(\xi_1^2)$, $E(\xi_1^3)$, $E(\xi_1^4)$ and bias $B(R)$.

\[ B(\xi_1^2) = E(R - E(R))^2 = E(R^2 - E(R))^2 = E(R_1^2 - \mu_1^2)^2 \]

\[ B(\xi_1^3) = \text{bias terms} \]

\[ B(\xi_1^4) = \text{bias terms} \]
Axially Loaded Columns:

\[ R = c_1 f_g + c_2 f_y \]  \hspace{1cm} (G.43)

As \( f_g \) & \( f_y \) are normal, \( R \) is also normal.

With mean \( R = c_1 p_c + c_2 p_y \)  \hspace{1cm} (G.44)

Bias = 0

Variance = \( \sigma^2_R = c_1^2 \sigma^2_c + c_2^2 \sigma^2_y \)  \hspace{1cm} (G.45)

\[ \mu_{3R} = 0 \]

\[ \mu_{4R} = 3 \sigma^4_R \]  \hspace{1cm} (G.46)

6.3.2.2. Moments of \( L \): \( L \) is given by:

\[ L_3 = c_3 (d + l) \]  \hspace{1cm} (G.47)

where \( d \) = dead load distributed as normal

\( l \) = live load distributed as lognormal

ie. \( d \sim N \left( \mu_d, \sigma^2_d \right) \)

\( \log l \sim N \left( m, \beta^2 \right) \)
using the properties of lognormal distribution, the moments of $L$
in terms of moments of log 1 are given by:
(Kendall, M.G., and Stuart, A, Vol.1, p.169)
\[
E(L) = \mu_L = \omega^{1/2} \mu
\]

There $\omega = e^\beta$ and $\mu = e^\mu$

\[
P_{2L} = \sigma_L^2 = \omega^2 (\omega - 1)
\]

\[
P_{3L} = \omega^3 \mu^3 (\omega - 1)^2 (\omega + 2)
\]

\[
P_{4L} = \omega^4 \mu^4 (\omega - 1)^2 (\omega + 2)(2\omega^2 + 3\omega - 3)
\]

\[
P_{2d} = \sigma_d^2
\]

\[
P_{3d} = 0
\]

\[
P_{4d} = 3 \sigma_d^4
\]

\[\text{Moments of } L \text{ are given by:}
\]

\[
\text{Bias (L) = 0}
\]

\[
E(L) = c_3 (P_d + P_L)
\]

\[
P_{2L} = c_3 (\sigma_d^2 + \sigma_L^2)
\]

\[
P_{3L} = c_3 \cdot P_{3L}
\]

\[
P_{4L} = c_3 (3 \sigma_d^4 + 6 \sigma_d^2 P_{2L} + P_{4L})
\]
G.3.2.3. Moments of $S_F$

$$S_F = \frac{R}{L}$$ \hspace{1cm} \ldots \hspace{1cm} (G.58)

Let $R = p_R (1 + \epsilon_R)$

$$L = p_L (1 + \epsilon_L)$$

\[ S_F = \left( \frac{p_R}{p_L} \right) (1 + \epsilon_R) (1 + \epsilon_L)^{-1} \] \hspace{1cm} \ldots \hspace{1cm} (G.59)

Defining $P_{RL} = \frac{p_R}{p_L}$

Bias ($S_F$) = $E(S_F) - P_{RL} = P_{RL} \cdot E(S_F)$ \hspace{1cm} \ldots \hspace{1cm} (G.60)

Expanding $S_F$ upto 4th powers of $\epsilon_R$ & $\epsilon_L$.

$$\delta S_F = (1 + \epsilon_R) (1 + \epsilon_L)^{-1} - 1$$

$$= (1 + \epsilon_R) \left(1 - \epsilon_L + \epsilon_L^2 - \epsilon_L^3 + \epsilon_L^4 - \ldots\right) - 1$$

$$= \epsilon_R - \epsilon_L - \epsilon_R \epsilon_L^2 + \epsilon_L^2 + \epsilon_R \epsilon_L^2 - \epsilon_L^3 - \epsilon_R^2 \epsilon_L^3 + \epsilon_L^4$$ \hspace{1cm} (G.61)

$$\delta^2 S_F = \epsilon_R + \epsilon_R^2 + 3 \epsilon_R \epsilon_L + 3 \epsilon_L^2 + 2 \epsilon_R^2 \epsilon_L - 2 \epsilon_R \epsilon_L^2 - 2 \epsilon_R^2 \epsilon_L^2 + 4 \epsilon_R \epsilon_L^2 \epsilon_L - 4 \epsilon_R^2 \epsilon_L^3 + 6 \epsilon_R \epsilon_L^3 - 6 \epsilon_R^2 \epsilon_L^4$$ \hspace{1cm} (G.62)

$$\delta^3 S_F = \epsilon_R^3 + 3 \epsilon_R^2 \epsilon_L + 9 \epsilon_R \epsilon_L^2 + 9 \epsilon_L^3 - 3 \epsilon_R^2 \epsilon_L^3 - 3 \epsilon_R \epsilon_L^4 + 3 \epsilon_R \epsilon_L^4$$ \hspace{1cm} (G.63)

$$\delta^4 S_F = \epsilon_R^4 + 4 \epsilon_R^3 \epsilon_L + 6 \epsilon_R^2 \epsilon_L^2 + 4 \epsilon_R \epsilon_L^3 + 4 \epsilon_R^3 \epsilon_L^3 + 4 \epsilon_R^2 \epsilon_L^4 + 4 \epsilon_R \epsilon_L^4 + 4 \epsilon_R^3 \epsilon_L^4 + 4 \epsilon_R^2 \epsilon_L^5$$ \hspace{1cm} (G.64)

Thus $E(\delta S_F) = \frac{\sigma^2}{P_{RL}^2} - \frac{P_{3L}}{P_{RL}^3} + \frac{P_{4L}}{P_{RL}^4}$ \hspace{1cm} (G.65)
Thus, the bias and moments of $S_F$ can be obtained using:

$$\text{Bias} (S_F) = \frac{\mu_{RL}}{\mu^2_R} \cdot E(\delta_{SF}^2),$$

$$E(S_F^\gamma) = \frac{\mu_{RL}^\gamma}{\mu^2_R} \cdot E(\delta_{SF}^\gamma), \quad \gamma = 2, 3, 4,$$

G.3.2.4. Moments of $S_L$:

$$S_L = R - L$$

It is easy to note that:

$$\delta_{S_L} = \delta_S - E(S_S) = R - E(R) - (L - E(L))$$

$$\text{Bias} (\delta_{S_L}) = \text{Bias} (R) - \text{Bias} (L)$$

$$E(\delta_{S_L}^2) = \mu_{3R} \cdot \mu_{3L},$$

$$E(\delta_{S_L}^3) = \mu_{4R} - \mu_{3L}^2,$$

$$E(\delta_{S_L}^4) = \mu_{4R}^2 - 6 \mu_{2R} \cdot \mu_{2L} \cdot \mu_{4L},$$

G.3.2.5. Coefficient of Skewness & Kurtosis:

The coefficient of skewness $\beta_1$, and kurtosis $\beta_2$ for a random variable $X$ can be obtained from:

$$\beta_1 = \frac{\mu_3X}{\sigma_X^3},$$

$$\beta_2 = \frac{\mu_4X}{\sigma_X^4} - 3.$$