Chapter 2

Nonlinear stability of viscous film flowing down an inclined plane with linear temperature variation

2.1 Introduction

Review of the literature in chapter 1 reveals that most of the previous investigations that dealt with the fluid flow down an inclined/vertical plane are either isothermal or on a uniformly heated plane. The corresponding flow on a non-uniform heated wall so far received much less attention, although non-uniform heating is more compatible with the practical situation.

In this chapter the researcher shall present the analysis of finite amplitude long wave instability of a thin liquid film falling down an inclined heated plane with linear temperature variation. It is assumed that the liquid is non-volatile, so that evaporation effects can be neglected while the film is sufficiently thin so that buoyancy effects can also be neglected, but it is not so thin that intermolecular forces come into play. However the heating on the inclined plane produces a surface tension gradient which induces thermocapillary stresses on the free surface, that effects on the flow dynamics
considerably. The study of Miladinova et al. [1] is based on a numerical solution of the evolution equation and they have taken small Marangoni numbers (weak heating) to search permanent waves but to characterize the instability mechanism a large temperature difference (large Marangoni number) is required. In the following problem the researcher has considered a wide span of Marangoni numbers.

2.2 Formulation of the problem

Let us consider the two-dimensional flow of a thin viscous liquid film flowing down an inclined heated plane of inclination $\theta$ ($0 < \theta \leq \pi/2$), with the horizon. A rectangular coordinate system is chosen such that the $x$–axis is along the flow and $z$–axis is normal to the inclined plane.

The governing equations are the conservation of mass, momentum for the fluid flow and energy equation for the temperature field

$$\nabla \cdot \mathbf{v} = 0 \quad (2.2.1)$$

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla p + \rho \nu \nabla^2 \mathbf{v} + \rho \mathbf{g} \quad (2.2.2)$$

$$\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla)T = \kappa \nabla^2 T, \quad (2.2.3)$$

where $\nabla = (\partial/\partial x, 0, \partial/\partial z)$, $\mathbf{v} = (u, 0, v)$ is the velocity vector, $\mathbf{g} = (g \sin \theta, 0, -g \cos \theta)$ is the gravitational acceleration, and $T$, $\rho$, $\nu$, denote absolute temperature, density and kinematic viscosity of the liquid, respectively. Also $\kappa = k_T/(\rho C_p)$ denote the thermal diffusivity, $k_T$ is the thermal conductivity and $C_p$ is the specific heat at constant pressure of the fluid. Dynamic influence of air flow above the film is ignored.

The boundary conditions are the usual no-slip condition and a linear temperature
distribution as the thermal boundary condition on the inclined plane \((z = 0)\):

\[
\mathbf{v} = 0, \quad \text{(2.2.4)}
\]

\[
T = T_g + Ax. \quad \text{(2.2.5)}
\]

where \(T_g\) denotes the temperature in the gas phase, \(A = (T_H - T_C)/l_0\), where \(T_H\) and \(T_C\) denote the temperatures at the hotter part and the colder part, respectively, along the inclined plane and \(l_0\) the longitudinal length scale. The temperature \(T\) increases (decreases) in the stream-wise direction with positive (negative) \(A\). But in the present study \(A\) is taken as positive.

At the free surface \((z = h(x, t))\), dynamic and kinematic conditions along with Newton’s law of cooling as the thermal boundary condition are,

\[
n \cdot \tau \cdot t = \nabla \sigma \cdot t, \quad \text{(2.2.6)}
\]

\[
p_a + n \cdot \tau \cdot n = -\sigma(T) \nabla \cdot n, \quad \text{(2.2.7)}
\]

\[
\partial_t h + \mathbf{v} \cdot \nabla (h - z) = 0. \quad \text{(2.2.8)}
\]

\[
k_T \nabla T \cdot n + k_g(T - T_g) = 0, \quad \text{(2.2.9)}
\]

where \(n\) and \(t\) are unit vectors, normal (outward pointing) and tangential to the interface, respectively, and \(\tau = -pI + 2\mu \mathbf{e}\) is the stress tensor with the rate of strain tensor \(\mathbf{e} = (\nabla \mathbf{v} + \nabla \mathbf{v}^T)/2\), \(\sigma\) is the surface tension of the fluid, \(p_a\) is the pressure of the ambient gas phase and \(k_g\) is the heat transfer coefficient between the liquid and air and \(\mu\) is the dynamical viscosity of the fluid.

The above equations are quite general regarding various coefficients \((k_T, \kappa, \mu, \sigma\) etc). It is well known that temperature variation in the fluid can cause dramatic changes in the above coefficients, but approximations can be made depending on the
type of the problem being examined. In the foregoing analysis we have assumed the variation of surface tension as

\[\sigma(T) = \sigma_0 - \gamma(T - T_g)\]  \hspace{1cm} (2.2.10)

where \(\sigma_0\) is the surface tension at \(T_g\), the temperature in the gas phase, which is taken as the reference temperature and \(\gamma = -\partial\sigma/\partial T\big|_{T=T_g}\) is a positive constant for most common fluids. The assumption of linear variation of surface tension with temperature is very much compatible with the experimental data. For example, in the range of 0 - 100°C, a clean air-water interface has a nearly linear variation of surface tension with temperature:

\[\sigma(Nm^{-1}) \approx 0.076 - 0.00017T(^0C)\]

with an accuracy of ±1% [2] (See table in appendix 1).

We are now interested in yielding a non-linear evolution equation in terms of film thickness \(h(x,t)\). We assume \(l_0\) as the characteristic longitudinal length scale whose order may be considered same as the wave length \(\lambda\), the mean film thickness \(h_0\) as the length scale in the transverse direction and the Nusselt velocity \(u_0 = gh_0^2 \sin \theta / 3\nu\) as the characteristic velocity. We define the dimensionless quantities as

\[x = l_0 x^*, \hspace{1cm} (h, z) = h_0(h^*, z^*), \hspace{1cm} t = (l_0/u_0) t^*\]

\[u = u_0 u^*, \hspace{1cm} v = (h_0/l_0) u_0 v^*, \hspace{1cm} p = \rho u_0^2 p^*, \hspace{1cm} T = T_g + T^*(T_H - T_C). \]  \hspace{1cm} (2.2.11)

Using the dimensionless quantities (2.2.11) in the governing equations and boundary conditions (2.2.1) - (2.2.9) we arrive after dropping the asterisk as

\[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0, \]  \hspace{1cm} (2.2.12)
\[
\varepsilon \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial z} \right) = -\varepsilon \frac{\partial p}{\partial x} + \sin \theta \frac{1}{\text{Fr}} + \varepsilon \left( \varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right), \tag{2.2.13}
\]

\[
\varepsilon^2 \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial z} - \cos \theta \frac{1}{\text{Fr}} + \varepsilon \left( \varepsilon^2 \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} \right), \tag{2.2.14}
\]

\[
\varepsilon \text{RePr} \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial z} \right) = \varepsilon^2 \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \tag{2.2.15}
\]

\[u = v = 0 \quad \text{and} \quad T = x \quad \text{at} \quad z = 0, \tag{2.2.16}\]

\[
\left[ \frac{\partial u}{\partial z} + \varepsilon^2 \frac{\partial v}{\partial x} \right] \left( 1 - \varepsilon^2 \left( \frac{\partial h}{\partial x} \right)^2 \right) - 2\varepsilon^2 \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial z} \right) \frac{\partial h}{\partial x} \left( 1 + \varepsilon^2 \left( \frac{\partial h}{\partial x} \right)^2 \right)^{-1/2}
\]

\[
= -\varepsilon \text{RePr} \left( \frac{\partial T}{\partial x} + \frac{\partial h}{\partial x} \frac{\partial T}{\partial z} \right) \quad \text{at} \quad z = h(x, t) \tag{2.2.17}\]

\[
p_a - p + \frac{2\varepsilon}{\text{Re}} \left( \varepsilon^2 \frac{\partial u}{\partial x} \frac{\partial h}{\partial x}^2 - \left( \frac{\partial u}{\partial z} + \varepsilon^2 \frac{\partial v}{\partial x} \right) \frac{\partial h}{\partial x} + \frac{\partial v}{\partial z} \right) \left( 1 + \varepsilon^2 \left( \frac{\partial h}{\partial x} \right)^2 \right)^{-3/2}
\]

\[
= \varepsilon^2 \text{We}(1 - \text{CaT}) \frac{\partial^2 h}{\partial x^2} \left( 1 + \varepsilon^2 \left( \frac{\partial h}{\partial x} \right)^2 \right)^{-3/2} \quad \text{at} \quad z = h(x, t) \tag{2.2.18}\]

\[
v = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} \quad \text{at} \quad z = h(x, t) \quad \text{(2.2.19)}
\]

\[
\left( \frac{\partial T}{\partial z} - \varepsilon^2 \frac{\partial h}{\partial x} \frac{\partial T}{\partial x} \right) \left( 1 + \varepsilon^2 \left( \frac{\partial h}{\partial x} \right)^2 \right)^{-1/2} + \text{BiT} = 0 \tag{2.2.20}
\]

where \(\text{Re}(\equiv \nu h_0/\nu)\) is the Reynolds number, \(\text{We}(\equiv \sigma_0/\rho \nu^2 h_0)\) is the Weber number, \(\text{Fr}(\equiv \nu^2/gh_0)\) is the Froude number, \(\text{Pr}(\equiv \nu/\kappa)\) is the Prandtl number, \(\text{Mn}(\equiv 3\gamma(T_H - T_C)/\rho g h_0 \sin \theta)\) is the Marangoni number, \(\text{Ca}(\equiv \gamma(T_H - T_C)/\sigma_0)\) is the capillary number *, \(\text{Bi}(\equiv k_g h_0/k_T)\) is the Biot number and \(\varepsilon(\equiv h_0/l_0 << 1)\) is the aspect ratio for long wave approximation. It can be shown by using the definition of the Nusselt velocity that

\[
\sin \theta \frac{\text{Fr}}{\text{Re}} = \frac{3}{\text{Re}}. \tag{2.2.21}\]

*\(\text{Ca} = \text{CrMa}, \text{Mn} = \varepsilon \text{Ma}/\text{RePr}\) where \(\text{Cr} = \mu \kappa_0/\sigma_0 h_0\) and \(\text{Ma} = \gamma(T_H - T_C) h_0/\mu \kappa\) are respectively the Crispation/capillary and Marangoni numbers in usual definition.
We have assumed that the Reynolds number is $O(1)$ and Weber number is $O(1/\varepsilon^2)$, Marangoni number is of order $O(1)$, Capillary number is $O(\varepsilon)$, the Prandtl number is $O(1)$ and the Biot number is $O(\varepsilon^2)$.

We are now interested in yielding a non-linear evolution equation in terms of film thickness $h(x, t)$. Expanding the velocity components, pressure and the temperature in powers of $\varepsilon$ as,

\[
\begin{align*}
    u &= u_0 + \varepsilon u_1 + \ldots, \quad v = v_0 + \varepsilon v_1 + \ldots, \quad p = p_0 + \varepsilon p_1 + \ldots \quad \text{and} \quad T = T_0 + \varepsilon T_1 + \ldots
\end{align*}
\]

and substituting the above into the governing equations (2.2.12)-(2.2.15) and the boundary conditions (2.2.16)-(2.2.20) we obtain a set of PDEs for a different order of $\varepsilon$. The velocity profile and its depth averaged velocities for the zeroth-order and the first-order problem is obtained as

at $O(1)$ :

\[
\begin{align*}
    u_0 &= 3\left(hz - \frac{z^2}{2}\right) - Mnz \\
    \bar{u}_0 &= h^2 - \frac{1}{2}Mnh
\end{align*}
\]

and at $O(\varepsilon)$:

\[
\begin{align*}
    u_1 &= \text{Re}[\left(\frac{3\cot \theta}{Re} h_x - \varepsilon^2 \text{We} h_{xxx} \right)\left(\frac{z^2}{2} - hz\right) \\
    &\quad + \frac{1}{2}(3h - Mn)(\frac{z^4}{4} + 2h^3z - h^2z^2)h_x + \frac{1}{2}\text{PrMn}(5h - 2Mn)h^2zh_x] \\
    \bar{u}_1 &= \text{Re}[\left(-\frac{\cot \theta}{Re} h_x + \frac{1}{3}\varepsilon^2 \text{We} h_{xxx}\right)h^2 + \frac{2}{5}(3h - Mn)h^4h_x \\
    &\quad + \frac{1}{4}\text{PrMn}(5h - 2Mn)h^3h_x].
\end{align*}
\]

Integrating the continuity equation (2.2.12) with respect to $z$ from 0 to $h$ by using Leibnitz’s rule and boundary conditions (2.2.16) and (2.2.19), we have

\[
\begin{align*}
    \frac{\partial h}{\partial t} + \frac{\partial (\bar{u}_0h)}{\partial x} + \varepsilon \frac{\partial (\bar{u}_1h)}{\partial x} + O(\varepsilon^2) = 0
\end{align*}
\]
Substituting $\bar{u}_0$ and $\bar{u}_1$ from (2.2.23) and (2.2.25), we get

$$h_t + A(h)h_x + \varepsilon \left( B(h)h_x + \varepsilon^2 C(h)h_{xxx} \right)_x + O(\varepsilon^2) = 0$$

(2.2.27)

where the suffixes denote the differentiation with respect to the corresponding variables and

$$A(h) = 3h^2 - Mn,$$

$$B(h) = - \cot \theta h^3 + \frac{2}{3} \text{Re} (3h - Mn) h^5 + \frac{1}{4} \text{RePrMn} (5h - 2Mn) h^4,$$

$$C(h) = \frac{1}{3} \text{ReWe} h^3.$$

### 2.3 Stability analysis

To study the instability, the film thickness may be written as

$$h(x,t) = 1 + \eta(x,t)$$

(2.3.1)

where $\eta << 1$ are the dimensionless perturbation of the film thickness.

Setting the transformation

$$t = \epsilon \tilde{t} \quad \text{and} \quad x = \epsilon \tilde{x}$$

(2.3.2)

and using (2.3.1) and (2.3.2) in (2.2.27), and retaining the terms up to the second order fluctuations after dropping the tilde sign can be written as

$$\eta_t + A\eta_x + B\eta_{xx} + C\eta_{xxxx} + A'\eta_x + B' \left( \eta_{xx} + \eta_x^2 \right) + C' \left( \eta_{xxxx} + \eta_x \eta_{xxx} \right) + \frac{1}{2} A'' \eta_x^2 + B' \left( \frac{1}{2} \eta_x^2 \eta_{xx} + \eta_x^2 \right) + C' \left( \frac{1}{2} \eta_x^2 \eta_{xxx} + \eta_x \eta_{xxx} \right) + O(\eta^4) = 0,$$

(2.3.3)

where $A, B, C$ and their corresponding derivatives are evaluated at $h = 1$. 
2.3.1 Linear stability analysis

In this section, we are interested to study the linear response for a sinusoidal perturbation of the film by assuming the perturbation of the form

\[ \eta = \Gamma [\exp\{i(kx - \omega t)\}] + \text{c.c.} \tag{2.3.4} \]

where \( \Gamma \) is the amplitude of the disturbance and c.c. represents complex conjugate. Here the wave number \( k \) is real and \( \omega = \omega_r + i\omega_i \) is the complex frequency. Using (2.3.4) in linearized part of (2.3.3), we get the dispersion relation as

\[ D(\omega, k) = -i\omega + iAk - Bk^2 + Ck^4 = 0. \tag{2.3.5} \]

Equating the real and imaginary parts of (2.3.5), we get

\[ \omega_r = Ak, \quad \text{and} \quad \omega_i = Bk^2 - Ck^4. \tag{2.3.6} \]

Therefore, the phase velocity

\[ c_r = \omega_r / k = 3 - Mn. \tag{2.3.7} \]

It is to be noted here that the phase speed is independent of \( k \), implying the wave is non-dispersive in nature. The sign of \( \omega_i \) which is a sum of two terms, determines the stability characteristics of the flow. It is clear that the second term in the right hand side of the second equation of (2.3.6) is related to the surface tension, which is always positive and a negative sign in front of this term results from the attenuation of perturbation implying the stabilizing role of the surface tension. The flow will be linearly stable if \( \omega_i < 0 \), i.e. \( Bk^2 - Ck^4 < 0 \).

Further it is clear from the expression of \( B \) that the first term in the right hand side of the second equation of (2.3.6) may have any sign, but when
$\text{Re} > \frac{20 \cot \theta}{8(3 - Mn) + 5\text{Pr}Mn(5 - 2Mn)}$,
then $B$ is positive and it enhances the perturbation resulting in instability. The minimum $\text{Re}$, at which instability sets in may be denoted as the critical Reynolds number $\text{Re}_c$ for the wave formation and obtained from the second term of (2.3.6) as

$$\text{Re}_c = \frac{20 \cot \theta}{8(3 - Mn) + 5\text{Pr}Mn(5 - 2Mn)} \quad (2.3.8)$$

As $\text{Mn} \to 0$, $\text{Re}_c = (5/6) \cot \theta$, which is the critical $\text{Re}$ for the isothermal case as obtained by Benjamin [3] and Yih [4].

In the neutral state $\omega_i = 0$ gives two relations

$$k = 0, \quad (2.3.9)$$

$$k_c = \sqrt{B/C}, \quad (2.3.10)$$

which correspond to two branches of the neutral curves and the flow instability takes place in between them. Further the neutral curves intersect at the bifurcation point $\text{Re} = \text{Re}_c$, $k = 0$. The wave number of the waves with a maximum growth is obtained from the relation $d\omega_i/dk = 0$, which gives $k_m = k_c/\sqrt{2}$

### 2.3.2 Weakly nonlinear stability analysis

To study the growth of weakly nonlinear waves, we shall use the method of multiple scales and expand the surface elevation $\eta$ as

$$\eta(x, x_1, ... \ t, t_1, t_2, ...) = \zeta \eta_1 + \zeta^2 \eta_2 + \zeta^3 \eta_3 + ..., \quad (2.3.11)$$

where the scalings $x, x_1, ... \ t, t_1, t_2,...$ are related according to

$$x_1 = \zeta x, \quad t_1 = \zeta t, \quad t_2 = \zeta^2 t, ..., \quad (2.3.12)$$
Using (2.3.11), (2.3.12) in (2.3.3) we get

\[(L_0 + \zeta L_1 + \zeta^2 L_2 + \ldots)(\zeta \eta_1 + \zeta^2 \eta_2 + \zeta^3 \eta_3 + \ldots) = -\zeta^2 N_2 - \zeta^3 N_3 - \ldots \quad \text{(2.3.13)}\]

where \(L_0, L_1, L_2\) etc. are the operators and \(N_2, N_3\) are the nonlinear terms of equation (2.3.13) are given in the appendix 2.

In the lowest order of \(\zeta\), we have

\[L_0 \eta_1 = 0 \quad \text{(2.3.14)}\]

which has a solution of the form

\[\eta_1 = \Gamma(x_1, t_1, t_2)[\exp i\Theta] + \text{c.c.} \quad \text{(2.3.15)}\]

where \(\Theta = kx - \omega_r t\). and c.c. denotes the complex conjugate. It is to be noted here that the above solution given in (2.3.15) is already obtained in connection with the linear stability analysis except \(\omega\) is replaced by \(\omega_r\), since in the vicinity of the neutral curve \(\omega_i = O(\varepsilon^2)\), so that the function \(\exp(\omega_i t)\) is slowly varying and may be absorbed in \(\Gamma(x_1, t_1, t_2)\).

In the second order, the perturbation system yields

\[L_0 \eta_2 = -L_1 \eta_1 - N_2. \quad \text{(2.3.16)}\]

Invoking (2.3.15) in (2.3.16), we have

\[L_0 \eta_2 = -i \left[ \frac{\partial D(\omega_r, k)}{\partial \omega_r} \frac{\partial \Gamma}{\partial t_1} - \frac{\partial D(\omega_r, k)}{\partial k} \frac{\partial \Gamma}{\partial x_1} \right] e^{i\theta} - \Omega \Gamma^2 e^{2i\theta} + \text{c.c.} \quad \text{(2.3.17)}\]

where \(D(\omega_r, k)\) is given by (2.3.5), and

\[\Omega = -2\alpha_3 k^2 + 2\alpha_6 k^4 + i\alpha_4 k.\]

The uniform valid solution for \(\eta_2\) is obtained from (2.3.17) as

\[\eta_2 = -\frac{\Omega \Gamma^2 e^{2i\theta}}{D(2\omega_r, 2k)} + \text{c.c.} \quad \text{(2.3.18)}\]
Introducing the co-ordinate transformation $\xi = (x_1 - c_gt_1)$, where $c_g = -D_k/D_{\omega_1}$ is the group velocity, and using the solvability condition on the third order equation, we get

$$\frac{\partial^2 \Gamma}{\partial t_2^2} + J_1 \frac{\partial^2 \Gamma}{\partial \xi^2} - \zeta^{-2} \omega_1 \Gamma + (J_2 + iJ_4)|\Gamma|^2 \Gamma = 0,$$

(2.3.19)

where,

$$J_1 = B - 6Ck^2,$$

$$J_2 = \frac{1}{2} \left( -B''k^2 + C''k^4 \right) + \frac{(A')^2k^2 - 2(B'k^2 - 7C'k^4)(B'k^2 - C'k^4)}{16Ck^4 - 4Bk^2}$$

and

$$J_4 = \frac{1}{2} A''k + \left[ \frac{A'k(B'k^2 - 7C'k^4) + 2A'k(B'k^2 - C'k^4)}{16Ck^4 - 4Bk^2} \right]$$

For filtered waves there is no spatial modulation and the diffusion term vanishes, we get

$$\frac{\partial \Gamma}{\partial t_2} - \zeta^{-2} \omega_1 \Gamma + (J_2 + iJ_4)|\Gamma|^2 \Gamma = 0.$$  

(2.3.20)

Solution of this equation may be written as

$$\Gamma = a e^{-ib(t_2)t_2},$$

(2.3.21)

which gives

$$\frac{\partial a}{\partial t_2} = [\zeta^{-2} \omega_1 - J_2a^2] a,$$

(2.3.22)

and

$$\frac{\partial (b(t_2)t_2)}{\partial t_2} = J_4a^2$$

(2.3.23)

The equation (2.3.22) is nothing but the Landau equation. The second term on the right hand side of the equation (2.3.22) is due to nonlinearity and may moderate or accelerate the exponential growth of the linear disturbance. The perturbed wave
speed caused by the infinitesimal disturbances appearing in the nonlinear system can be modified using equation (2.3.23). The threshold amplitude will be

\[ \zeta a = \left[ \frac{\omega_i}{J_2} \right]^{1/2}. \]  

(2.3.24)

and the non-linear wave speed will be

\[ Nc_r = c_r + c_i \left( \frac{J_4}{J_2} \right), \quad \text{where} \quad c_i = \frac{\omega_i}{k}. \]  

(2.3.25)

2.4 Results and discussion

The object of the present study is to quantify the effect of thermocapillarity on a viscous film flowing down an inclined plane with linear temperature variation in the finite amplitude regime. Physical parameters chosen for this investigation are (1) Reynolds number ranging from 0 to 6, (2) Marangoni number ranging from 0 to 0.6, (3) Prandtl numbers 1, 7, and 10. Experimental findings by Kabov et al.[5], Kabov[6] and Scheid et al.[7] clearly indicate that there is certain critical value of the heat flux for the onset of instability. To achieve such a critical value a large temperature difference would be required as pointed out by Kalliadasis et al.[8] and the agreement becomes much better for a large temperature difference, for example a temperature difference as large as 10K or even larger were observed in the experiments. To estimate the value of the Marangoni number we have considered, water at 25°C with \( \gamma = 5 \times 10^{-5} Kgs^{-2}K^{-1}, \mu = 10^{-2} gcm^{-1}s^{-1}, \rho = 1 gcm^{-3} \) gives \( \text{Pr} \approx 7 \) and with temperature difference 11K we obtain \( \text{Mn} = 0.5 \) (here we have taken \( \epsilon = 0.01 \) such that our long-wave assumption is satisfied). Therefore to capture the instability mechanism the estimated value of the Marangoni numbers should be wide-ranging. For this reason we have considered the value of Mn from 0 to 0.6.
With the help of long wave expansion method an evolution equation of the free surface (Benney type) is derived which includes the effect of viscosity, gravity, mean surface tension and thermocapillarity in terms of different nondimensional numbers. It is well known that the Prandtl number is always important in convective heat transfer problems. The product RePr in the conduction term of the energy equation is often called the Peclet number Pe that expresses the relative importance of convection and conduction. Finally we like to mention here the Marangoni number, which expresses the relative importance of thermocapillarity and viscous stress, defined by the longitudinal temperature gradient is taken within the above mentioned range and therefore the plate temperature increases in the downstream direction.

The linear stability analysis renders the neutral stability curve which separates the stable and unstable regions and the critical Reynolds number below which the flow is always stable to infinitesimal disturbances regardless the wavelength of the disturbance. Fig.1 depicts the variation of the critical Reynolds number with the Marangoni number, for different Prandtl numbers. It is clear from the figure that Re_c decreases as Mn increases and the rate of decrease is more for large Prandtl numbers. Therefore Mn gives a destabilizing effect and this effect is more for large Prandtl numbers.

The neutral stability curves for different Marangoni numbers are presented in fig.2. The figure demonstrated that by increasing Marangoni number, this instability region becomes larger. This result is in good agreement with the result reported by Kalliadasis et al. [8], whose derivation of the evolution equation is based on the integral boundary layer method.

As the perturbed wave grows to finite amplitude, the linear stability theory is no
longer valid for an accurate prediction of the flow behaviour. The non-linear stability analysis is employed to investigate whether the finite amplitude disturbance in the linear stable region can create instability (subcritical instability) and to examine whether subsequent nonlinear growth of disturbance in the linear unstable region will configure a new equilibrium state with a finite amplitude (supercritical stability) or grow to be unstable (explosive) state, which could be considered as solutions of complex patterns. From equation (2.3.22) it is clear that the sign of $J_2$ predicts the ultimate behavior of the system. In other words the negative value of $J_2$ makes the system unstable. Therefore, the condition of subcritical instability is $\omega_i < 0, J_2 < 0$ and that of supercritical stability is $\omega_i > 0, J_2 > 0$. The condition of unconditional stability and explosive states are $\omega_i < 0, J_2 > 0$ and $\omega_i > 0, J_2 < 0$ respectively. Fig.3 shows that both the zones of supercritical and unconditional stability will exist for different ranges of $k$. It is clear that the unconditional stable zone reduces as the Reynolds number $Re$ increases for fixed $Mn$ and $Pr$. Fig.4 shows the variation of $Mn$ with $k$ for fixed value of other parameters. It is apparent from this figure that as $Mn$ increases unconditional stable zone gradually reduces and ultimately vanishes after a cutoff Marangoni number, whereas subcritical unstable, supercritical stable and explosive zones increase.

In the subcritical unstable region the linear amplification rate is negative, while the nonlinear amplification rate is positive i.e if the disturbance amplitude is larger than the threshold amplitude, then the amplitude will increase although the prediction by linear theory is stable. On the other hand if the initial finite amplitude disturbance is less than the threshold amplitude, then the system will become conditionally stable. Fig.5 displays the threshold amplitude in the subcritical region with different
Marangoni numbers. It is found that threshold amplitude will become smaller for an increase in the Marangoni numbers, and then the flow will be more unstable.

In the supercritical stable region the linear amplification rate is positive, while the nonlinear amplification rate is negative. Therefore an infinitesimal disturbance in the linear unstable region will not grow infinitely but rather reaches an equilibrium amplitude as given in equation (2.3.24). Fig.6 displays the supercritical stable amplitude for different Marangoni numbers. It is found that the threshold amplitude will become larger for an increase in the Marangoni numbers, and then the flow will be more unstable.

It is interesting to note that the linear wave speed $c_r$ given in equation (2.3.7) is non-dispersive while the non-linear wave speed $Nc_r$ given in equation (2.3.25) is dispersive and is always larger than the linearized one. The variation of non-linear wave speed $Nc_r$ with respect to wave numbers for different Marangoni numbers $Mn$ are shown in fig.7. It is found that the non-linear wave speed increases with the increase in Marangoni numbers. This result also agrees well with that reported by Miladinova et al. [1], their analysis is based on the numerical solution of the evolution equation. Further, it is clear from equation (2.3.7) that the linear phase speed $c_r$ decreases (increases) with the increase (decrease) of Marangoni numbers, which is just the opposite for a uniformly heated plane as reported by Joo et al. [9], which states that the linear phase speed remains unchanged by the thermocapillary force.

In summary, thermocapillarity influence the span of supercritical/subcritical regions and it also has a strong effect on the amplitude and speed of the non-linear waves.
Bibliography


Appendix 1

Surface-tension co-efficient for an air water interface

<table>
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<th>T°C</th>
<th>( \sigma \ Nm^{-1} )</th>
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<tr>
<td>0</td>
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<tr>
<td>20</td>
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<tr>
<td>100</td>
<td>0.0589</td>
</tr>
</tbody>
</table>

Appendix 2

\[
L_0 \equiv \frac{\partial}{\partial t} + A \frac{\partial}{\partial x} + B \frac{\partial^2}{\partial x^2} + C \frac{\partial^4}{\partial x^4},
\]

\[
L_1 \equiv \frac{\partial}{\partial t_1} + A \frac{\partial}{\partial x_1} + 2B \frac{\partial^2}{\partial x_1 \partial x} + 4C \frac{\partial^4}{\partial x_1^2 \partial x},
\]

\[
L_2 \equiv \frac{\partial}{\partial t_2} + B \frac{\partial^2}{\partial x_1^2} + 6C \frac{\partial^4}{\partial x^2 \partial x_1^2},
\]

\[
N_2 = A' \eta_1 \frac{\partial \eta_1}{\partial x} + B' \left[ \eta_1 \left( \frac{\partial^2 \eta_1}{\partial x^2} + \left( \frac{\partial \eta_1}{\partial x} \right)^2 \right) \right] + C' \left[ \eta_1 \frac{\partial^4 \eta_1}{\partial x^4} + \frac{\partial \eta_1}{\partial x} \frac{\partial^3 \eta_1}{\partial x^3} \right],
\]

\[
N_3 = A' \left[ \eta_1 \left( \frac{\partial^2 \eta_2}{\partial x^2} + \frac{\partial \eta_1}{\partial x} \frac{\partial \eta_2}{\partial x} \right) + \frac{\partial \eta_1}{\partial x} \left( \frac{\partial^3 \eta_2}{\partial x^3} + 2 \frac{\partial^2 \eta_1}{\partial x \partial x_1} \right) + \frac{\partial \eta_1}{\partial x} \frac{\partial^3 \eta_1}{\partial x^3} \right] + C' \left[ \eta_2 \left( \frac{\partial^4 \eta_2}{\partial x^4} + 4 \frac{\partial^4 \eta_1}{\partial x^4 \partial x_1} \right) + \frac{\partial \eta_1}{\partial x} \frac{\partial^3 \eta_1}{\partial x^3} \right]
\]

\[
+ D' \left[ \eta_2 \frac{\partial^6 \eta_2}{\partial x^6} + B'' \left[ \frac{1}{2} \eta_1 \eta_2 \frac{\partial^2 \eta_1}{\partial x^2} + \eta_1 \left( \frac{\partial \eta_1}{\partial x} \right)^2 \right] \right] + C'' \left( \frac{1}{2} \eta_1 \eta_2 \frac{\partial^4 \eta_1}{\partial x^4} + \frac{\partial \eta_1}{\partial x} \frac{\partial^3 \eta_1}{\partial x^3} \right).
\]
Figure 2.1: Variation of $Re_c$ with $Mn$ for different $Pr$ and for fixed $\theta = \pi/3$. 
Figure 2.2: Linear stability curve for different Mn and for fixed We = 450, Pr = 7.
Figure 2.3: Stability curve Re versus $k$ for a fixed values of the parameter $\text{We} = 450$, $\text{Mn} = 0.6$, $\theta = \pi/3$ and $\text{Pr} = 7$. Zones I, II, III and IV denote unconditional, subcritical, explosive and supercritical zone respectively.
Figure 2.4: Stability curve $\text{Mn}$ versus $k$ for a fixed values of the parameter $\text{We} = 450$, $\text{Re} = 2$, $\theta = \pi/3$ and $\text{Pr} = 7$. Zones I, II, III and IV denote unconditional, subcritical, explosive and supercritical zone respectively.
Figure 2.5: Amplitude of disturbances in the subcritical region for different Mn and for Re = 2, We = 450, $\theta = \pi/3$ and Pr = 7.
Figure 2.6: Amplitude of disturbances in the supercritical region for different Mn and for $Re = 1$, $We = 450$, $\theta = \pi/3$ and $Pr = 7$. 
Figure 2.7: Nonlinear wave speed for different Mn in the supercritical region for $Re = 2$, $We = 450$, $\theta = \pi/3$ and $Pr = 7$. 