Chapter 3

Stability of conducting liquid flowing down an inclined plane at moderate Reynolds number in the presence of constant electromagnetic field

3.1 Introduction

In the previous chapter the researcher has discussed different aspects of thin film instability mechanism flowing on an inclined plane, where the substrate is being heated non-uniformly. But it may also be interesting to see how the wave characteristics change for conducting thin liquid film flowing down an inclined plane in the presence of electromagnetic field. It is a well established fact that the effect of the Lorentz’s force \( j \times B \) (\( j \) and \( B \) are the current density and the magnetic field, respectively) causes a loss of energy in the MHD flow, but due to the presence of electromagnetic field the complex interaction between the flowing fluid and the electromagnetic field
through Lorentz force introduces significant additional complication into the theoretical analysis. Therefore it may essentially affect the critical conditions for the onset of instability in conducting film.

The flow of electrically conducting liquid film has several practical applications in nuclear energy equipments [1], different cooling systems [2] and in other technological applications, e.g. in laser cutting process, where the surface waves are undesirable at the molten interface and by applying a magnetic field to counteract the inertia force, the instability could be prevented to maintain the smooth flow. Moreover, it is more compatible in the context of practical situation as the electromagnetic field has no mechanical contact with the fluid flow.

As discussed in the review of literature in chapter 1, all the finite amplitude studies of thin film flow of conducting liquid in the presence of constant electromagnetic field were based on weakly nonlinear analysis of the system in the regime of small Reynolds number by using long-wave expansion method. In this chapter, an attempt is made to deduce the evolution equation using momentum integral method, which is appropriate in the regime of moderate Reynolds number.

### 3.2 Mathematical formulation of the problem

Consider a layer of an incompressible conducting liquid flowing down an inclined non-conducting plane of inclination $\theta$ with the horizon under the action of gravity in presence of electromagnetic field. The coordinate system is chosen such that $x$-axis lies along the flow and $z$-axis is normal to the inclined plane. The magnetic field acts parallel to the $z$-axis and the electric field acts normal to the $xz$-plane. Therefore, the unperturbed electric and magnetic fields are taken as $\mathbf{E}_0 = (0, E_0, 0)$ and $\mathbf{B}_0 =$...
(0, 0, \mathbf{B}_0), \text{ respectively.}

The equations of motion, are the conservation of mass and Navier-Stokes equation with the electromagnetic body force \( \mathbf{j} \times \mathbf{B} \) i.e.

\[
\nabla \cdot \mathbf{v} = 0
\]

(3.2.1)

\[
\rho (\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) = - \nabla p + \rho \nu \nabla^2 \mathbf{v} + \mathbf{j} \times \mathbf{B} + \rho g
\]

(3.2.2)

where \( \mathbf{j} \) and \( \mathbf{B} \) are the current density and the magnetic field respectively. \( \mathbf{g} = (g \sin \theta, 0, -g \cos \theta) \) is the gravitational acceleration and \( \rho, \nu \) denote density and kinematic viscosity of the liquid, respectively. The current density is defined from Ohm’s law as

\[
\mathbf{j} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B})
\]

(3.2.3)

where \( \sigma \) is the electrical conductivity and \( \mathbf{E} \) the electric field. The fields \( \mathbf{E} \) and \( \mathbf{B} \) are defined by the Maxwell’s equation

\[
\nabla \cdot \mathbf{B} = 0
\]

(3.2.4)

\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{j}
\]

(3.2.5)

\[
\nabla \times \mathbf{E} = 0
\]

(3.2.6)

where \( \mu_0 \) is the permeability. It is to be noted here that we ignore the displacement currents in Maxwell’s equations as we are not concerned with the effects which are related in any way to the propagation of electromagnetic waves [3]. Dynamic influence of air flow above the film is ignored.

For the determination of the magnetic field we make an assumption that the magnetic Reynolds number to be small so that the induced magnetic field can be neglected compared to the applied magnetic field (induction-free approximation). As
a consequence we replace B by the steady magnetic field $B_0$ in all the MHD equations and by this approximation we may ignore the equation (3.2.5) in calculating magnetic field [4] and then $B = B_0$.

Now for electric field we have from (3.2.5) and (3.2.6)

$$E = E_0 - \nabla \phi$$  \hspace{1cm} (3.2.7)

$$\nabla^2 \phi = B_0 \cdot (\nabla \times v)$$  \hspace{1cm} (3.2.8)

where $\phi$ is the potential of the electric field. As we restrict the consideration in two dimensional case, so $v = (u, 0, v)$, $\partial / \partial y = 0$ and then $\phi = \text{constant}$, $E = E_0$

Therefore, under above approximations we get the solutions of the electromagnetic field equations (3.2.4)-(3.2.6) as:

$$B = B_0$$ \hspace{1cm} (3.2.9)

$$E = E_0$$ \hspace{1cm} (3.2.10)

The above approximations have been made by several authors, see for example Gordeev and Murzenko [5], Korsunsky [6], Dandapat and Mukhopadhyay [7] and the references therein.

Now to solve equations (3.2.1)-(3.2.2) we shall use the solutions (3.2.9) and (3.2.10) and it also requires some pertinent boundary conditions, which are, usual no-slip condition on the inclined plane, dynamic and kinematic conditions at the free surface:

$$v = 0 \quad \text{at} \quad z = 0,$$  \hspace{1cm} (3.2.11)

$$n \cdot \tau \cdot t = 0 \quad \text{at} \quad z = h(x, t),$$  \hspace{1cm} (3.2.12)

$$p_a + n \cdot \tau \cdot n = -\sigma_0 \nabla \cdot n \quad \text{at} \quad z = h(x, t),$$  \hspace{1cm} (3.2.13)

$$\partial_t h + v \cdot \nabla (h - z) = 0 \quad \text{at} \quad z = h(x, t).$$  \hspace{1cm} (3.2.14)
where \( \mathbf{n} \) and \( \mathbf{t} \) are unit vectors, normal (outward pointing) and tangential to the interface respectively and \( \tau = -p\mathbf{I} + 2\mu \mathbf{e} \) is the stress tensor with the rate of strain tensor \( \mathbf{e} = (\mathbf{v} + \mathbf{v}^T)/2; \) \( \sigma_0, \mu \) and \( p_a \) are the surface tension, dynamic viscosity of the fluid and the pressure of the ambient gas phase respectively. Before solving the problem we want to rewrite the problem precisely in dimensionless form. We define the dimensionless quantities as

\[
x = l_0 x^*, \quad (h, z) = h_0(h^*, z^*), \quad t = (l_0/u_0) t^* \\
u = u_0 u^*, \quad v = (h_0/l_0) u_0 v^*, \quad p = \rho u_0^2 p^*,
\]

where we assume \( l_0 \) as the characteristic longitudinal length scale whose order may be considered same as the wave length \( \lambda \), the mean film thickness \( h_0 \) as the length scale in transverse direction and the Nusselt velocity \( u_0 = gh_0^2 \sin \theta / 3\nu \) as the characteristic velocity.

Using the above dimensionless quantities, the governing equations and the boundary conditions reduce the following form under usual boundary layer approximation and after dropping the asterisk as

(I) Governing equations

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0, \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\sin \theta}{\epsilon \text{Fr}} + \frac{E_p M^2}{\epsilon \text{Re}} - \frac{M^2}{\epsilon \text{Re}} u + \frac{1}{\epsilon \text{Re}} \frac{\partial^2 u}{\partial z^2},
\]

\( 0 = -\frac{\partial p}{\partial z} - \frac{3 \cot \theta}{\text{Re}}. \)

(II) Boundary conditions at the wall (\( z = 0 \))

\( u = 0, \quad v = 0. \)
(III) Boundary conditions at the free surface \((z = h(x, t))\)

\[
\frac{\partial u}{\partial z} = 0, \tag{3.2.20}
\]

\[
p = p_a - \varepsilon^2 \text{We} \frac{\partial^2 h}{\partial x^2}, \tag{3.2.21}
\]

and

\[
v = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x}, \tag{3.2.22}
\]

where \(\text{Re}(\equiv u_0 h_0/\nu)\) is the Reynolds number, \(\text{We}(\equiv \sigma_0/(\rho \mu_0^2 h_0))\) is the Weber number, \(M(\equiv B_0 h_0 \sqrt{\sigma/\langle \rho \nu \rangle})\) is the Hartmann number, \(E_p(\equiv E_0/(B_0 u_0))\) is the electric parameter and \(\varepsilon(\equiv h_0/l_0 << 1)\) is the aspect ratio for long wave approximation. Let us point it out here that our interest in this chapter is to study the hydrodynamic stability of the conducting thin film flow in the presence of electromagnetic field within the range of large \(\text{Re}(\sim O(1/\varepsilon))\), small \(M(\sim O(1))\) and \(E_p(\sim O(1))\) with very large \(\text{We}(\sim O(1/\varepsilon^2))\).

**Steady basic flow**

Classical solution for the steady uniform flow satisfying the system obtained as

\[
U_s = \frac{\varpi}{M^2} \left[ 1 - \frac{\cosh\{M(1 - z)\}}{\cosh M} \right], \quad \text{where} \quad \varpi = 3 + E_p M^2 \tag{3.2.23}
\]

and the corresponding depth averaged velocity becomes

\[
U_{av} = \int_0^1 U_s(z) \, dz = \frac{\varpi}{M^2} \left[ 1 - \frac{\tanh M}{M} \right].
\]

which is in complete correspondence with Korsunsky [6] (considering the angle of inclination of the magnetic field tending to \(\pi/2\) in that analysis). It is clear from figure 3.1 that the average velocity varies with \(E_p\) and \(M\). It can be seen that \(U_{av}\)
either decreases or increases with the small or large values of $E_p$ respectively for a particular value of $M$. This is due to the fact that the Lorentz force which is instrumental in driving the $U_{av}$ becomes the union of two mechanisms. When the electric field $E$ is small, the Lorentz force is produced mainly by the combination of velocity and magnetic fields. In this case Lorentz force opposes the basic flow field, on the other hand for strong $E$, Lorentz force acts favourably to the basic flow field. However, for negative $E_p$, which can be obtained by changing the orientation of the electric field, the value of $\varpi$ will be either zero, positive or negative for a particular value of $E_p$. This implies that, for a suitable negative $E_p$, the basic flow may cease and even, can turn back when $\varpi \leq 0$. In other words, negative $E_p$ hinders the liquid flow. Let us point it out here that the researcher will consider $E_p$ to be positive in this study unless stated otherwise.

**Transient and non-uniform flow**

Integrating (3.2.18) by using the normal stress condition (3.2.21) we get pressure $p$ as

$$p = p_a + \varepsilon^2 \text{We} \frac{\partial^2 h}{\partial x^2} + \frac{3 \cot \theta}{\text{Re}} (h - z).$$

(3.2.24)

Integrating (3.2.16) and (3.2.17) with respect to $z$ from 0 to $h$ by Leibnitz rule and using the boundary conditions (3.2.19)-(3.2.22), we have

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \int_0^h u dz = 0,$$

(3.2.25)

$$\frac{\partial}{\partial t} \int_0^h u dz + \frac{\partial}{\partial x} \int_0^h u^2 dz = \varepsilon^2 \text{We} h \frac{\partial^3}{\partial x^3} - \frac{3 \cot \theta}{\text{Re}} h \frac{\partial h}{\partial x} + \frac{\varpi}{\varepsilon \text{Re}} h - \frac{M^2}{\varepsilon^2 \text{Re}} 1 \frac{\partial u}{\partial z} \bigg|_{z=0}.$$  

(3.2.26)
To deduce the equation (3.2.26) we have eliminated the pressure term from (3.2.17) by using the equation (3.2.24) and then integrated as stated above. A specific velocity profile for transient and non-uniform flow must now be imposed to deduce required information from equations (3.2.25) and (3.2.26). Corresponding to the basic uniform flow a velocity profile, which is valid for transient and non-uniform flow, is assumed as

\[ u = \frac{1}{\delta_1} \left( \frac{\tilde{q}}{h} \right) f(Z), \quad f(Z) = \frac{\varpi}{M^2} \left[ 1 - \frac{\cosh M(1 - Z)}{\cosh M} \right], \quad Z = \frac{z}{h}, \quad (3.2.27) \]

where

\[ \tilde{q} = \int_0^h uzdz, \quad \delta_1 = \int_0^1 f(Z)dZ = \frac{\varpi(1 - \alpha)}{M^2}, \quad \alpha = \frac{\tanh M}{M}. \]

Further we have,

\[ \int_0^h u^2dz = \delta \left( \frac{\tilde{q}^2}{h} \right), \quad \delta = \frac{\delta_2}{\delta_1}, \quad \delta_2 = \int_0^1 f^2(Z)dZ, \]

which gives

\[ \delta = \frac{[3(1 - \alpha) - M^2\alpha^2]/[2(1 - \alpha)^2]} \]

and

\[ \frac{\partial u}{\partial z} \bigg|_{z=0} = \frac{M^2\alpha}{1 - \alpha} \left( \frac{\tilde{q}}{h^2} \right). \]

Let us note that the shape factor \( \delta \) characterizing the velocity profile is independent of the electric parameter \( E_p \) and \( \delta \) decreases as Hartmann number \( M \) increases. For \( M \to 0, \delta = 6/5 \) and when \( M \to \infty, \delta \to 1 \). Thus the range of \( \delta \) is \( 1 \leq \delta \leq 6/5 \).

Using above in equations (3.2.25)-(3.2.26), we have

\[ h_t + \tilde{q}_z = 0, \quad (3.2.28) \]

\[ \tilde{q}_t + \left( \delta \frac{\tilde{q}^2}{h} + \gamma \frac{h^2}{2} \right) = \text{We} \varepsilon h h_{xxx} - \frac{M^2}{\varepsilon \text{Re}} \tilde{q} + \frac{\varpi}{\varepsilon \text{Re}} \left[ h - \frac{\alpha}{\delta_1} \left( \frac{\tilde{q}}{h^2} \right) \right], \quad (3.2.29) \]
where
\[ \gamma = \frac{3 \cot \theta}{\text{Re}}. \quad (3.2.30) \]

Here the subscripts denote the derivative of the respective variables with respect to the indicated variables. The system \((3.2.28)-(3.2.29)\) has a trivial solution
\[ \tilde{q} = q_0 = \delta_1 = \frac{\omega (1 - \alpha)}{M^2}, \quad h = 1 \quad (3.2.31) \]

which is nothing but the Nusselt flat-film solution in the presence of electromagnetic field which was also obtained by Korsunsky [6]. Merit of the above integral boundary layer method is that the effects of inertia are fully incorporated in contrast to the traditional long-wave expansion method [6] which captures the inertia effects partially, limiting its validity range to the small vicinity of the critical Reynolds number.

### 3.3 Evolution equation and stability of the film flow

To derive the evolution equation for slightly nonlinear waves, let us assume that
\[ h = 1 + \eta(x, t), \quad \tilde{q} = q_0 + q(x, t) \quad (3.3.1) \]

where \(\eta \ll 1\) and \(q \ll 1\) are the dimensionless perturbations of the film thickness and the flow rate, respectively. Substituting (3.3.1) in (3.2.28) and (3.2.29) and retaining the terms up to the second order fluctuations, and eliminating \(q\) from these resulting equations by using quasi-stationary approximation method for the nonlinear terms which was used by Jurman and McCready [8], Alekseenko et al. [9],
Mukhopadhyay and Dandapat [7]. we finally obtain

\[ \eta_t + \alpha_1 \eta_x + \alpha_2 \eta_{tt} + \alpha_3 \eta_{xx} + \alpha_4 \eta_{xxx} + \alpha_5 \eta_{xxxx} + \alpha_6 \eta_x + \alpha_7 \eta_t + \alpha_8 (\eta_{tt})_x + \alpha_9 (\eta_{tt})_t + \alpha_{10} (\eta_{xxx})_x = 0 \]  

(3.3.2)

where,

\[ \begin{align*}
\alpha_1 &= (1 + 2\alpha) \delta_1, \\
\alpha_2 &= \text{Re}(1 - \alpha)/M^2, \\
\alpha_3 &= 2\alpha_2 \delta \delta_1, \\
\alpha_4 &= \alpha_2 (\delta \delta_1^2 - \gamma), \\
\alpha_5 &= \alpha_2 \text{We}, \\
\alpha_6 &= 2(2 + \alpha) \delta_1, \\
\alpha_7 &= 4(1 - \alpha), \\
\alpha_8 &= -3\alpha_2 \gamma, \\
\alpha_9 &= -2\alpha_2 (\delta - 1), \\
\alpha_{10} &= 3\alpha_5.
\end{align*} \]  

(3.3.3)

In quasi-stationary approximation the basic assumption used in confirmation with the experimental observation is that the waves generally evolve in shape rather slowly with the downstream distance [8]. That is we have used the relations

\[ q = c \eta, \quad \text{and} \quad \partial_t = -c \partial_x \]  

(3.3.4)

where \( c \) is the phase velocity which is assumed to be constant for quasi-stationary wave in the interval \( \Delta t \), and also made a transformation

\[ t = \varepsilon \hat{t}, \quad x = \varepsilon \hat{x} \]  

(3.3.5)

and finally dropped the tildes.

### 3.3.1 Linear stability analysis

In this section, we are interested to study the linear response for a sinusoidal perturbation of the film by assuming the perturbation of the form

\[ \eta = \Gamma[\exp\{i(kx - \omega t)\}] + \text{c.c.} \]  

(3.3.6)
where $\Gamma$ is the amplitude of the disturbance and c.c. represents complex conjugate. Here the wave number $k$ is real and $\omega = \omega_r + i\omega_i$ is the complex frequency. Using (3.3.6) in linearized part of (3.3.2), we get the dispersion relation as

$$D(\omega, k) \equiv -i\omega + i\alpha_1 k - \alpha_2 \omega^2 + \alpha_3 k \omega - \alpha_4 k^2 + \alpha_5 k^4 = 0. \quad (3.3.7)$$

Solution of (3.3.7) gives

$$\omega^\pm = \frac{1}{2} \alpha_2^{-1} (\alpha_3 k - i) \pm (a + ib)^{1/2}, \quad (3.3.8)$$

where

$$a = \alpha_2^{-1} \left[ \alpha_5 k^4 + \left( \frac{1}{4} \alpha_2^{-1} \alpha_3^2 - \alpha_4 \right) k^2 - \frac{1}{4} \alpha_2^{-1} \right] \quad \text{and} \quad b = \alpha_2^{-1} \left( \alpha_1 - \frac{1}{2} \alpha_2^{-1} \alpha_3 \right) k. \quad (3.3.9)$$

In other words $a + ib$ lies in the upper complex plane, and the principal value of its square root must lie in the first quadrant of the complex plane. Separation of the real and imaginary parts of (3.3.8) gives

$$\omega_r^\pm = \frac{1}{2} \alpha_2^{-1} \alpha_3 k \pm \left( \frac{1}{2} [a + (a^2 + b^2)^{1/2}] \right)^{1/2} \quad \text{and} \quad \omega_i^\pm = -\frac{1}{2} \alpha_2^{-1} \pm \left( \frac{1}{2} [-a + (a^2 + b^2)^{1/2}] \right)^{1/2}. \quad (3.3.10)$$

It is obvious from above that $\omega_i^-$ gives stability while $\omega_i^+$ will ensure stability provided

$$\left( \frac{1}{2} \left[ -a + (a^2 + b^2)^{1/2} \right] \right)^{1/2} < \frac{1}{2} \alpha_2^{-1}. \quad (3.3.11)$$

In other words, for $k \neq 0$,

$$\gamma > \delta^2 \left[ (1 + 2\alpha)^2 - \delta (1 + 4\alpha) \right] - \text{Re} k^2. \quad (3.3.11)$$

Using the relation (3.2.30) for $\gamma$ in (3.3.11), we arrive at the stability criterion

$$\text{Re} < \text{Re}_{\text{linear}} = 3 \cot \theta [\delta_1^2 \left( (1 + 2\alpha)^2 - \delta (1 + 4\alpha) \right) - \text{Re} k^2]^{-1}. \quad (3.3.11)$$
This shows that the surface tension renders stability to the flow under long wavelength approximation. Though its influence is weak as $k$ is small for long waves. The minimum $Re$, at which instability sets in may be denoted as critical Reynolds number $Re_c$ for wave formation and obtained from (3.3.10) as

$$Re_c = \frac{3M^4 \cot \theta}{\omega^2(1 - \alpha)^2} \left[ (1 + 2\alpha)^2 - \delta(1 + 4\alpha) \right]^{-1}.\quad (3.3.12)$$

In the neutral state $\omega_i = 0$ gives phase speed

$$c = \alpha_1.\quad (3.3.13)$$

It is to be noted here that the phase speed is independent of $k$ implying the wave is non-dispersive. Also by using the above relation we get in the neutral state

$$k = 0,\quad (3.3.14)$$

$$k_N = \left[ (\alpha_1^2 + \alpha_2 - \alpha_1\alpha_3 + \alpha_4) / \alpha_5 \right]^{1/2},\quad (3.3.15)$$

which correspond to two branches of the neutral curves and the flow instability takes place between them.

Let us draw our attention for a particular case: viz. for $\lim M \to 0$, i.e when electromagnetic field is absent, we have form equation (3.3.12) and equation (3.3.13) $Re_c \to \cot \theta$ and $c \to 3$, respectively, which were obtained by Benjamin [10], Yih [11] and Prokopiou et al. [12] for Newtonian fluid flowing down an inclined plane under the action of gravity only. Therefore, the above results are in good agreement with the known previous results.
Results and discussion for linear stability

Numerical investigation shows that $Re_c$ in (3.3.12) has a point of discontinuity at $M = M_c$ (say) = 4.629 (approx), which is large enough for the flow of thin film, even in the presence of very strong magnetic field. It is to be pointed out that, for large values of $M$ shear wave instability is likely to be important, but here we restricted our analysis to the surface wave instability which is only valid for small values of $M$. To estimate the value of the Hartmann number, let us consider mercury with electrical conductivity $\sigma = 0.1336 \times 10^6 \Omega^{-1} m^{-1}$, dynamic viscosity $\mu = 1.54 \times 10^{-3} N.s/m^2$, mean film thickness $h_0 = 0.73 \times 10^{-4} m$ ($h_0$ can be computed from $h_0 = (3\nu^2Re/g \sin \theta)^{1/3}$ with $Re = 10^2$, $\theta = 5\pi/12$ and $\nu = 1.11 \times 10^{-7} m^2/s$) gives $M \approx 0.27$ when $B_0 = 0.4T$ and $M \approx 0.88$ when $B_0 = 1.3T$. Our prediction for the flow of thin conducting liquid film, $M_c$ is unattainable as $M$ is proportional to the mean film thickness $h_0$ which is very small and most of the common liquids are poorly conducting. A close scrutiny shows that the factor $[(1+2\alpha)^2 - \delta(1+4\alpha)] \to 0$ as $M \to M_c$. The relation (3.3.12) is valid so long $M \leq M_c$; $Re_c$ becomes negative for $M > M_c$, which is meaningless. Therefore, in the forthcoming discussion we will consider $M \ll M_c$. We have to remember here that the above result is true for linear stability theory. One may expect instability if finite amplitude disturbances are introduced into the system.

It is clear from figure 3.2 that as $E_p$ increases for any fixed value of $M$, $Re_c$ decreases implying the destabilizing role of $E_p$, while $Re_c$ increases with the increase of $M$ exhibiting the stabilizing role of $M$. The destabilizing and stabilizing role of $E_p$ and $M$ can be explained in the following way: The basic flow has only one downstream component. In the perturbed state the downstream velocity component is larger in
order of magnitude than the transverse velocity component. One part of the Lorentz force connected with the electric field, helps to accelerate the flow in the downstream direction while the other part of the Lorentz force, due to the interaction of the velocity and the magnetic fields, is directed upstream to oppose the downstream flow. The magnetic lines of force act like elastic string which tend to resist any deviation from the mean flow due to perturbation. When Hartmann number $M$ increases, the field strength increases to provide more restoring force in suppressing disturbances. We conclude that an electrical field destabilizes the film flow whereas magnetic field stabilizes it at not too large $E_p$.

However by changing the orientation of the electric field one may have $E_p$ be negative. In this case one should change $E_p$ to $-\bar{E}_p$ where $\bar{E}_p > 0$ and this technical substitution raises new result. By inspection from equation (3.3.12) we can see that the value of $\varpi$ depends on $M$ for a particular value of $\bar{E}_p$. This implies that for a particular value of $\bar{E}_p = \bar{E}_{pc}$ (say), $\varpi$ vanishes implying complete stability of the flow at $\bar{E}_{pc}$. Consequence of this complete stability can be clear if we remember that the basic flow $U_s$, given in equation (3.2.23), becomes zero at $\bar{E}_{pc}$. Now for fixed $M$, as $\bar{E}_p$ increases $Re_c$ increases so long $\bar{E}_p < \bar{E}_{pc}$ implying stabilizing role of $\bar{E}_p$. However, for $\bar{E}_p > \bar{E}_{pc}$, $Re_c$ decreases with the increase of $\bar{E}_p$ implying destabilizing role of $\bar{E}_p$.

It may be inferred from equation (3.3.12) that one can obtain a mirror image of the curve in figure 3.2 about the $Re_c$ axis, if the orientation of the magnetic field be changed with the result remaining same. This is due to the fact that $Re_c$ is an even function of $M$. 
3.3.2 Weakly nonlinear stability analysis

As the perturbed wave grows to finite amplitude, the linear stability theory is no longer valid for accurate prediction of the flow behaviour. The weakly nonlinear stability analysis is employed to investigate whether the finite amplitude disturbance in the linear stable region can create instability and to examine whether subsequent nonlinear growth of disturbance in the linear unstable region will configure a new equilibrium state with a finite amplitude or grow to be unstable state. Now to study the growth of weakly nonlinear waves, we have used the method of multiple scales and expand the surface elevation $\eta$ as

$$\eta(x, x_1, ... t, t_1, t_2...) = \zeta \eta_1 + \zeta^2 \eta_2 + \zeta^3 \eta_3 + ..., \quad (3.3.16)$$

where the scaling $x, x_1, ... t, t_1, t_2...$ are related according to

$$x_1 = \zeta x, \quad t_1 = \zeta t, \quad t_2 = \zeta^2 t, .... \quad (3.3.17)$$

Using (3.3.16), (3.3.17) in (3.3.2) we get

$$(L_0 + \zeta L_1 + \zeta^2 L_2 + ...) (\zeta \eta_1 + \zeta^2 \eta_2 + \zeta^3 \eta_3 + ...) = -\zeta^2 N_2 - \zeta^3 N_3 - ... \quad (3.3.18)$$

where $L_0, L_1, L_2$ etc. are the operators and $N_2, N_3$ the nonlinear terms of equation (3.3.18) are given in the appendix.

In the lowest order of $\zeta$, we have

$$L_0 \eta_1 = 0 \quad (3.3.19)$$

which has a solution of the form

$$\eta_1 = \Gamma(x_1, t_1, t_2) [\exp i \Theta] + c.c. \quad (3.3.20)$$
where $\Theta = kx - \omega t$ and c.c. denotes the complex conjugates. It is to be noted here that the above solution given in (3.3.20) is already obtained in connection with the linear stability analysis except $\omega$ is replaced by $\omega_r$, since in the vicinity of the neutral curve $\omega = O(\varepsilon^2)$, so that the function $\exp(\omega t)$ is slowly varying and may be absorbed in $\Gamma(x, t)$. In the second order, the perturbation system yields

$$L_0\eta_2 = -L_1\eta_1 - N_2. \quad (3.3.21)$$

Invoking (3.3.20) in (3.3.21), we have

$$L_0\eta_2 = -i \left[ \frac{\partial D(\omega_r, k)}{\partial \omega_r} \frac{\partial \Gamma}{\partial t_1} - \frac{\partial D(\omega_r, k)}{\partial k} \frac{\partial \Gamma}{\partial x_1} \right] e^{i\Theta} - \Omega^2 \Gamma^2 e^{2i\Theta} + \text{c.c.} \quad (3.3.22)$$

where $D(\omega_r, k)$ is given by (3.3.7), and

$$\Omega = 2\alpha_{10}k^4 - 2\alpha_8k^2 - 2\alpha_9\omega + i(\alpha_1k - \alpha_7\omega_r).$$

The uniform valid solution for $\eta_2$ is obtained from (3.3.22) as

$$\eta_2 = \frac{-\Omega^2 \Gamma^2 e^{2i\Theta}}{D(2\omega_r, 2k)} + \text{c.c.} \quad (3.3.23)$$

Introducing the co-ordinate transformation $\xi = (x - c_gt_1)$ and $\tau = t_2$, where $c_g = -D_k/D_{\omega_r}$ is the group velocity, and using the solvability condition on the third order equation, we get

$$\frac{\partial \Gamma}{\partial \tau} - i\frac{\partial c'_g(k)}{\partial \xi} \frac{\partial^2 \Gamma}{\partial \xi^2} - \gamma^{-2}(F_r + iF_i)\omega_i\Gamma + (J_2 + iJ_4)|\Gamma|^2\Gamma = 0, \quad (3.3.24)$$

where

$$c'_g(k) = -(D_{kk} + 2c_gD_{\omega_r,k} + c_g^2D_{\omega_r,\omega_r})/D_{\omega_r}, F_r = 1/X, F_i = Y/X,$$
\[ J_2 = \frac{N E_r + M E_i}{N^2 + M^2}, J_4 = \frac{N E_i - M E_r}{N^2 + M^2}, \]
\[ E_r = \frac{P_r - \gamma P_i}{\chi}, E_i = \frac{P_i + \gamma P_r}{\chi} \]

\[ \chi = 1 + (\alpha_1 \alpha_2 - \alpha_3)^2 k^2, \gamma = (2\alpha_1 \alpha_2 - \alpha_3)k, \]
\[ N = 4(\alpha_3 k \omega_r + 4\alpha_5 k^4 - \alpha_2 \omega_r^2 - \alpha_4 k^2), M = 2(\alpha_1 k - \omega_r), \]
\[ P_r = k^2 [(\alpha_6 - \alpha_1 \alpha_7)^2 - 2k^2(\alpha_1^2 \alpha_9 + \alpha_8 - \alpha_{10} k^2)(\alpha_1^2 \alpha_9 + \alpha_8 - 7\alpha_{10} k^2)], \]

and
\[ P_i = 3k^3 [(\alpha_6 - \alpha_1 \alpha_7)(\alpha_1^2 \alpha_9 + \alpha_8 - 3\alpha_{10} k^2)]. \]

For filtered wave
\[ \frac{\partial \Gamma}{\partial \tau} - \zeta^{-2}(F_r + i F_i)\omega_i \Gamma + (J_2 + i J_4)|\Gamma|^2 \Gamma = 0. \quad (3.3.25) \]

Solution of this equation may be written as
\[ \Gamma = Ae^{-ib\tau}, \]
which gives
\[ \frac{\partial A}{\partial \tau} = [\zeta^{-2} F_r \omega_i - J_2 A^2] A. \quad (3.3.26) \]

and
\[ \frac{\partial (b(\tau)\tau)}{\partial \tau} = J_4 A^2 - \zeta^{-2} F_i \omega_i \quad (3.3.27) \]

The equation (3.3.26) is nothing but the Landau equation [13]. The second term on the right hand side of the equation (3.3.26) is due to nonlinearity and may moderate or accelerate the exponential growth of the linear disturbance. The perturbed wave speed caused by the infinitesimal disturbances appearing in the nonlinear system can be modified using equation (3.3.27). The threshold amplitude will be
\[ \zeta A = \left[ F_r \omega_i / J_2 \right]^{1/2} \quad (3.3.28) \]
and the non-linear wave speed will be

\[ \mathcal{N} c_r = c_r + c_i \left( \frac{J_1 F_r}{J_2} - F_i \right), \quad \text{where} \quad c_i = \frac{\omega_i}{k} \quad (3.3.29) \]

**Results and discussion for weakly nonlinear stability**

From equation (3.3.26) it is clear that the sign of \( J_2 \) predicts the ultimate behaviour of the system. In other words the negative value of \( J_2 \) makes the system unstable. Therefore, the condition of subcritical instability is \( \omega_i < 0, J_2 < 0 \) and that of supercritical stability is \( \omega_i > 0, J_2 > 0 \). The condition of unconditional stability and explosive states are \( \omega_i < 0, J_2 > 0 \) and \( \omega_i > 0, J_2 < 0 \) respectively.

By studying the span of different zones from the Figs. 3.3-3.5 it is clear that the influence of Hartmann number on the flow is very strong. The increase of Hartmann number enlarges and shrinks the subcritical and explosive unstable zones respectively, keeping the other regions almost same. It is interesting to see that when the Hartmann number is very large (near to 1.0), it has a significant effect on the unconditional and supercritical zones, respectively. For large Hartmann number the supercritical stable zone decreases whereas unconditional stable zone increases.

By observing the span of different zones from the Figs. 3.6-3.8 it is clear that the influence of electric field on the flow is very feeble but from Fig. 3.9 we can say that the change of electric parameter almost preserve the supercritical and unconditional stable zones, while the increase of electric field enlarges and shrinks the explosive and subcritical unstable zones, respectively.

In the supercritical stable region the linear amplification rate is positive, while nonlinear amplification rate is negative. Therefore an infinitesimal disturbance in the linear unstable region will not grow infinitely but rather reaches an equilibrium
amplitude as given in equation (3.3.28). Figs. 3.10-3.11 displays the supercritical stable amplitude for different Hartmann numbers and for different electric parameters, respectively. It is found that the threshold amplitude will become smaller (larger) for the increase of the Hartmann numbers (electric parameters), then the flow will be more stable (unstable).

In the subcritical unstable region the linear amplification rate is negative, while non-linear amplification rate is positive, i.e., disturbance amplitude is larger than the threshold amplitude, then the amplitude will increase although the prediction by linear theory is stable. On the other hand, if the initial finite amplitude disturbance is less than the threshold amplitude, then the system will become conditionally stable. Figs. 3.12 and 3.13 display the threshold amplitude in the subcritical region with different Hartmann numbers and for different electric parameters, respectively. It is found that threshold amplitude will become larger (smaller) for increase of the Hartmann numbers (electric parameters), then the flow will be more stable (unstable).

It is interesting to note that the linear wave speed $c_r$ given in equation (3.3.13) is non-dispersive while the non-linear wave speed $Nc_r$ given in equation (3.3.29) is dispersive. The variation of non-linear wave speed $Nc_r$ with respect to wave numbers for different Hartmann numbers $M$ and electric parameters $E_p$ are shown in Figs. 3.14 and 3.15, respectively. It is found that the non-linear wave speed is decreasing (increasing) with the increase of Hartmann numbers (electric parameters). Further, it is clear from equation (3.3.13) that the linear phase speed $c_r$ decreases (increases) with the increase (decrease) of Hartmann numbers.
3.4 Conclusion

The object of the present study is to quantify the effect of electromagnetic field on a conducting liquid film flowing down an inclined plane at moderate Reynolds number under the assumption of small magnetic Reynolds number. Using momentum integral method a nonlinear evolution equation for the development of the free surface is derived. A normal mode approach and the method of multiple scales are used to obtain the linear and nonlinear stability solution for the film flow. Physical parameters chosen for this investigation are (1) Reynolds number ranging from 0.0 to 50.0, (2) Hartmann number ranging from 0.0 to 1.0, (3) electric parameter ranging from -1.0 to 3.5.

The linear stability analysis renders the neutral stability curve which separates stable and unstable region and the critical Reynolds number below which the flow is always stable to infinitesimal disturbances regardless of the wave length of the disturbance. It is found that the magnetic field stabilizes the flow whereas the electric field stabilizes or destabilizes the flow depending on its orientation with the flow.

The nonlinear stability of the study reveals that both the supercritical stability and subcritical instability are possible for this type of thin film flow. The influence of electromagnetic field (taking electric parameter to be positive) on the span of supercritical/subcritical region is examined. The influence of magnetic field is very significant, while the impact of electric field is very feeble in comparison. The change of Hartmann numbers (electric parameters) almost preserve the supercritical and unconditional stable zones, while the increase of Hartmann numbers (electric parameters) shrinks (enlarges) and enlarges (shrinks) the explosive and subcritical
unstable zones, respectively.

Finally we also scrutinize the effect of Hartmann numbers (electric parameters) on the threshold amplitude and speed of the waves. In the supercritical region, threshold amplitude and speed of the nonlinear waves decrease (increase) with the increase of Hartmann numbers (electric parameters), while in the subcritical region threshold amplitude increases (decreases) with the increase of Hartmann numbers (electric parameters).
Appendix

\[ L_0 \equiv \frac{\partial}{\partial t} + \alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial^2}{\partial t^2} + \alpha_3 \frac{\partial^2}{\partial t \partial x} + \alpha_4 \frac{\partial^2}{\partial x^2} + \alpha_5 \frac{\partial^4}{\partial x^4}, \]

\[ L_1 \equiv \frac{\partial}{\partial t_1} + \alpha_1 \frac{\partial}{\partial x_1} + 2 \alpha_2 \frac{\partial^2}{\partial t_1 \partial t_2} + \alpha_3 \left( \frac{\partial^2}{\partial t_1 \partial x_1} + \frac{\partial^2}{\partial t_1 \partial x} \right) + 2 \alpha_4 \frac{\partial^2}{\partial x_1 \partial x} + 4 \alpha_5 \frac{\partial^4}{\partial x^3 \partial x_1}, \]

\[ L_2 \equiv \frac{\partial}{\partial t_2} + \alpha_2 \left( \frac{\partial^2}{\partial t_1^2} + 2 \frac{\partial^2}{\partial t_1 \partial t_2} \right) + \alpha_3 \left( \frac{\partial^2}{\partial t_1 \partial x_1} + \frac{\partial^2}{\partial t_2 \partial x} \right) + \alpha_4 \frac{\partial^2}{\partial x_1^2} + 6 \alpha_5 \frac{\partial^4}{\partial x^2 \partial x_1}, \]

\[ N_2 = \alpha_1 \eta_1 \frac{\partial \eta_1}{\partial x} + \alpha_7 \eta_1 \frac{\partial \eta_1}{\partial t} + \alpha_8 \left[ \eta_1 \frac{\partial^2 \eta_1}{\partial x^2} + \left( \frac{\partial \eta_1}{\partial x} \right)^2 \right] + \alpha_9 \left[ \left( \frac{\partial \eta_1}{\partial t} \right)^2 + \eta_1 \frac{\partial^2 \eta_1}{\partial t^2} \right] + \alpha_{10} \left[ \frac{\partial \eta_1}{\partial x} \eta_3 + \eta_1 \frac{\partial^4 \eta_1}{\partial x^3} \right], \]

\[ N_3 = \alpha_6 \left[ \eta_1 \left( \frac{\partial \eta_2}{\partial x} + \frac{\partial \eta_1}{\partial x_1} \right) + \eta_2 \frac{\partial \eta_1}{\partial x} \right] + \alpha_7 \left[ \eta_1 \left( \frac{\partial \eta_2}{\partial t} + \frac{\partial \eta_1}{\partial t_1} \right) + \eta_2 \frac{\partial \eta_1}{\partial t} \right] + \alpha_8 \left[ 2 \frac{\partial \eta_1}{\partial x} \left( \frac{\partial \eta_1}{\partial x_1} + \frac{\partial \eta_2}{\partial x} \right) + \eta_1 \left( 2 \frac{\partial^2 \eta_1}{\partial x \partial x_1} + \frac{\partial^2 \eta_2}{\partial x^2} \right) + \eta_2 \frac{\partial^2 \eta_1}{\partial x^2} \right] + \alpha_9 \left[ 2 \frac{\partial \eta_1}{\partial t} \left( \frac{\partial \eta_1}{\partial t_1} + \frac{\partial \eta_2}{\partial t} \right) + \eta_1 \left( 2 \frac{\partial^2 \eta_1}{\partial t \partial t_1} + \frac{\partial^2 \eta_2}{\partial t^2} \right) + \eta_2 \frac{\partial^2 \eta_1}{\partial t^2} \right] + \alpha_{10} \left[ \frac{\partial^3 \eta_1}{\partial x^3} \left( \frac{\partial \eta_1}{\partial x_1} + \frac{\partial \eta_2}{\partial x} \right) + \frac{\partial \eta_1}{\partial x} \left( 3 \frac{\partial^3 \eta_1}{\partial x^3 \partial x_1} + \frac{\partial^3 \eta_2}{\partial x^3} \right) + \eta_1 \left( \frac{\partial^4 \eta_2}{\partial x^4} + 4 \frac{\partial^4 \eta_1}{\partial x^2 \partial x_1} \right) + \eta_2 \frac{\partial^4 \eta_1}{\partial x^4} \right]. \]
Bibliography


Figure 3.1: Depth averaged velocity $U_{av}$-vs-$M$ for fixed values of the parameters $We = 450$, $\theta = 75^\circ$, and $Re = 10$. 
Figure 3.2: Re\textsubscript{c}-vs-cot \( \theta \) for different values of \( E_p \) and for fixed values of the parameters \( \text{We} = 430, \theta = 75^\circ \) and \( \text{Re} = 10 \).
Figure 3.3: Stability curve Re-vs-k for fixed values of the parameters $E_p = 0$, $M = 0.03$, $We = 450$ and $\theta = 75^\circ$. (1 $\Rightarrow$ Explosive zone, 2 $\Rightarrow$ Supercritical stable zone, 3 $\Rightarrow$ Unconditional stable zone, 4 $\Rightarrow$ Subcritical unstable zone.)
Figure 3.4: Stability curve Re-vs-k for fixed values of the parameters $E_p = 0$, $M = 0.27$ $We = 450$ and $\theta = 75^\circ$. (1 ⇒ Explosive zone, 2 ⇒ Supercritical stable zone, 3 ⇒ Unconditional stable zone, 4 ⇒ Subcritical unstable zone.)
Figure 3.5: Stability curve Re-vs-k for fixed values of the parameters $E_p = 0$, $M = 0.88$, $We = 450$ and $\theta = 75^\circ$. (1 ⇒ Explosive zone, 2 ⇒ Supercritical stable zone, 3 ⇒ Unconditional stable zone, 4 ⇒ Subcritical unstable zone.)
Figure 3.6: Stability curve Re-vs-$k$ for fixed values of the parameters $E_p = 0.5$, $M = 0.27$, $We = 450$ and $\theta = 75^\circ$. (1 $\Rightarrow$ Explosive zone, 2 $\Rightarrow$ Supercritical stable zone, 3 $\Rightarrow$ Unconditional stable zone, 4 $\Rightarrow$ Subcritical unstable zone.)
Figure 3.7: Stability curve Re-vs-κ for fixed values of the parameters $E_p = 1.0$, $M = 0.27$, $We = 450$ and $\theta = 75^\circ$. (1 ⇒ Explosive zone, 2 ⇒ Supercritical stable zone, 3 ⇒ Unconditional stable zone, 4 ⇒ Subcritical unstable zone.)
Figure 3.8: Stability curve Re-vs-k for fixed values of the parameters $E_p = 2.0$, $M = 0.27$, $We = 450$ and $\theta = 75^\circ$. (1 ⇒ Explosive zone, 2 ⇒ Supercritical stable zone, 3 ⇒ Unconditional stable zone, 4 ⇒ Subcritical unstable zone.)
Figure 3.9: Stability curve $E_p$-vs-$k$ for fixed values of the parameters $Re = 20$, $M = 0.27$, $We = 450$ and $\theta = 75^\circ$. (1 $\Rightarrow$ Explosive zone, 2 $\Rightarrow$ Supercritical stable zone, 3 $\Rightarrow$ Unconditional stable zone, 4 $\Rightarrow$ Subcritical unstable zone.)
Figure 3.10: Threshold amplitude in the supercritical region for different values of $M$ and for fixed values of the parameters $E_p = 1$, $Re = 30$, $We = 450$ and $\theta = 75^\circ$. 
Figure 3.11: Threshold amplitude in the supercritical region for different values of $E_p$ and for fixed values of the parameters $M = 0.27$, $Re = 30$, $We = 450$ and $\theta = 75^\circ$. 
Figure 3.12: Threshold amplitude in the subcritical region for different values of M and for fixed values of the parameters $E_p = 1, \text{Re} = 30, \text{We} = 450$ and $\theta = 75^\circ$. 
Figure 3.13: Threshold amplitude in the subcritical region for different values of $E_p$ and for fixed values of the parameters $M = 0.8$, $Re = 10$, $We = 450$ and $\theta = 75^\circ$. 
Figure 3.14: Non-linear phase speed in the supercritical region for different values of \( M \) and for fixed values of the parameters \( E_p = 1 \), \( Re = 30 \), \( We = 450 \) and \( \theta = 75^\circ \).
Figure 3.15: Non-linear phase speed in the supercritical region for different values of $E_p$ and for fixed values of the parameters $M = 0.27$, $Re = 30$, $We = 450$ and $\theta = 75^\circ$. 