Chapter 8

Multisection Technique for Multiobjective Optimization with Interval Objectives

8.1 Introduction

In some situations, the single objective optimization problems are not enough to handle the difficulties arisen from the confliction between various objectives and goals. Especially, in the different branches of engineering, management sciences and economics, most of the problems are having conflicting objectives with large extent and it is only natural to formulate the mathematical models and also the compromise solutions of those to be as good as possible to the decision maker. On the other hand, consideration of nonlinearity in the structure of model formulation is also inevitable. In this context, we need new mathematical formulations and solution methodologies – nonlinear uncertain multiobjective optimization. An extensive theoretical development on the multiobjective programming (MOP) problem with crisp coefficients can be seen in the books written by Collette and Siarry (2003), Miettenen (2004), Rangaiah (2009). In the last few decades, the MOP problems have been solved by Evolutionary computation methods (Deb, 2001; Abraham et al., 2005; Engelbrecht, 2005) – such as different types of Genetic Algorithms (GA), Particle Swarm Optimization (PSO) etc.
Normally, the uncertainty is controlled with the help of fuzzy or stochastic techniques. The possibility theory has also been applied to solve the conventional MOP problems. Inuiguchi and Sakawa (1996) have extended the Pareto optimal solutions of the conventional linear MOP problems to two different types of efficiencies – one is possibilistic and the other is necessary efficiencies. A modified possibility programming approach has been applied by Iskander (2004) for the MOP problems with fuzzy linear fractional objectives and stochastic fuzzy constraints. Ida (2005) applied the extreme ray generation method to find the set of extreme solutions of the linear MOP. Hladik (2008) also considered the linear MOP problems. The additive and multiplicative tolerance limits have been calculated for the objective function coefficients in case of Pareto optimality. The solutions of quadratic MOP problems with fuzzy coefficients and fuzzy decision variables has been investigated by Ammar (2009) using the $\alpha$-level sets. Inuiguchi and Kume (1994) considered the problems with interval valued objective functions. They transformed each interval objective function into one or several objective functions to get the compromise solutions. This is known as satisficing approach. The same goal achieved by Inuiguchi and Sakawa (1995) have followed a different manner called optimizing approach. It is already stated that by extending the conventional MOP problems to its interval form we can obtain the fruitful theoretical development in solving interval extended MOP. In this regard, the work of Wu (2009) can be mentioned in which an extensive theoretical progress has been given including the Karush-Kuhn-Tucker optimality conditions for interval valued MOP problems. Besides these, many experts have prescribed different solution techniques for interval valued MOP by extending the concept of traditional solution methodologies. An illustrated overview of the state-of-the-art about this topic has been given by Oliveira and Antunes (2007). Among these, most of the techniques are restricted only for linear MOP problems.

In this chapter, the interval multiobjective optimization problems (IMOP), where the uncertainties of the coefficients of objective function are represented by interval numbers, are considered. We know that the method of global criterion (GCM or sometimes known as compromise programming method) is relatively a simpler method among the non preference methods for solving MOP problems with crisp parameters. Here, we have extended the said method for interval parameters, i.e., the interval extension of global criterion method has been introduced to solve the proposed IMOP. Applying this extended method, the interval valued multiobjective problem is reduced to
several single objective interval valued optimization problems. Then we have applied the interval oriented optimization technique ICCGO based on the division criteria of prescribed/accepted search region proposed in Chapter 4 to solve those optimization problems with the help of finite interval arithmetic and interval order relations mentioned in Chapter 2. Considering the individual objective functions, the ideal interval objective vector is formed. We take this ideal vector as a reference point and then different distance metrics between the feasible objective vector and this reference point are constructed. At the final stage of the global criterion method, this distance metric is minimized and we obtain the required compromise solution. Here, the metrics $L_1$, $L_2$ and $L_\infty$ have been considered. Finally, the proposed technique has been applied on some problems to test the efficiency of the proposed technique.

The following notations are used throughout the chapter.

**Notations**

$f_i (x) = [f_{il}(x), f_{ir}(x)]$ \hspace{1cm} Interval valued objective function

$z_i^* = [z_{il}^*, z_{ir}^*]$ \hspace{1cm} $i$th component of interval valued ideal objective vector

$C_i = [C_{il}, C_{ir}]$ \hspace{1cm} Interval valued cost coefficients

$n$ \hspace{1cm} Number of decision variables

$m$ \hspace{1cm} Number of constraints

$g_k (x)$ \hspace{1cm} Constraint functions

$x_i$ \hspace{1cm} Decision variables

$S$ \hspace{1cm} Feasible region

**8.2 Interval Valued Multiobjective Optimization**

Over the last few years some remarkable developments have been done in the area of multiobjective optimization theory. In this context, a number of theoretical developments have been obtained to solve the linear as well as the nonlinear multiobjective optimization problems. However, the researchers have not considered the uncertainty or inexactness of the parameters in the formulation of the said problems. No such developments are seen for interval valued multiobjective optimization. In this chapter, we have discussed a solution methodology for solving multiobjective optimization problems with interval valued objective functions.

Here the following interval valued multiobjective optimization problem is considered,
Minimize \( f(x) = (f_1(x), f_2(x), \ldots, f_n(x)) \)
subject to
\[
\begin{align*}
g_i(x) & \leq 0, \quad i = 1, 2, \ldots, m \\
h_k(x) & = 0, \quad k = 1, 2, \ldots, q;
\end{align*}
\]
where \( f_i(x) = f_i(x_1, x_2, \ldots, x_n) = [f_{iL}(x), f_{iR}(x)], r = 1, 2, \ldots, \kappa \).

Before going to discuss about the solution procedures of the above optimization problem, first of all, we consider the definitions of Pareto optimality and ideal objective vector in interval form. Wu (2009) have already discussed the Karush-Kuhn-Tucker optimality conditions for interval valued multiobjective optimization problems. Some related theoretical results have been given by him using the Ishibuchi and Tanaka’s (1990) interval ranking definitions. However, we have modified those by applying Mahato and Bhunia’s (2006) interval ranking relations with respect to pessimistic decision maker’s point of view.

**Definition 8.2.1** A decision vector \( x^* \in S \) is Pareto optimal if there does not exist another decision vector \( x \in S \) such that \( f_k(x) \prec_{\min} f_k(x^*) \) for any \( k \), i.e., for Type-1 and Type-2 intervals
\[
f_{kL}(x) + f_{kR}(x) < f_{kL}(x^*) + f_{kR}(x^*)
\]
and for Type-3 intervals
\[
[(f_{kL}(x) + f_{kR}(x) \leq f_{kL}(x^*) + f_{kR}(x^*)) \text{ and } (f_{kR}(x) - f_{kL}(x) < f_{kR}(x^*) - f_{kL}(x^*))].
\]
In case of \([f_{kL}(x) + f_{kR}(x) \leq f_{kL}(x^*) + f_{kR}(x^*)) \text{ and } (f_{kR}(x) - f_{kL}(x) > f_{kR}(x^*) - f_{kL}(x^*))\], the optimistic decision will be considered, i.e., \( f_{kL}(x) \leq f_{kL}(x^*) \).

**Definition 8.2.2** Let \( X \) be a metric space. The (Open) ball of radius \( \delta > 0 \) centered at a point \( x^* \) in \( X \) is defined as
\[
B(x^*, \delta) = \{ x \in X : d(x, x^*) < \delta \},
\]
where \( d \) is the distance function or metric. If the less than \(<\) symbol is replaced by a less than or equal to \(\leq\) symbol, the above definition becomes the same of a Closed ball:
\[
B[x^*, \delta] = \{ x \in X : d(x, x^*) \leq \delta \}.
\]

**Definition 8.2.3** A decision vector \( x^* \in S \) is locally Pareto optimal if there exists \( \delta > 0 \) such that \( x^* \) is Pareto optimal in \( S \cap B(x^*, \delta) \).

**Definition 8.2.4** A decision vector \( x^* \in S \) is weakly Pareto optimal if there does not exist another decision vector \( x \in S \) such that \( f_k(x) \prec_{\min} f_k(x^*) \) for all \( k \).
Definition 8.2.5 An objective vector maximizing each of the objective functions is called an ideal or perfect objective vector.

Definition 8.2.6 An utopian objective vector $z^{**} \in \mathbb{R}^k$ is an infeasible objective vector whose components are formed by $z_i^{**} = z_i^* - \varepsilon_i$ for all $i = 1, 2, ..., k$, where $z_i^*$ is the component of the ideal objective vector and $\varepsilon_i > 0$ is a relatively small but computationally significant scalar.

Definition 8.2.7 Let $X = \mathbb{R}^n$ and suppose that $\xi = \{\xi_1, \xi_2, ..., \xi_n\}$ and $\eta = \{\eta_1, \eta_2, ..., \eta_n\}$ be any two points in $\mathbb{R}^n$. Define the mapping $d_p: X \times X \rightarrow \mathbb{R}^n$ and $d_\infty: X \times X \rightarrow \mathbb{R}^n$ as follows:

$$d_p(\xi, \eta) = \left\{ \sum_{i=1}^{n} |\xi_i - \eta_i|^p \right\}^{\frac{1}{p}} \quad \text{and} \quad d_\infty(\xi, \eta) = \max_{1 \leq i \leq n} |\xi_i - \eta_i|.$$ 

Then $d_p$ and $d_\infty$ are two metrics on the same set $X = \mathbb{R}^n$.

Definition 8.2.8 Let $X = l_p$, $1 \leq p < \infty$, be the set of all sequences $\xi = \{\xi_i\}$ of real scalars such that $\sum_{i=1}^{n} |\xi_i|^p < \infty$. Define the mapping $d_p: X \times X \rightarrow \mathbb{R}^n$ by

$$d(\xi, \eta) = \left\{ \sum_{i=1}^{n} |\xi_i - \eta_i|^p \right\}^{\frac{1}{p}}$$

where $\xi = \{\xi_i\}$ and $\eta = \{\eta_i\}$ are in $l_p$.

Theorem: $l_p$ is a metric space.

Proof: This theorem can easily be proved.

8.3 Solution by Interval Extension of GCM

It is well known that the method of Global Criterion is relatively a simpler non preference method to solve MOP problems with non interval coefficients (Collette and Siarry (2003), Miettenen (2004), Rangaiah (2009)). Where the decision maker does not have any extra preference to the objectives or where we need the solution giving same weight to the objectives then it is the best solution technique. Here, the said method is extended for MOP problems with interval objectives. The related interval extensions of the useful definitions and the distance metrics have been given in the previous section.

To illustrate the method, let us first take the interval valued multiobjective optimization problem (8.1). At first we need a reference point (in interval form) in the
feasible objective region. Here, the optimal values $Z_r^*$ of the individual objective functions $Z_r = f_r(x)$ for $r = 1, 2, \ldots, \kappa$ considering the same constraints are found. This is also known as ideal objective vector. It will be the required reference point. Now the basic of the said method is to minimize the distance between this reference point and the feasible objective region. To serve the purpose several distance metrics in interval form has been defined. Among these the $L_p$-metrics have been taken for different values of $p$ in the range $1 \leq p < \infty$. The $L_p$-problem to be solved is

$$\begin{align*}
\text{Min} & \left( \sum_{r=1}^{\kappa} |f_r(x) - Z_r|^{1/p} \right)^p \\
\text{subject to} & \quad \text{the same constraints as given in the original problem.}
\end{align*}$$

with respect to the same constraints as given in the original problem.

If $p \to \infty$, we get the $L_\infty$-metric, also can be known as Tchebycheff metric in interval form is given by

$$\begin{align*}
\text{Min} & \quad \max_{r=1,2,\ldots,\kappa} \left( |f_r(x) - Z_r| \right) \\
\text{subject to} & \quad \text{the same constraints.}
\end{align*}$$

The solution can be obtained by applying $L_p$-metric with any $p$ in the range $1 \leq p < \infty$ or $p \to \infty$. By the interval extension of the global criterion method an interval valued multiobjective optimization problem can be reduced to several interval valued single objective optimization problems. Then those single objective optimization problems can be solved by interval oriented multisection method ICCGO discussed in Chapter 4.

**Table 8.1: Ideal objective vectors of the IMOP problems**

<table>
<thead>
<tr>
<th>Problems</th>
<th>$m$</th>
<th>Ideal objective vector or Reference point ($z^*$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>10</td>
<td>$z^* = (z_1^<em>, z_2^</em>) = ([27.5, 40.0], [21.0, 33.0])$</td>
</tr>
<tr>
<td>$F_2$</td>
<td>10</td>
<td>$z^* = (z_1^<em>, z_2^</em>) = ([21.0, 33.0], [36.0, 51.0])$</td>
</tr>
<tr>
<td>$F_3$</td>
<td>10</td>
<td>$z^* = (z_1^<em>, z_2^</em>, z_3^*) = ([48.0, 64.0], [32.0, 48.0], [48.0, 64.0])$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>10</td>
<td>$z^* = (z_1^<em>, z_2^</em>) = ([0.902938, 0.929417], [52.532585, 84.508941])$</td>
</tr>
<tr>
<td>$F_5$</td>
<td>10</td>
<td>$z^* = (z_1^<em>, z_2^</em>) = ([237.3776, 361.398101], [5.0, 5.0])$</td>
</tr>
<tr>
<td>$F_6$</td>
<td>10</td>
<td>$z^* = (z_1^<em>, z_2^</em>) = ([2315.03685, 2869.78065], [494.692717, 1528.181004])$</td>
</tr>
<tr>
<td>$F_7$</td>
<td>10</td>
<td>$z^* = (z_1^<em>, z_2^</em>, z_3^*) = ([2.355561, 8.81432], [0.0, 0.0], [0.0, 0.0])$</td>
</tr>
</tbody>
</table>

In short, the overall procedure can be presented at a glance. The first step of the proposed interval oriented global criterion method is to form the single objective interval optimization problems considering the individual objective function separated from the given problem. Those are solved by the aforesaid interval oriented multisection
algorithm ICCGO and obtained solutions form the ideal objective vector or the reference point. Now, the distance metric is constructed by applying any of the $L_p$-metrics stated earlier. At the final step, the distance metric is minimized with the same constrained satisfaction. This is again an interval valued single objective optimization problem and is solved by applying the previous algorithm. Here we have used $p = 1, 2$ or $p \to \infty$ separately.

8.4 Numerical Experiments and Comparative Discussions

In order to evaluate the capability of our proposed technique for finding the compromise solutions of the interval oriented MOP problems, we have performed some numerical experiments on a set of newly constructed seven test problems. These test problems with different complexity level are given in Appendix VI. Each problem has been reformulated as single objective interval valued optimization problems with the help of proposed interval oriented GCM. The single objective optimization problems have been solved by the ICCGO taking suitable value of $m$ (here $m = 10$) and the error tolerance $\varepsilon = 10^{-6}$. The list of ideal objective vectors for the test problems have been given in Table 8.1 and the summary of the final compromised solutions with the optimizer points (considering the distance metrics $L_1$, $L_2$ and $L_\infty$) have been shown in Table 8.2. From the Table 8.2, we can get a comparative study of the solutions obtained for different distance metrics. The ICCGO algorithm has been coded in C programming language and implemented on a PC with INTEL® CORE ™ 2 Duo CPU @ 2.00 GHz and 1 GB RAM in LINUX Operating System.

8.5 Concluding Remarks

Nonlinearity and imposing uncertainty in different optimization problems are two indispensable facts to be handled very efficiently while solving real world problems influenced by inexact data or insufficient information. In this chapter, the uncertain MOP problems are viewed in terms of interval oriented optimization problems. The main advantage of using interval oriented technique is to solve the uncertain problems by deterministic process. To reduce the multiobjective to several single objective optimization problems the interval extension of global criterion method has been used. The corresponding theoretical developments are also extended for interval numbers. Again, the interval valued single objective optimization problems are tackled by interval
oriented multisection method. This technique neither requires any type of derivative information nor it involves any stochastic or heuristic/meta heuristic methods. Basically, it depends on the multisection division criteria of the search space and interval order relations with respect to decision maker's point of view. Further, the proposed method possesses the merits of fast convergence as the feasible search space reduced exponentially in each iteration and tends very quickly to the solution point of the problem.
Table 8.2: Summary of the solutions of the IMOP problems

<table>
<thead>
<tr>
<th>Problems</th>
<th>m</th>
<th>( L_1 )-metric</th>
<th>( L_\infty )-metric</th>
<th>( L_\infty )-metric</th>
</tr>
</thead>
<tbody>
<tr>
<td>F_1</td>
<td>10</td>
<td>[x^* = (5.0, 6.75)]</td>
<td>[z_{1\text{min}} = (20.125, 31.875)]</td>
<td>[x^* = (1.30411, 8.990802)]</td>
</tr>
<tr>
<td>F_2</td>
<td>10</td>
<td>[x^* = (6.0, 6.0)]</td>
<td>[z_{1\text{min}} = (1.30411, 8.990802)]</td>
<td>[x^* = (6.0, 6.0)]</td>
</tr>
<tr>
<td>F_3</td>
<td>10</td>
<td>[x^* = (7.0, 0.0, 0.0, 9.0, 0.0, 0.0, 0.0)]</td>
<td>[z_{1\text{min}} = (30.0, 46.0)]</td>
<td>[x^* = (0.0, 0.0, 0.0, 16.0, 0.0, 0.0, 0.0)]</td>
</tr>
<tr>
<td>F_4</td>
<td>10</td>
<td>[x^* = (1.44804, 1.4, 1.4, 2.2, 4.60044)]</td>
<td>[z_{1\text{min}} = (0.684806, 0.745162)]</td>
<td>[x^* = (2.6, 2.24, 3.0, 3.00123, 2.6)]</td>
</tr>
<tr>
<td>F_5</td>
<td>10</td>
<td>[x^* = (5.0, 1.0, 1.0, 1.0, 1.0, 1.0)]</td>
<td>[z_{1\text{min}} = (195.182, 314.9725)]</td>
<td>[x^* = (5.0, 1.0, 4.2, 0.6, 5.0, 1.0)]</td>
</tr>
<tr>
<td>F_6</td>
<td>10</td>
<td>[x^* = (3.50717, 0.7, 6.7, 4.3, 8.5, 0)]</td>
<td>[z_{1\text{min}} = (2528.49555, 3113.047)]</td>
<td>[x^* = (3.50717, 7.7, 3.7, 4.2, 9.5)]</td>
</tr>
<tr>
<td>F_7</td>
<td>10</td>
<td>[x^* = (0.1998, 0.1, 0.7)]</td>
<td>[z_{1\text{min}} = (2.354305, 8.812488)]</td>
<td>[x^* = (0.1998, 0.5, 0.5)]</td>
</tr>
</tbody>
</table>