Chapter 4

Stability of conducting viscous film flowing down an inclined plane with linear temperature variation in the presence of a uniform normal electric field

4.1 Introduction

In chapter 2 the researcher has discussed a thin film flow along a non-uniformly heated plane for wide span of Marangoni number. But the review of the literature in chapter 1 confirms that, there is no such study addressing the effect of thermocapillary force in the presence of a uniform normal electric field. In this chapter an attempt is made to consider the combined effect of uniform normal electric field that at infinity on the flow of conducting viscous film on an inclined heated plane with linear temperature variation.
4.2 Formulation of the problem

Consider a layer of a conducting thin liquid film flows down an inclined heated plane of inclination $\theta$ with the horizon under the action of gravity and an uniform electric field that at infinity and perpendicular to the unperturbed interface. The co-ordinate system is chosen such that $x$-axis along the flow and $z$-axis normal to the inclined plane. We assume that the electrical permittivities are constant but takes different value in different medium. Due to constant permittivity, the fluid is not coupled to the electric field in the bulk [1]. However, electrical parameters suffer discontinuities at the interfacial region only, so interface experiences the effect of electric field.

The governing equations consist of the continuity equation, Navier-Stokes equation for the flow of the liquid layer, energy equation for the temperature field and Laplace equation for the electric field. The governing equations in dimensional form can be written as:

\begin{align*}
\nabla \cdot \mathbf{v} &= 0 \quad (4.2.1) \\
\rho (\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}) &= -\nabla p + \rho \nu \nabla^2 \mathbf{v} + \rho \mathbf{g} \quad (4.2.2) \\
\partial_t T + (\mathbf{v} \cdot \nabla)T &= \kappa \nabla^2 T, \quad (4.2.3) \\
\nabla^2 \Phi &= 0 \quad (4.2.4)
\end{align*}

where $\mathbf{v} = (u, 0, v)$ is the velocity vector, $\mathbf{g} = (g \sin \theta, 0, -g \cos \theta)$ is the acceleration due to gravity vector and $p, \rho, \nu, T$ denote the pressure, density, kinematic viscosity and absolute temperature respectively and $\nabla = (\partial/\partial x, 0, \partial/\partial z)$. Also $\kappa = k_T/(\rho C_p)$ denote the thermal diffusivity, $k_T$ thermal conductivity, $C_p$ the specific heat at constant pressure of the fluid and $\Phi$ denotes the electric potential.

The pertinent boundary conditions on the inclined plane ($z = 0$) and at the free
surface \((z = h(x,t))\) are:

no-slip condition at the plane: \(\mathbf{v} = 0\) at \(z = 0\),

\[ (4.2.5) \]

law of temperature variation of the plane: \(T = T_g + \mathcal{A} r\) at \(z = 0\).

\[ (4.2.6) \]

kinematic boundary condition: \(\partial_t h + (\mathbf{v} \cdot \nabla)(h - z) = 0\) at \(z = h(x,t)\),

\[ (4.2.7) \]

condition that the liquid is grounded perfect conductor: \(\Phi = 0\) at \(z = h(x,t)\),

\[ (4.2.8) \]

continuity of the shear stress: \([\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{t}] = \nabla \sigma(T) \cdot \mathbf{t}\) at \(z = h(x,t)\),

\[ (4.2.9) \]

continuity of the normal stress: \([\mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n}] - [p] = -\sigma(T) \nabla \cdot \mathbf{n}\) at \(z = h(x,t)\),

\[ (4.2.10) \]

Newton’s law of cooling: \(k_T \nabla T \cdot \mathbf{n} + k_g(T - T_g) = 0\) at \(z = h(x,t)\),

\[ (4.2.11) \]

where \(T_g\) denotes the temperature in the gas phase, \(\mathcal{A} = (T_H - T_C)/l_0\), where \(T_H\) and \(T_C\) denote the temperatures at hotter part and the colder part respectively along the inclined plane and \(l_0\) the characteristic longitudinal length scale whose order may be considered same as the wave length \(\lambda\). In this study we have taken the temperature \(T\) is increasing in the stream-wise direction and hence \(\mathcal{A}\) is positive. Also \(\sigma(T)\) is the surface tension of the liquid, \(k_g\) is the heat transfer coefficient between the liquid and air and \([\ast]\) denotes a jump in the quantity as the interface is crossed from the liquid to vacuum region. \(\mathbf{n}\) and \(\mathbf{t}\) are the normal and tangent vectors pointing outward to
the interface respectively and the stress tensor \( \tau \) is given by

\[
\tau = \tau^f + \tau^e
\]

where the viscous stress tensor

\[
\tau^f_{ij} = \rho \nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

and the electrical (Maxwell) stress tensor

\[
\tau^e_{ij} = \varepsilon_m \left( E_i E_j - \frac{1}{2} E_k E_k \delta_{ij} \right),
\]

where \( \varepsilon_m \) is the permittivity of the concerned medium.

Further we have uniform normal electric field far from the surface, which gives

\[
\Phi_z \to E_0 \quad \text{as} \quad z \to \infty, \quad (4.2.12)
\]

where \( E_0 \) is the basic uniform normal applied electric field and at the free surface \((z = h(x, t))\) as another boundary condition.

The above equations are quite general regarding various coefficients \((k_T, \kappa, \mu, \sigma\) etc). It is well known that temperature variation in the fluid can cause dramatic changes in the above coefficients, but approximations can be made depending on the type of the problem being examined. In the foregoing analysis we have assumed the variation of surface tension as

\[
\sigma(T) = \sigma_0 - \gamma(T - T_g) \quad (4.2.13)
\]

where \( \sigma_0 \) is the surface tension at \( T_g \), the temperature in the gas phase, which is taken as the reference temperature and \( \gamma = -\partial \sigma/\partial T|_{T=T_g} \) is a positive constant for most common fluids. The assumption of linear variation of surface tension with
temperature is very much compatible with the experimental data. Apart from water [2] there are many liquids [3] which follow the linear variation of surface tension with the temperature scales. For example, the molten tin (Sn) in the range of 520 - 1670K and molten Zirconium (Zr) in the range 2000 - 2250K follow the law

\[ \sigma(T) = 561.6 - 0.103(T - 505) \text{ Nm}^{-1} \quad \text{and} \quad \sigma(T) = 1543 - 0.66(T - 2128) \text{ Nm}^{-1} \]

respectively. It is to be noted here that both the liquids are highly conducting.

To express the governing equations and boundary conditions in non-dimensional form, we shall assume two length scales \( l_0 \) and \( h_0 \) as the characteristic measure for the length in longitudinal and transverse direction respectively; \( l_0 \) may be assumed as one wave length and \( h_0 \) is the mean depth of the film, which gives \( l_0 \gg h_0 \). Further to measure the transverse length in the vacuum, which extend to infinity, \( h_0 \) is not a proper scale, so we shall consider \( l_0 \) as the measure of the transverse length in the vacuum. The Nusselt velocity \( u_0 = gh_0^2 \sin \theta / 3\nu \) will be assumed as the characteristic velocity along the longitudinal direction. We define the dimensionless quantities as

\[
x = l_0 x^*, \quad h = h_0 h^*, \quad z = h_0 z^* \text{(in liquid)}, \quad z = (l_0 / h_0) \zeta \text{(in vacuum)},
\]

\[
t = (l_0 / u_0) t^*, \quad u = u_0 u^*, \quad v = (h_0 / l_0) u_0 v^*, \quad p = \rho_0^2 p^*,
\]

\[
T = T_g + T^*(T_H - T_C), \quad \Phi = E_0 h_0 \Phi^*. \quad (4.2.14)
\]

Using the above dimensionless quantities in the governing equations (4.2.1)-(4.2.12) and in the boundary conditions (4.2.5)-(4.2.12) after using the relation (4.2.13) reduces to the form, after dropping the asterisk as:

I. Equations in the liquid \((0 < z < h)\)

\[
ux + vz = 0, \quad (4.2.15)
\]
\[ u_t + uu_x + vu_z = -p_x + \frac{\sin \theta}{\varepsilon \text{Fr}} + \frac{1}{\varepsilon \text{Re}} (\varepsilon^2 u_{xx} + u_{zz}), \quad (4.2.16) \]

\[ \varepsilon^2 (u_t + vu_x + vu_z) = -p_z - \frac{\cos \theta}{\varepsilon \text{Fr}} + \frac{\varepsilon}{\varepsilon \text{Re}} (\varepsilon^2 v_{xx} + v_{zz}). \quad (4.2.17) \]

\[ \varepsilon \text{Re} \text{Pr} (T_t + uT_x + uT_z) = \varepsilon^2 T_{xx} + T_{zz} \quad (4.2.18) \]

II. Equation in the vacuum (\( \varepsilon h < \zeta < \infty \))

\[ \phi_{xx} + \phi_{\zeta\zeta} = 0. \quad (4.2.19) \]

III. Boundary conditions at the wall (\( z = 0 \))

\[ u = 0, \quad v = 0, \quad T = x. \quad (4.2.20) \]

IV. Boundary conditions at the free surface (\( z = h, \zeta = \varepsilon h \))

\[ v = h_t + uh_x, \quad (4.2.21) \]

\[ \phi = 1 - h, \quad (4.2.22) \]

\[ \left[ (1 - \varepsilon^2 h_x^2)(u_z + \varepsilon^2 v_z) + 4\varepsilon^2 h_x v_z \right] (1 + \varepsilon^2 h_x^2)^{-1/2} = -\text{Mn}(T_x + h_x T_z), \quad (4.2.23) \]

\[ p_a - p + \frac{2\varepsilon}{\text{Re}} \left( v_z \frac{1 - \varepsilon^2 h_x^2}{1 + \varepsilon^2 h_x^2} - h_x \frac{u_z + \varepsilon^2 v_z}{1 + \varepsilon^2 h_x^2} \right) = \varepsilon^2 \text{We}(1 - \text{Ca}T) \frac{h_{xx}}{(1 + \varepsilon^2 h_x^2)^{3/2}} \]

\[ + \frac{1}{2} \text{Ew}(1 + \varepsilon^2 h_x^2)(1 + \varepsilon \phi_{\zeta})^2, \quad (4.2.24) \]

\[ (T_z - \varepsilon^2 h_x T_x) (1 + \varepsilon^2 h_x^2)^{-1/2} + \text{Bi}T = 0. \quad (4.2.25) \]

V. Boundary condition at infinity (\( \zeta \to \infty \))

\[ \phi_{\zeta} \to 0, \quad (4.2.26) \]

where \( \phi \) is the perturbed electric potential i.e \( \Phi = E_0(z - h_0) + \phi \), \( \text{Fr}(= u_0^2/gh_0) \) is the Froude number, \( \text{Re}(= u_0 h_0/\nu) \) is the Reynolds number, \( \text{Pr}(= \nu/\kappa) \) is the
Prandtl number, \( \text{Mn} (\equiv 3\gamma(T_H - T_C)/\rho g l_0 h_0 \sin \theta) \) is the Marangoni number \(^\star\), \( \text{We} (\equiv \sigma_0/\rho u_0^2 h_0) \) is the Weber number, \( \text{Ca} (\equiv \gamma(T_H - T_C)/\sigma_0) \) is the capillary number, \( \text{Ew} (\equiv \epsilon_0 E_0^2/\rho u_0^2) \) is the electric Weber number (\( \epsilon_0 = 8.854 \times 10^{-12} \) farads/meter is the permittivity of the vacuum), \( \text{Bi} (\equiv k_T h_0/k_T) \) is the Biot number and \( \varepsilon (\equiv h_0/l_0) \) is the aspect ratio for long wave expansion. In deriving the above sets of equations we have used that the Maxwell stress vanishes inside the liquid layer while the viscous stress are null in the air. Also the dynamic influence of the air above the liquid film is ignored.

### 4.2.1 Nonlinear evolution equation

The equation (4.2.19) is solved using (4.2.22) and (4.2.26), which gives

\[
\phi = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{z(h(x') - 1)}{z^2 + (x-x')^2} dx'.
\]  

Solving equation (4.2.17) for lowest order, by using (4.2.24) and (4.2.27), we have,

\[
\rho_0 = \rho_a + \frac{3\cot \theta}{\text{Re}} (h - z) + \frac{1}{2} \text{Ew} (-1 + 2\varepsilon (\mathcal{H}(h - 1))_x) - \varepsilon^2 \text{We} h_{xx},
\]

where \( \mathcal{H}(h) \) represents the Hilbert transform operator given by

\[
\mathcal{H}(h(x)) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{h(x')}{x' - x} dx',
\]

with \( \mathcal{P} \) the principal value of the integral. It can be shown by using the definition of the Nusselt velocity that

\[
\frac{\sin \theta}{\text{Fr}} = \frac{3}{\text{Re}}.
\]

\(^\star\text{Mn} = \varepsilon \text{Ma}/\text{RePr} \), \( \text{Ca} = \text{Cr} \text{Ma} \), where \( \text{Ma} = \gamma(T_H - T_C)h_0/\mu \kappa \) and \( \text{Cr} = \mu \kappa/\sigma_0 h_0 \) are respectively the Marangoni numbers and Crispation/capillary in usual definition.
We have assumed that the Reynolds number is $O(1)$ and Weber number is $O\left(1/\varepsilon^2\right)$, Marangoni number is of order $O(1)$, Capillary number is $O(\varepsilon)$, the Prandtl number is $O(1)$, Biot number is $O(\varepsilon^2)$ and the electric Weber number is $O(1/\varepsilon)$. We are now interested in yielding a non-linear evolution equation in terms of film thickness $h(x,t)$. Expanding the velocity components, pressure and the temperature in powers of $\varepsilon$ as,

$$u = u_0 + \varepsilon u_1 + ..., \quad v = v_0 + \varepsilon v_1 + ..., \quad p = p_0 + \varepsilon p_1 + ... \quad \text{and} \quad T = T_0 + \varepsilon T_1 + ...$$

and substituting the above into the governing equations (4.2.15), (4.2.16), (4.2.18) and the boundary conditions (4.2.20), (4.2.23), (4.2.25) and also using (4.2.28) we obtain a set of PDEs for a different order of $\varepsilon$. The velocity profile and its depth averaged velocities for the zeroth-order and the first-order problem is obtained as

at $O(1)$:

$$u_0 = 3 \left(hz - \frac{z^2}{2}\right) - Mn z$$

$$\overline{u}_0 = h^2 - \frac{1}{2}Mnh$$

(4.2.30) (4.2.31)

and at $O(\varepsilon)$:

$$u_1 = \text{Re}\left(\frac{3 \cot \theta}{\text{Re}} h_x - \varepsilon^2 \text{We} h_{xxx} + \varepsilon E_w (H(h-1))_{xx} \right) \left( \frac{z^2}{2} - hz \right)$$

$$+ \frac{1}{2} (3h - Mn) \left( \frac{z^4}{4} + 2h^3 z \right) h_x + \frac{1}{2} h_1 z^3 + \frac{1}{2} \text{Pr}Mn (5h - 2Mn) h^2 z h_x \right]$$

(4.2.32)

$$\overline{u}_1 = \text{Re}\left[ - \frac{\cot \theta}{\text{Re}} h_x + \frac{1}{3} \varepsilon^2 \text{We} h_{xxx} - \frac{1}{3} \varepsilon E_w (H(h-1))_{xx} \right] h^2$$

$$+ \frac{3}{5} (3h - Mn) h^4 h_x + \frac{1}{4} \text{Pr}Mn (5h - 2Mn) h^3 h_x \right].$$

(4.2.33)

Integrating the continuity equation (4.2.15) with respect to $z$ from 0 to $h$ by using
Leibnitz's rule and boundary conditions (4.2.20) and (4.2.21), we have

\[
\frac{\partial h}{\partial t} + \frac{\partial (\bar{u}_0 h)}{\partial x} + \varepsilon \frac{\partial (\bar{u}_1 h)}{\partial x} + O(\varepsilon^2) = 0
\]

(4.2.34)

Substituting \(\bar{u}_0\) and \(\bar{u}_1\) from (4.2.31) and (4.2.33) in (4.2.34), we get

\[
h_t + A(h)h_x + \varepsilon \left( B(h)h_x + \varepsilon C(h) (\mathcal{H}(h - 1))_{xx} + \varepsilon^2 D(h)h_{xxx} \right)_x = 0
\]

(4.2.35)

where the suffixes denote the differentiation with respect to the corresponding variables and

\[
A(h) = 3h^2 - Mn h
\]

\[
B(h) = -\cot \theta h^3 + \frac{2}{5} \text{Re} (3h - Mn) h^5 + \frac{1}{4} \text{RePrMn} (5h - 2Mn) h^4
\]

\[
C(h) = -\frac{1}{3} \text{ReEw} h^3
\]

\[
D(h) = \frac{1}{3} \text{ReWe} h^3
\]

4.3 Stability analysis

To study the instability, the film thickness may be written as

\[
h(x, t) = 1 + \eta(x, t)
\]

(4.3.1)

where \(\eta << 1\) are the dimensionless perturbation of the film thickness.

Setting the transformation

\[
t = \varepsilon \tilde{t} \quad \text{and} \quad x = \varepsilon \tilde{x}
\]

(4.3.2)

and using (4.3.1) and (4.3.2) in (4.2.35), and retaining the terms up to the third order
fluctuations after dropping the tilde sign can be written as

\[\eta_t + A\eta_x + B\eta_{xx} + C(\mathcal{H}(\eta))_{xxx} + D\eta_{xxxx} + A'\eta_x + B'(\eta\eta_x)_x + C' (\mathcal{H}''(\eta))_{xx} + D'' (\mathcal{H}''(\eta))_{xxx} + O(\eta^4) = 0. \quad (4.3.3)\]

where \(A, B, C, D\) and their corresponding derivatives are evaluated at \(h = 1\).

4.3.1 Linear stability analysis

In this section, we are interested to study the linear response for a sinusoidal perturbation of the film by assuming the perturbation of the form

\[\eta = \Gamma[\exp\{i(kx - \omega t)\}] + \text{c.c.} \quad (4.3.4)\]

where \(\Gamma\) is the amplitude of the disturbance and c.c. represents complex conjugate. Here the wave number \(k\) is real and \(\omega = \omega_r + i\omega_i\) is the complex frequency. Using (4.3.4) in linearized part of (4.3.3), we get the dispersion relation as

\[D(\omega, k) = -i\omega + iAk - Bk^2 + Ck^3 + Dk^4 = 0. \quad (4.3.5)\]

Equating the real and imaginary parts of (4.3.5), we get

\[\omega_r = Ak \quad \text{and} \quad \omega_i = Bk^2 - Ck^3 - Dk^4 \quad (4.3.6)\]

Therefore, the phase speed

\[c_r = \omega_r/k = 3 - Mn, \quad (4.3.7)\]

it is to be noted here that the phase speed is independent of \(k\), implying the wave is non-dispersive in nature. Also it is clear that linear phase speed is independent of
electric Weber number but it depends on Marangoni numbers. The minimum Re at which instability sets in may be denoted as the critical Reynolds number \( Re_c \) for the wave formation and obtained from (4.3.6) as

\[
Re_c = \frac{60We \cot \theta}{5E_w^2 + 3We[8(3 - Mn) + 5PrMn(5 - 2Mn)]}
\]  

(4.3.8)

As \( Mn \to 0, E_w \to 0, Re_c = (5/6) \cot \theta \), which is the critical Re for isothermal case as obtained by Benjamin [4] and Yih [5]. Also as \( E_w \to 0 \), the \( Re_c \) coincides as obtained by Mukhopadhyay and Mukhopadhyay [2]. Again as \( Mn \to 0 \), the \( Re_c \) is compatible with that of Mukhopadhyay and Dandapat [6] except the possible difference as that was calculated by momentum integral method.

In the neutral state \( \omega_i = 0 \) gives two relations

\[
k = 0, \quad k_c = \frac{ReE_w \pm \sqrt{Re^2E_w^2 + 36BD}}{6D}
\]

(4.3.9)

(4.3.10)

which correspond to two branches of the neutral curves and the flow instability takes place in between them. Further the neutral curves intersect at the bifurcation point \( Re = Re_c, k = 0 \).

### 4.3.2 Non Linear stability analysis

To study the growth of weakly nonlinear waves, we shall use the method of multiple scales and expand the surface elevation \( \eta \) as

\[
\eta(x, x_1, ... t, t_1, t_2, ...) = \varsigma \eta_1 + \varsigma^2 \eta_2 + \varsigma^3 \eta_3 + ..., \quad (4.3.11)
\]

where the scalings \( x, x_1, ... t, t_1, t_2, ... \) are related according to

\[
x_1 = \varsigma x, \quad t_1 = \varsigma t, \quad t_2 = \varsigma^2 t, ..., \quad (4.3.12)
\]
Using (4.3.11), (4.3.12) in (4.3.3) we get

\[(L_0 + \zeta L_1 + \zeta^2 L_2 + \ldots)(\zeta \eta_1 + \zeta^2 \eta_2 + \zeta^3 \eta_3 + \ldots) = -\zeta^2 N_2 - \zeta^3 N_3 - \ldots \quad (4.3.13)\]

where \(L_0, L_1, L_2\) etc. are the operators and \(N_2, N_3\) are the nonlinear terms of equation (4.3.13) that are given in the appendix.

In the lowest order of \(\zeta\), we have

\[L_0 \eta_1 = 0 \quad (4.3.14)\]

which has a solution of the form

\[\eta_1 = \Gamma(x_1, t_1, t_2)\exp i\Theta] + \text{c.c.} \quad (4.3.15)\]

where \(\Theta = kx - \omega rt.\) and c.c. denotes the complex conjugates. It is to be noted here that the above solution given in (4.3.15) is already obtained in connection with the linear stability analysis except \(\omega\) is replaced by \(\omega_r\), since in the vicinity of the neutral curve \(\omega_t = O(\varepsilon^2)\), so that the function \(\exp(\omega_t t)\) is slowly varying and may be absorbed in \(\Gamma(x_1, t_1, t_2)\).

In the second order, the perturbation system yields

\[L_0 \eta_2 = -L_1 \eta_1 - N_2. \quad (4.3.16)\]

Invoking (4.3.15) in (4.3.16), we have

\[L_0 \eta_2 = -i \left[ \frac{\partial D(\omega_r, k)}{\partial \omega_r} \frac{\partial \Gamma}{\partial t_1} - \frac{\partial D(\omega_r, k)}{\partial k} \frac{\partial \Gamma}{\partial x_1} \right] \exp i\Theta - \Omega r^2 e^{2i\Theta} + \text{c.c.} \quad (4.3.17)\]

where \(D(\omega_r, k)\) is given by (4.3.5), and

\[\Omega = iA'k - 2B'k^2 + 2C'k^2|k| + 2D'k^4.\]

The uniform valid solution for \(\eta_2\) is obtained from (4.3.17) as

\[\eta_2 = -\frac{\Omega r^2 e^{2i\Theta}}{D(2\omega_r, 2k)} + \text{c.c.} \quad (4.3.18)\]
Introducing the co-ordinate transformation $\xi = (x_1 - c_g t_1)$, where $c_g = -D_k/D_\omega$, is the group velocity, and using the solvability condition on the third order equation, we get

$$\frac{\partial \Gamma}{\partial t_2} + J_1 \frac{\partial^2 \Gamma}{\partial \xi^2} - \zeta^{-2} \omega_i \Gamma + (J_2 + i J_4) |\Gamma|^2 \Gamma = 0,$$

(4.3.19)

where,

$$J_1 = B + 3Ck - 6Dk^2,$$

$$J_2 = \frac{1}{2} \left( D''k^4 - 3C''k^3 - B''k^2 \right)$$

$$+ \left[ \frac{9D'k^4 + 7C'k^3 - 3B'k^2}{16Dk^4 + 8Ck^3 - 4Bk^2} \right]$$

and

$$J_4 = \frac{1}{2} A''k - A'k \left[ \frac{9D'k^4 + 7C'k^3 - 3B'k^2}{16Dk^4 + 8Ck^3 - 4Bk^2} \right]$$

For filtered waves there is no spatial modulation and the diffusion term vanishes, we get

$$\frac{\partial \Gamma}{\partial t_2} - \zeta^{-2} \omega_i \Gamma + (J_2 + i J_4) |\Gamma|^2 \Gamma = 0.$$

(4.3.20)

Solution of this equation may be written as

$$\Gamma = a e^{-i b(t_2) t_2},$$

(4.3.21)

which gives

$$\frac{\partial a}{\partial t_2} = \left[ \zeta^{-2} \omega_i - J_2 a^2 \right] a,$$

(4.3.22)

and

$$\frac{\partial (b(t_2) t_2)}{\partial t_2} = J_4 a^2$$

(4.3.23)

The equation (4.3.22) is nothing but the Landau equation. This equation is used to characterize the nonlinear behaviour of the travelling film flow. The second term on
the right hand side of the equation (4.3.22) is due to nonlinearity and may moderate or accelerate the exponential growth of the linear disturbance. For the existence of a supercritical stable zone in the linear unstable region ($\omega_i > 0$), the second Landau constant $J_2$ should be positive and the threshold amplitude will be

$$\zeta a = [\omega_i/J_2]^{1/2}. \quad (4.3.24)$$

On the other hand, in the linear stable zone ($\omega_i < 0$) if $J_2 < 0$ the flow will be subcritically unstable and $\zeta a$ is the threshold amplitude.

The non-linear wave speed in the supercritical stable zone is obtained by

$$Nc_r = c_r + c_i (J_4/J_2), \quad \text{where} \quad c_i = \omega_i/k. \quad (4.3.25)$$

### 4.4 Side-band stability analysis

In this section we examine whether a filtered finite-amplitude wave of film flow with phase change is stable with respect to side-band disturbance.

Spatially uniform solution of (4.3.20) in the form

$$\Gamma_\infty(t_2) = |\Gamma_\infty| \exp(-iQt_2)$$

gives

$$Q = \zeta^{-2}\omega_i J_4 / J_2, \quad \Gamma_\infty^2 = \zeta^{-2}\omega_i J_2 / J_2$$

This solution is perturbed by spatial side-band disturbances in the form

$$\Gamma = \Gamma_\infty(t_2) + [\delta\Gamma_+(t_2) \exp(iKX) + \delta\Gamma_-(t_2) \exp(-iKX)] \exp(-iQt_2),$$
with $K$ being the modulation wave number and it is substituted in (4.3.20). Neglecting the terms containing nonlinearity of $\delta \Gamma_+, \delta \Gamma_-$ one obtains

$$\frac{\partial}{\partial t_2} \begin{pmatrix} \delta \Gamma_+ \\ \delta \Gamma_- \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \delta \Gamma_+ \\ \delta \Gamma_- \end{pmatrix}$$

(4.4.1)

where

$$A_{11} = -2(J_2 + iJ_4)\Gamma_\infty^2 + iQ + \zeta^{-2}\omega_i + K^2J_1, \quad A_{12} = -(J_2 + iJ_4)\Gamma_\infty^2, \quad A_{21} = \overline{A}_{12}, \quad A_{22} = -2(J_2 - iJ_4)\Gamma_\infty^2 - iQ + \zeta^{-2}\omega_i + K^2J_1.$$

Considering only the linear stability of the supercritical wave, one write the solution of (4.4.1) in the form

$$\begin{pmatrix} \delta \Gamma_+ \\ \delta \Gamma_- \end{pmatrix} = \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \exp(\lambda t_2).$$

The eigen value of $\lambda$ in the above solution, after solving the eigen value problem (4.4.1) are

$$\lambda_1 = J_1K^2, \quad \lambda_2 = J_1K^2 - 2\zeta^{-2}\omega_i.$$

From (4.4.1) the trace of the matrix $= A_{11} + A_{22}$ is real and negative for $\omega_i > 0$, therefore at least one of the eigen value, in this case, is real and negative, which corresponds to the linearly sideband stable mode. The other eigen value can, however, have either positive or negative real part depending the values of the problem parameters. If both the eigen values are negative then the spatially uniform solution is stable with respect to the side-band disturbances for $\omega_i > 0$. 
4.5 Results and discussion

The object of this study is to quantify the thermocapillary effect on a viscous film flowing down an inclined plane with linear temperature variation in presence of normal electric field in the finite amplitude regime. Physical parameters chosen for this investigation are (1) Reynolds number ranging from 0 to 2, (2) Marangoni number ranging from 0 to 0.6, (3) Electric Weber number ranging from 0 to 50, (4) Prandtl number 7 and (5) Weber number 450. The ongoing analysis is completely theoretical one. Since, a complete set of data regarding the necessary physical properties for a particular fluid that can perfectly match with our problem concerned are not available, therefore it is very difficult to estimate the value of the parameters involved in this problem for a particular fluid. To estimate the electric Weber number $E_w$ we have considered a highly viscous liquid glycerine of mean film thickness $h_0 = 2.0 \times 10^{-3} m$, kinematic viscosity $\nu = 1.19 \times 10^{-3} m^2/s$, density $\rho = 1260 kg/m^3$, the inclination angle $\theta = \pi/3$ and a very high electric field $E_0 = 800 kV/m$ (although it is very strong but is still almost one-third of the dielectric breakdown of the field in air) which gives $E_w = 49.8$. For estimation of the other parameter, like $M_n$ we refer [2].

Fig. 1 shows the variation of $Re_c$ with $E_w$ for different $M_n$. It is clear from the figure that as $M_n$ increases $Re_c$ decreases showing the destabilizing effect of $M_n$ on the other hand $Re_c$ also decreases with $E_w$, confirming the destabilizing effect of $E_w$. Therefore we conclude that the influence of both the parameters $M_n$ and $E_w$ are qualitatively same but there is a quantitative difference between the parameters. It is clear from the figure that the effect of electric field is very feeble compare to the thermocapillary effect. The reason behind the fact is that the electric parameter does not appear in the first order correction of the evolution equation (4.2.35) whereas the
Marangoni number appears from the first order terms.

Numerically analyzing the nature of \( J_2 \) and \( \omega_i \), we have found that supercritical stable, subcritical unstable, unconditional and explosive zones are possible with the variation of the Marangoni number \( \text{Mn} \). As for example for a fixed \( k = 0.145 \) the variation of \( J_2 \) and \( \omega_i \) with the Marangoni number \( \text{Mn} \) are depicted in the fig. 2. It reveals from the figure that for Marangoni number \( \text{Mn} \leq 0.2472 \text{(approx)} \) subcritical unstable and for \( \text{Mn} \geq 0.3007 \text{(approx)} \) supercritical stable zones are possible and in the intermediate range of \( \text{Mn} \) unconditional stable zone exist. Similarly for a fixed wave number and for a proper chosen parameters other than \( E_w \) it is found that all the above stability zones can exist with the variation of \( E_w \) [6]. Thus both Marangoni number \( \text{Mn} \) and electric Weber number \( E_w \) control the stability criteria.

Variation of the threshold amplitude with the Marangoni number \( \text{Mn} \) in the supercritical stable and subcritical unstable zones for a fixed wave number and for different electric parameters \( E_w \) are shown in fig. 3 and fig. 4 respectively. Fig. 3 reveals that the threshold amplitude increases with both \( \text{Mn} \) and \( E_w \) in the supercritical stable region, while fig. 4 shows that in the subcritical unstable region it decreases with the increase of the parameters \( \text{Mn} \) and \( E_w \) confirming the destabilizing role of the parameters \( \text{Mn} \) and \( E_w \). It is observed from fig. 3 that in the supercritical stable region the threshold amplitude becomes larger rapidly for the initial change in \( E_w \) but as \( E_w \) increases the rate of increase diminishes.

Fig. 5 shows the variation of nonlinear wave speed with the Marangoni number \( \text{Mn} \) in the supercritical stable region for a fixed wave number and for different values of electric parameters \( E_w \). It is clear from the figure that nonlinear wave speed increases with both \( \text{Mn} \) and \( E_w \) as expected.
Finally, by computing the eigen value problem for the side-band instability we found that for considered parameter range both the eigen values $\lambda_1$ and $\lambda_2$ are negative, so the supercritical waves are stable with regard to side-band disturbance.
Bibliography


Appendix

\[ L_0 = \frac{\partial}{\partial t} + A \frac{\partial}{\partial x} + B \frac{\partial^2}{\partial x^2} + C \mathcal{H} \frac{\partial^3}{\partial x^3} + D \frac{\partial^4}{\partial x^4}, \]

\[ L_1 = \frac{\partial}{\partial t_1} + A \frac{\partial}{\partial x_1} + 2B \frac{\partial^2}{\partial x_1 \partial x} + 3C \mathcal{H}_0 \frac{\partial^3}{\partial x_1^2 \partial x} + 4D \frac{\partial^4}{\partial x_1^3 \partial x_1}, \]

\[ L_2 = \frac{\partial}{\partial t_2} + B \frac{\partial^2}{\partial x_1^2} + 3C \mathcal{H}_0 \frac{\partial^3}{\partial x_1^2 \partial x_1} + 6D \frac{\partial^4}{\partial x_1^3 \partial x_1}, \]

\[ N_2 = A' \eta \frac{\partial \eta}{\partial x} + B' \left[ \eta \frac{\partial^2 \eta}{\partial x^2} + \left( \frac{\partial \eta}{\partial x} \right)^2 \right] + C' \left[ \frac{\partial \eta}{\partial x} \mathcal{H} \left( \frac{\partial^2 \eta}{\partial x^2} \right) + \eta \mathcal{H} \left( \frac{\partial^3 \eta}{\partial x^3} \right) \right] + D' \left[ \eta \frac{\partial^4 \eta}{\partial x^4} + \frac{\partial \eta}{\partial x} \frac{\partial^3 \eta}{\partial x^3} \right], \]

\[ N_3 = A' \left[ \eta \left( \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial x_1} \right) + \eta \frac{\partial \eta}{\partial x_1} \right] + B' \left[ \eta \left( \frac{\partial^2 \eta}{\partial x^2} + 2 \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial x_1} + \eta \frac{\partial^2 \eta}{\partial x \partial x_1} \right) \right] + \eta \frac{\partial^3 \eta}{\partial x^4} \]

\[ + \frac{\partial \eta}{\partial x} \left( \frac{\partial^2 \eta}{\partial x} + \frac{\partial \eta}{\partial x_1} \right) + C' \left[ \frac{\partial \eta}{\partial x} \mathcal{H} \left( 2 \frac{\partial^2 \eta}{\partial x \partial x_1} + \frac{\partial^2 \eta}{\partial x^2} \right) + \frac{\partial \eta}{\partial x} \frac{\partial^2 \eta}{\partial x_1^2} \right] + \eta \mathcal{H} \left( \frac{\partial^3 \eta}{\partial x^3} \right) + D' \left[ \eta \left( \frac{\partial^4 \eta}{\partial x^4} + 4 \frac{\partial^4 \eta}{\partial x^3 \partial x_1} \right) \right] \]

\[ + \eta \frac{\partial^4 \eta}{\partial x_1^4} \left( \frac{\partial^2 \eta}{\partial x^3} + 3 \frac{\partial^2 \eta}{\partial x^2 \partial x_1} \right) + \eta \frac{\partial^3 \eta}{\partial x_1^3} \left( \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial x_1} \right) + \frac{1}{2} A'' \eta \frac{\partial^2 \eta}{\partial x^2} \]

\[ + B'' \left[ \frac{1}{2} A'' \eta \frac{\partial^2 \eta}{\partial x^2} + \eta \left( \frac{\partial \eta}{\partial x} \right)^2 \right] + C'' \eta \left[ \frac{2}{2} \frac{\partial \eta}{\partial x} \mathcal{H} \left( \frac{\partial^2 \eta}{\partial x^2} \right) + \eta \mathcal{H} \left( \frac{\partial^3 \eta}{\partial x^3} \right) \right] + D'' \left[ \frac{1}{2} \eta \frac{\partial^4 \eta}{\partial x^4} + \eta \frac{\partial \eta}{\partial x} \frac{\partial^3 \eta}{\partial x^3} \right]. \]
Figure 4.1: Variation of $Re_c$ for different $E_w$ and for $\theta = \pi/3$, $We = 450$ and $Pr = 7$. 
Figure 4.2: Variation of $J_2$ and $\omega_i$ for different $Mn$ and for fixed values of $k = 0.145$, $Re = 2$, $We = 450$, $\theta = \pi/3$, $Pr = 7$ and $E_w = 1$. 
Figure 4.3: Amplitude of disturbances in the supercritical region for different Mn and for $k = 0.132$, Re = 2, We = 450, $\theta = \pi/3$ and Pr = 7.
Figure 4.4: Amplitude of disturbances in the subcritical region for different Mn and for $k = 0.132$, Re = 2, We = 450, $\theta = \pi/3$ and Pr = 7.
Figure 4.5: Nonlinear wave speed for different Mn in the supercritical region for $k = 0.08$, $Re = 2$, $We = 450$, $\theta = \pi/3$, $Pr = 7$ and $E_w = 1$. 