Chapter 2

Complete Integrability of some integrable systems (CIHS)

In this chapter, in the first two sections all the basics on Poisson structure on Euclidean spaces and general manifolds and Hamiltonian structures are developed and the properties of them needed for later use are given. In particular, these will be used in chapter 4, chapter 5 and chapter 6. Then, we study various integrable systems such as the Toda lattice, the Lagrange top, the geodesic motion on an ellipsoid, the KdV equations and the generalized KdV equations (or the Gel’fand-Dikii system). We describe the Hamiltonian function, the Poisson structure and the first integrals of motion of these systems. We explain in chapter 4 how these systems obey the Adler-Kostant-Symes geometric principle.

2.1 SOME PRELIMINARIES: In this section, we give an introduction to the various aspects of Poisson brackets used in different parts (contexts) of the thesis.

2.1.1: The basic geometry of a physical system is described by a configuration space \( M \), a \( C^\infty \) manifold. Elements \( q \) of \( M \) represent instantaneous configurations of the physical system. For instance, for \( k \) particles in 3-dimensional space \( \mathbb{R}^3 \), the configuration space is \( M = (\mathbb{R}^3)^k \cong \mathbb{R}^{3k} \). Let \( I \) be an interval in \( \mathbb{R} \). A curve \( \mu : t \to q(t) \) (where \( t \) runs over the time interval \( I \)) with values in \( M \) describes the motion, i.e., change in time, of the physical system. The velocity \( v(t_o) \) at time \( t_o \) (also denoted by \( \dot{q}(t_o) \)) is defined as: \( v(t_o) = \frac{d}{dt}|_{t=t_o} q(t) \). Thus \( v(t_o) \) belongs to the tangent space \( T_{q(t_o)}M \) at \( q(t_o) \). The collection of spaces \( T_qM \) for \( q \) in \( M \) is the tangent bundle \( TM \) over \( M \). Hence to describe the positions and velocities of the various parts of the system at a certain time \( t_o \), we
need to specify a point in TM. We can interpret TM as the kinematical space. Let \( N \) be the dimension of \( M \). Every system of local coordinates \( q^1, \ldots, q^N \) on \( M \) gives rise to a system of local coordinates \( q^1, \cdots, q^N, v^1, \cdots, v^N \) on the tangent bundle TM by differentiation. If the transition functions between two coordinate systems \( q \) and \( q' \) are given by

\[
\tilde{q}^j = \phi^j(q^1, \ldots, q^N), \quad 1 \leq j \leq N,
\]

the corresponding transition functions between \( v^j \) and \( \tilde{v}^j \) are

\[
\tilde{v}^j = \sum_{i=1}^{N} \partial_i \phi^j(q^1, \ldots, q^N)v^i,
\]

where \( \partial_i = \frac{\partial}{\partial q_i} \). The coordinate changes on TM corresponding to coordinate changes on \( M \) are linear in the \( v_i \)'s; in more abstract terms, TM is not only a fibration over \( A/ \), but even a vector bundle. Velocity vectors at a point can be added. For \( q \) in \( A/ \), we denote by \((q, v)\) a point in \( T_qM \), and by \((q, \xi)\) a point in \( T^*_qM \), where \( T^*M = (T_qM)^* \) is the algebraic linear dual of \( T_qM \). The collection of the vector spaces \( T^*_qM \), for \( q \) running over \( M \), is another vector bundle \( T^*M \) over \( A/ \), called the co-tangent bundle.

A force field \( F \) is called conservative if the work integral

\[
\int_a^b F \, dS = \int_{t_a}^{t_b} F(q(t)) \dot{q}(t) \, dt
\]

only depends on the end points \( a \) and \( b \), and not on the particular choice of the curve \( q(t) \) joining them. In the case of a conservative field there exists a function \( V \), unique upto an additive constant, such that

\[
F_i = -\partial_i V \quad \text{or} \quad F = -dV.
\]

In fact, we define \( V(a) - V(b) = \int_{t_a}^{t_b} F(q(t)) \cdot \dot{q}(t) \, dt \) for an arbitrary curve \( q(t) \) between \( a \) and \( b \). In summary, the potential \( V \) is a function on \( M \), the force field \( F \) is a differential form of degree one on \( M \), that is, a section of the cotangent bundle \( T^*M \) over \( M \).
2.1.2: In the case of a conservative force field, acting on a particle of mass \( m \), the law of conservation of energy

\[
\frac{1}{2}m|v_a|^2 + V(a) = \frac{1}{2}m|v_b|^2 + V(b)
\]

is a consequence of the differential equation (Newton's 2nd law)

\[ F = ma \]

Here \( V \) is the potential energy and \( v_a \) (resp. \( v_b \)) is the velocity at time \( t_a \) (resp. \( t_b \)), where the particle is at the point \( a \) (resp. \( b \)).

Given a system of various particles with positions \( q(i) \), velocities \( v(i) \) and masses \( m(i) \), we can verify that this expression \( \sum_i \frac{1}{2}m(i)|v(i)|^2 \) defines a differential quadratic form on \( M \), called the kinetic energy \( T \). This is a function \( T: TM \to \mathbb{R} \) and \( T \) restricted to the vector space \( T_qM \) is a quadratic form for every \( q \) in \( M \). The potential \( V: M \to \mathbb{R} \) can be lifted to a function \( V: TM \to \mathbb{R} \). The total energy \( E = T + V \) is then a function defined on \( TM \).

The discovery of Lagrange is that the laws of motion can be formulated entirely in terms of the function \( L = T - V \) on \( M \) called the Lagrangian. The equations governing the motion are

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q^i} \quad \text{(Euler-Lagrange)}
\]

Here we denote the local coordinates on \( TM \) by \( q^1, \ldots, q^N, \dot{q}^1, \ldots, \dot{q}^N \) where \( \dot{q}^i \) is used as an alternative notation for \( v^i \). The Euler-Lagrange equations are derived from the variational principle of Hamiltonian \( \int L(q, \dot{q}) dt = 0 \). Another picture of the above was discovered by Hamilton which we give below.

Assume that a Lagrangian function \( L: TM \to \mathbb{R} \) is given, not necessarily of the form \( L = T - V \) as above. We define the Legendre transformation \( A: TM \to T^*M \) by
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\[ \Lambda(q, v) = (q, \Lambda_q(v)) \text{ where } \Lambda_q(v) \in T^*M \text{ is defined by} \]

\[ \langle \Lambda_q(v), w \rangle = \int_0^1 \langle \Lambda_q(v + cw), w \rangle \text{ for } q \in M, \ v, w \in T_qM \]

The energy function \( E : TM \to \mathbb{R} \) is then defined by

\[ E(q, v) = \langle \Lambda_q(v), v \rangle - L(q, v) \).

In terms of local coordinates, \( q^1, \ldots, q^N \) are a set of linear coordinates on the vector space \( T_qM \) for fixed \( q = (q^1, \ldots, q^N) \); we define \( p_1, \ldots, p_N \) as the dual system of coordinates on the vector space \( T^*M \) dual to \( T_qM \). Then \( q^1, \ldots, q^N, p_1, \ldots, p_N \) form a set of local coordinates on \( T^*M \) associated in a canonical way to the local coordinates \( q^1, \ldots, q^N \) on \( M \). The Legendre transformation is then given by \( p_i = \frac{\partial L}{\partial \dot{q}^i} \), and the energy function is \( E = \sum_i p_i \dot{q}^i - L(q, \dot{q}) \). The Lagrangian is non-degenerate iff the map \( A \) is a local diffeomorphism of \( TM \) into \( T^*M \), that is, iff the determinant \( \det \left( \frac{\partial \sigma L}{\partial \dot{q}^i \partial \dot{q}^j} \right) \) is nowhere 0 on \( TM \). The coordinates \( p_1, \ldots, p_N \) are called the generalized momenta. The Hamiltonian is the function \( H \) on \( T^*M \) such that \( H \circ A = E \). i.e., \( \Lambda^*H = E \). In coordinates:

\[ H(q^1, \ldots, q^N, p_1, \ldots, p_N) = \sum_{i=1}^N p_i \dot{q}^i - L(q, \dot{q}) \]

where the relation between \( p_i \) and \( \dot{q}^i \) is given by \( p_i = \frac{\partial L}{\partial \dot{q}^i} \) as above.

2.1.3: According to Hamilton, the Euler-Lagrange equations are equivalent, via the Legendre transformation, to the following system

\[
\begin{align*}
\frac{dq^i}{dt} &= \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}.
\end{align*}
\]

This is a system of first order differential equations on the cotangent bundle with symmetry.

Consider a mechanical quantity represented by a function \( F : T^*M \to \mathbb{R} \) on the phase space. Suppose that a motion \( \mu : I \to \mathbb{R} \) is given, where \( \mu(t) = q(t) \) is the moving point
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in \( M \). Lift \( \mu \) to \( TM \), with value \( (q(t), \dot{q}(t)) \) at time \( t \) and then define \( \tilde{\mu} : t \rightarrow T^*M \) by

\[
\tilde{\mu}(t) = \Lambda(q(t), \dot{q}(t))
\]

The time derivative \( \frac{d}{dt}(F \circ \tilde{\mu})(t) \) can be written as \( F \circ \tilde{\mu} \) where \( F \) is a new function on \( T^*M \), independent of the motion \( \mu \). Using Hamilton's equations (2.1.4), we obtain the dynamical law

\[
\dot{F} = \sum_i \left( \frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q^i} \right)
\]

This can be expressed as follows. On the cotangent bundle, we can introduce in an invariant way the Poisson bracket \( \{F, G\} \) of two functions \( F, G \) in \( C^\infty(T^*M) \) by

\[
\{F, G\} = \sum_i \left( \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q^i} - \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} \right)
\] (2.1.5)

This Poisson bracket is independent of the chosen Lagrangian \( L \) and of the corresponding Hamiltonian \( H \). The dynamical law can be written in a concise way as

\[
F = \{H, F\} \quad \text{for every} \quad F \quad \text{in} \quad C^\infty(T^*M)
\] (2.1.6)

This Poisson bracket \( \{F, G\} \) is a bilinear expression of \( F, G \) and enjoys the following properties:

1. \( \{F, G\} = -\{G, F\} \) (Skew symmetry)
2. \( \{F, G_1 G_2\} = \{F, G_1\} G_2 + G_1 \{F, G_2\} \) (Leibniz rule)
3. \( \{F_1, \{F_2, F_3\}\} + \{F_2, \{F_3, F_1\}\} + \{F_3, \{F_1, F_2\}\} = 0 \) (Jacobi identity)

For the coordinate functions we obtain

Hamilton's equations (2.1.4) are recovered from the dynamical law (2.1.6) by using the relations

\[
\{q^i, F\} = -\frac{\partial F}{\partial p_i}, \quad \{p_i, F\} = \frac{\partial F}{\partial q^i}
\]
2.1.7 Definition: In general, a Poisson bracket on a manifold $M$ is a bilinear mapping $\{.,.\}$ on the function space $C^\infty(M)$ of smooth functions satisfying the following conditions:

1. $\{F, G\} = -\{G, F\}$ (Skew symmetry)
2. $\{F, G_1G_2\} = \{F, G_1\}G_2 + G_1\{F, G_2\}$ (Leibniz rule)
3. $\{F_1, \{F_2, F_3\}\} = \{F_2, \{F_3, F_1\}\} = \{F_3, \{F_1, F_2\}\} = 0$ (Jacobi identity)

A Poisson manifold is by definition a manifold equipped with a Poisson bracket.

2.1.8 Remarks: (1) Let $P$ be a Poisson manifold. The mapping $\xi_F$ defined for $F$ in $C^\infty(P)$ by $\xi_F(G) = \{F, G\}$ is a derivation of $C^\infty(P)$ and therefore can be viewed as a vector field on $P$. It is called the Hamiltonian vector field with Hamiltonian $F$. From the Leibniz rule, we have the derivation, $\xi_{F_1F_2} = F_1\xi_{F_2} + F_2\xi_{F_1}$. Hence we can define a mapping $J : T^*P \rightarrow TP$ by

$$J(dF) = \xi_F \quad \text{or} \quad \langle dG, J(dF) \rangle = \{F, G\}.$$

For local coordinates $x_1, \cdots, x_n$ on $P, J$ is given by a matrix $(J_{\alpha\beta})$ of functions, namely, $J_{\alpha\beta} = \{x_\alpha, x_\beta\}$. The Poisson bracket then becomes:

$$\{F, G\} = \sum_{\alpha, \beta} J_{\alpha\beta} \partial^\alpha F \partial^\beta G,$$

where $\partial^\alpha = \frac{\partial}{\partial x_\alpha}$. Hence the vector field $\xi_F$ has components $\xi^\beta_F = \sum_\alpha J_{\alpha\beta} \partial^\alpha F$.

(2) In the canonical case where $P$ is the cotangent bundle of a manifold $M$ and the Poisson bracket is the usual one, we can choose a local coordinate system such that the matrix $J$ is of a simple form (in block form):

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

In fact, any coordinate system $(q^1, \cdots, q^N, p_1, \cdots, p_N)$ on $P = T^*M$ corresponding to a coordinate system $(q^1, \cdots, q^N)$ on $M$ does the trick.
(3) The Poisson bracket can be recovered from the mapping \( \mathbf{J} \) on the matrix \((\mathbf{J}_{\alpha\beta})\) and we can define a Poisson structure in terms of \(\mathbf{J}^\gamma_{\beta\alpha}\). The various conditions on the bracket become:

(i) \(\mathbf{J}_{\alpha\beta} = -\mathbf{J}_{\beta\alpha}\) (Skew symmetry)

(ii) \(\sum \lambda (\mathbf{J}_{\alpha\lambda} \partial^\lambda \mathbf{J}_{\beta\gamma} + \mathbf{J}_{\beta\lambda} \partial^\lambda \mathbf{J}_{\gamma\alpha} + \mathbf{J}_{\gamma\lambda} \partial^\lambda \mathbf{J}_{\alpha\beta}) = 0\) (Jacobi identity)

(4) Given a function \(H\) in \(C^\infty(P)\), we obtain the Hamiltonian vector field \(\mathbf{\xi}_H\) and hence a differential equation \(\dot{x}(t) = \mathbf{\xi}_H(x(t))\) or, in local coordinates

2.1.9 Definition: A symplectic manifold is a manifold \(P\) equipped with a differential 2-form \(\omega\) such that

(i) \(d\omega = 0\) and

(ii) \(\omega\) is non-degenerate, i.e., if \(v \in T_xP\) is such that \(\omega(v, w) = 0\) \(\forall w \in T_xP\), then \(v = 0\).

(2) Using local coordinates \(x^\alpha\), define the matrix \(A = (\Lambda^\alpha_{\beta})\) as the inverse of the matrix \(J = (\mathbf{J}_{\alpha\beta})\). The non-degeneracy of \(\omega\) proves that the mapping \(A : TP \rightarrow T^*P\) defined by

\[\omega(v, w) = <\Lambda w, v>\]

is invertible. The Poisson bracket on \(P\) is defined by

\[\{F, G\} = \omega(\Lambda^{-1}(dF), \Lambda^{-1}(dG))\]

and every symplectic manifold is a Poisson manifold (that is, a symplectic manifold is nothing but a Poisson manifold \(P\) for which the associated map \(J : T^*P \rightarrow TP\) is invertible).

2.1.10: We discuss below how the Poisson bracket is defined in a few examples:

(1) **On the cotangent bundle:** Let \(M\) be an arbitrary manifold of dimension \(n\) and let \(T^*M\) be the cotangent bundle of \(M\), which is of dimension \(2n\). The Liouville form \(\omega\) on \(T^*M\) is defined in local coordinates \(q^1, \cdots, q^n, p_1, \cdots, p_n\) as \(\omega = \sum p_i dq^i\). The global definition is as follows: We write \(P = T^*M\) and \(\pi_p : TP \rightarrow P\) for the canonical
projection. Let \( \pi_M^* : T^*M \to M \) be the canonical projection of the cotangent bundle of \( M \) into \( M \). Then \( T\pi_M^* : TP \to TM \) is the differential of \( \pi_M^* \). For \( v \in TP \), set \( \alpha(v) = \langle \pi_p(v), T\pi_M^*(v) \rangle \) and this is the local expression of the differential 1-form \( \alpha \) on \( P = T^*M \). The canonical symplectic form \( \omega \) on \( T^*M \) is defined by \( \omega = d\alpha = \sum dp_i \wedge dq^i \).

The Poisson bracket is expressed as

\[
\{F, G\} = \sum_{i=1}^{n} \left( \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q^i} - \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} \right) \quad \text{(cf. 2.1.5)}
\]

(2) On the dual of a Lie algebra: Let \( \mathcal{G} \) be a Lie algebra of finite dimension over \( \mathbb{R} \) and \( \mathcal{G}^* \) be its dual as a vector space. The space \( \mathcal{Q} \) can be embedded as a subspace of \( C^\infty(\mathcal{G}^*) \), namely to \( X \) in \( \mathcal{G} \) we associate the linear function \( F_X : \mathcal{Q}^* \to \mathbb{R} \) by \( \xi \mapsto \langle \xi, X \rangle \). Then there exists a unique Poisson bracket on \( \mathcal{G}^* \) such that

\[
\{F_X, F_Y\} = F_{[X,Y]} \quad \text{for} \quad X, Y \in \mathcal{G} \tag{2.1.11}
\]

In other words, the map \( X \mapsto F_X \) from \( \mathcal{G} \) into \( C^\infty(\mathcal{G}^*) \) is a homomorphism of Lie algebras.

There are three descriptions of this Poisson bracket:

(a) Let \( e_1, \cdots, e_n \) be a basis of \( \mathcal{G} \). The functions \( \xi_1 = F_{e_1}, \xi_2 = F_{e_2}, \cdots, \xi_n = F_{e_n} \) form a system of linear coordinates on \( \mathcal{G}^* \). Introduce the structure constants \( c_{\gamma}^{\alpha \beta} \) of the Lie algebra \( \mathcal{G} \) by \( [e_\alpha, e_\beta] = \sum_\gamma c_{\gamma}^{\alpha \beta} e_\gamma \).

Hence the first expression of the Poisson bracket on \( \mathcal{G}^* \):

\[
\{F_1, F_2\} = \sum_{\alpha, \beta, \gamma} c_{\gamma}^{\alpha \beta} \frac{\partial F_1}{\partial \xi_\alpha} \frac{\partial F_2}{\partial \xi_\beta} \xi_\gamma \tag{2.1.12}
\]

It is the unique Poisson bracket for which formula (2.1.11) holds for \( X, Y \) running over the given basis of \( \mathcal{G} \), that is, \( \{\xi_\alpha, \xi_\beta\} = \sum_\gamma c_{\gamma}^{\alpha \beta} \xi_\gamma \).

(b) For any function \( F \) in \( C^\infty(\mathcal{G}^*) \) we can consider its gradient \( \nabla F \) as a function on \( \mathcal{G}^* \) with values in \( \mathcal{G} \) characterized by the relation

\[
\langle \eta, \nabla F(\xi) \rangle = \left. \frac{d}{dt} \right|_{t=0} F(\xi + t\eta) \quad \text{for} \quad \xi, \eta \in \mathcal{G}^* .
\]
With the previous notations

$$\nabla F = \sum_{\alpha} \frac{\partial F}{\partial \xi_\alpha} e_\alpha$$

The invariant version of formula (2.1.12) is read as follows:

$$\{F_1, F_2\}(\xi) = \langle \xi, [\nabla F_1(\xi), \nabla F_2(\xi)] \rangle$$  \hspace{1cm} (2.1.13)$$

(c) Let $G$ be a Lie group with Lie algebra $\mathcal{G}$. For every $g$ in $G$, the left translation $\gamma_g : g' \to gg'$ is an automorphism of the manifold $G$, hence induces an automorphism $\rho_g$ of the cotangent bundle $T^*G$. We can identify $\mathcal{G}^*$ with the fibre of $T^*G$ at the unit element of $G$. The map defined by $\rho(g, \xi) = \rho_g(\xi), \ g \in G, \ \xi \in \mathcal{G}^*$ is a diffeomorphism $p$ of $G \times \mathcal{G}^*$ with $T^*G$. We define the projection $\pi^* : T^*G \to \mathcal{G}^*$ by $\pi^*(\rho(g, \xi)) = \xi$. On the cotangent bundle $T^*G$ there is defined a canonical Poisson bracket, invariant under the automorphism $\rho_g$ of $T^*G$. The map $F \mapsto F \circ \pi^*$ identifies $C^\infty(\mathcal{G}^*)$ with the subspace $C^\infty(T^*G)^G$ consisting of the functions $/ \in T^*G$ such that $/ \circ \rho_g = /$ for every $g$ in $G$. This space is closed under Poisson brackets. The Poisson bracket in $C^\infty(\mathcal{G}^*)$ is characterized by the property

$$\{F_1, F_2\} \circ \pi^* = \{F_1 \circ \pi^*, F_2 \circ \pi^*\}$$

for $F_1, F_2$ in $C^\infty(\mathcal{G}^*)$ (i.e., $\pi^*$ is a Poisson map from $T^*G$ to $\mathcal{G}^*$).

2.2 HAMILTONIAN STRUCTURE: We discuss in this section the various forms of Hamiltonian structures used in different contexts (or parts) of the thesis.

2.2.1: A Hamiltonian structure in an even dimensional phase space $M$ of dimension $n$ is defined by the Poisson bracket. The Poisson bracket $\{,\}$ is a mapping $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ which puts each pair of functions $f, g \in C^\infty(M)$ into correspondence a third function $h = \{f, g\}$. If $\xi_1, \cdots, \xi_n$ are local coordinates of a point $\xi \in M$, then the operation $\{,\}$, by definition, implies

$$\{f, g\} = \sum_{i,k=1}^n \omega_{ik}(\xi) \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_k}$$  \hspace{1cm} (2.2.2)$$
where $\omega$ is an $n \times n$ non-degenerate matrix. The conditions of Poisson bracket given in the previous section imply that the matrix $\omega^{ik}(\xi)$ is skew-symmetric:

$$W_{ik} W_{i,k} + W_{kl,i} = 0$$

(2.2.4)

The condition of Jacobi identity is equivalent to a certain system of equations for $\omega^{ik}(\xi)$ that are identical with the first pair of Maxwell's equations for the inverse matrix of $a,$ i.e., $W: W_{ik} = (\omega^{-1})_{ik}$:

$$W_{ik} W_{i,k} + W_{kl,i} = 0$$

(2.2.4)

It is defined in terms of a differential form as:

$$W = W_{ik} d\xi^i A \xi^k,$$

where $r = A d\xi^k = -d\xi^k A d\xi^i.$

The Poisson bracket maps a linear space of functions on a phase space into a Lie algebra. Having chosen some function $h(\xi)$ and terming it a Hamiltonian function, we can define a map (dependent on $t$) of this algebra into itself $f(\xi) \to f(\xi, t), f(\xi, 0) = f(\xi)$ by means of the following differential equation:

$$\dot{f} = \{f, h\}$$

(2.2.5)

In particular, taking $f$ in the form $f(\xi, t) = \prod_{i=1}^n \delta(\xi_i - \xi_i(t)),$ for $\xi_i(t),$ we obtain:

$$\frac{d\xi_i}{dt} = \sum_k \omega^{ik} \frac{\partial h}{\partial \xi_k}$$

which are Hamiltonian equations.

Using a linear transformation in the phase space, we can reduce every skew-symmetric non-degenerate constant matrix $\omega$ to a block form:

$$\omega = \begin{pmatrix}
J & 0 & \vdots & \vdots \\
0 & J & 0 & \vdots \\
0 & 0 & J & \vdots \\
\vdots & \vdots & \vdots & 0 \\
0 & 0 & \ldots & 0 & J
\end{pmatrix}, \quad J = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}$$
Denoting the coordinates in \(M\) space as \(\xi_{2k-1} = p_k, \xi_{2k} = q_k, \ k = i, \cdots n/2\) where \(i = 1, \cdots, n\), and from (2.2.6) we obtain
\[
\frac{dh}{dh}
\]
i.e., Hamiltonian equations in the conventional form. The pair of variables \(p_k\) and \(q_k\) are canonical conjugates. In these variables the Poisson bracket is written as follows:
\[
\{J, g\} = 2 \sum_{k} \frac{\partial q_k}{\partial \xi} \frac{\partial p_k}{\partial \xi} \cdot
\]
2.2.7: Any function \(H \in C^\infty(M)\) defines a derivation \(\{H, \cdot\}\), it defines a vector field \(X_H\) the Hamiltonian vector field, by definition, \(X_Hf = \{H, f\} = df(X_H)\). The function \(H\) (the Hamiltonian) defines a vector field \(X_H\) (the Hamiltonian vector field) which in turn defines a differential equation, \(x = X_H(x) = J_x dH_x\), the Hamiltonian system generated by the Hamiltonian \(H\).

By skew-symmetry, \(\{H, H\} = 0\) so that \(dH(X_H) = 0\) and \(H\) is constant along the trajectories of \(H\). Any function \(f\) having that property (remaining constant along the trajectories), that is such that \(df(X_H) = 0\) (or \(\{H, f\} = 0\)) is called a first integral.

Symplectic structures provide a special instance of Poisson structure. On a symplectic manifold \((M, \omega)\), the Poisson bracket \(\{f, g\}\) of two functions is defined in terms of their Hamiltonian vector fields by \(\{f, g\} = \omega(X_f, X_g) = dg(X_f)\), for \(f, g \in C^\infty(M)\). The Hamiltonian system generated by \(f\) can be written as \(\dot{q}_j = \{q_j, f\}, p_j = \{p_j, f\}\).

2.2.8 Examples: (a) The vector triple product defines a Poisson bracket in \(\mathbb{R}^3\) by 
\[
\{f, g\}(r) = r.(\nabla f(r) \times \nabla g(r))
\]
(b) A Poisson bracket on \(\mathbb{R}^2\) is defined by 
\[
\{f, g\}(x, y) = y \left( \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right)
\]
(c) Poisson bracket in \(G^*\): Let \(G\) be a Lie algebra, and let \(G^*\) be the dual vector space, with \(\langle \cdot, \cdot \rangle\) the pairing between them. For \(\xi \in G\), let \(\ell / \in C^\infty(G^*)\) be the function
Define $\{f_\xi, f_\eta\} = \{f_{[\xi, \eta]}\}$. This is a Poisson bracket on linear functions on $G^*$. For $f \in C_\infty(G^*)$, we introduce its gradient

$$\nabla f : G^* \to G \quad \text{by} \quad \lim_{\epsilon \to 0} \frac{f(\mu + \epsilon \nu) - f(\mu)}{\epsilon} = \langle \nu, \nabla f(\mu) \rangle$$

The Lie-Poisson bracket is given by

$$\{f, g\}(\mu) = \langle \mu, [\nabla f(\mu), \nabla g(\mu)] \rangle$$

The invariant functions, $f$ on $G^*$ are such that $\{f, g\} = 0 \ \forall g$ and these functions satisfy $f(Ad_\mu) = f(\mu)$.

### 2.3 THE TODA LATTICE:

The Toda lattice is a system of unit masses connected by non-linear springs governed by an exponential restoring force. The equations of motion of the system can be derived from the Hamiltonian function

$$H = H(x, y) = \frac{1}{2} \sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n} e^{x_i - x_{i+1}} \quad x_0 = x_{n+1} = 0$$

where $x_i$ is the displacement of $i^{th}$ mass from equilibrium (or $x_i$ denotes the position of the mass points, $i = 1, \cdots n$) and $y_i$ is the corresponding momentum. The equations of motion are given by

$$\dot{x}_i = y_i$$

The corresponding flow is given by

$$\dot{x}_i = \frac{\partial H}{\partial y_i} \quad \dot{y}_i = -\frac{\partial H}{\partial x_i} \quad i = 1, \cdots n$$

By the Flaschka's transformation,

$$a_i = \frac{1}{2} e^{\frac{1}{2}(x_i - x_{i+1})} \quad i = 1, \cdots n - 1$$

$$b_i = \frac{1}{2} y_i \quad i = 1, \cdots n$$
we can write the system as

\[
\begin{align*}
\dot{a}_i &= a_i(b_{i+1} - b_i) \\
\dot{b}_i &= 2(a_i^2 - a_{i-1}^2), \quad i = 1, \cdots, n-1, \quad a_0 = a_n = b_{n+1} = 0
\end{align*}
\]  

(2.3.1)

The above system (2.3.1) is a Hamiltonian system. It can be written in the form, \( i = J \nabla_z H, \) (2.3.2) where \( z = \begin{pmatrix} a \\ b \end{pmatrix}, \quad a \in \mathbb{R}^{n-1}, \quad b \in \mathbb{R}^n \) and \( J \) defines a Poisson bracket through the formula

\[
\{f(z), g(z)\} = (\nabla f, J \nabla g)
\]

where \((,)\) is the standard dot product in \( \mathbb{R}^{2n-1} \), \( V \) is the standard gradient. We take for \( J \), the \((2n - 1) \times (2n - 1)\) matrix

\[
J = J(a) = \begin{bmatrix}
O_{n-1} & S \\
-\tilde{S}^t & O_n
\end{bmatrix}
\]

where \(O_{n-1}, O_n\) are the \((n - 1) \times (n - 1)\) and \(n \times n\) zero matrices respectively. \( S = S(a) \) is a \((n - 1) \times n\) matrix function defined by

\[
S_{ij} = (-\delta_{ij} + E_{ij})a_i, \quad i = 1, \cdots, n-1, \quad j = 1, \cdots, n.
\]

\[
E_{ij} = 1 \text{ if } j = i + 1, \quad 0 \text{ if } j \neq i + 1 \text{ and}
\]

\[
\delta_{ij} = 1 \text{ if } i = j, \quad 0 \text{ if } i \neq j
\]

Then the Poisson bracket is given by

\[
\{g, f\} = \sum_{i=n}^{n-1} \{a_i f_{a_i}(g_{b_i} - g_{b_{i+1}}) + a_i g_{a_i}(f_{b_{i+1}} - f_{b_i})\}
\]

The function \( H \) of (2.3.2) is given by

\[
H = H(z) = \sum_{i=1}^{n} b_i^2 + \sum_{i=1}^{n-1} a_i^2
\]
2.4 THE LAGRANGE TOP:

A Lagrange top is an axially symmetric rigid body with centre of mass on the axis of symmetry, moving about a fixed point (the origin of $\mathbb{R}^3$) under the influence of gravity. Let $S$ be a rigid body moving about a fixed point $0 \in V = \mathbb{R}^3$. Let $G$ be its centre of gravity, and $\mu$ be the total mass. Let $\gamma$ be the unitary vector field on $S$ in the direction of gravity, say $z$-axis. The $(S, \gamma)$ is a dynamical system.

Let $M$ be the angular momentum of $S$ with respect to body coordinates. Let $\Omega = (p, q, r)$ be the angular velocity of the body $S$ (or the rotation vector of $S$ or the variable position vector of $S$). Let $I$ be the inertia matrix of $S$ which can be regarded as a positive definite symmetric automorphism of $V = \mathbb{R}^3$. Then $M = I(\Omega) = (I_1 p, I_2 q, I_3 r)$ (2.4.1)

where $I_1, I_2, I_3$ denote the principal moments of inertia, when the body frame is principal.

The total derivative with respect to time is given by

$$\frac{\mathcal{L}}{dt}(\cdot)_{\text{total}} = \frac{\mathcal{L}}{t}(\cdot) + \gamma \times (\cdot)$$

(2.4.2)

Then the torque exerted on the body by the (vertical) force of gravity is $I \times \text{gravity}$ where $I$ is the centre of gravity in the body coordinates and $\mu g \gamma$ is the downward gravity force.

The rotation version of Newton’s equations

$$\frac{d}{dt}(\text{angular momentum})_{\text{total}} = \text{torque} \quad \text{and} \quad \frac{d}{dt}(z\text{-axis})_{\text{total}} = 0$$

give the Euler-Poisson equations

Using the Lie algebra isomorphism,

$$\lambda : (\mathbb{R}^3, \times) \rightarrow (\mathfrak{so}(3), [\cdot, \cdot])$$

$$x = (x_1, x_2, x_3) \mapsto \hat{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$
we get the Lie algebra version of (2.4.2) as

\[
\dot{M} = \frac{dM}{dt} = [M, \Omega] + \mu g[\gamma, l], \quad \frac{d\gamma}{dt} = [\gamma, \Omega].
\] (2.4.4)

In the absence of gravity, we have the quadratic differential equation,

\[
\frac{dM}{dt} = [M, \Omega]
\] (2.4.5)

where \([M, \beta] = [\Omega, a], [\alpha, \beta] = 0, a = \beta^2\) with \(f_1 = \frac{1}{2}(I_1 + I_2 + I_3)I - \text{diag}(I_1, I_2, I_3)\).

Lagrange top corresponds to the case where \(I_1 = I_2\), and where the centre of gravity and fixed point of rotation belong to the principal axis of inertia. Let \(z_0\) be their respective distance and let \(l = (0, 0, z_0)\). Adjoin the relation

\[
[M, \beta] = [\Omega, a], \quad \beta = \mu g l, \quad a = I_1 \beta, \text{ and hence } [a, \beta] = 0
\] (2.4.6)

The equations (2.4.4) and (2.4.6) are the equations of the Lagrange top.

2.5 THE KdV EQUATION AND THE GENERALIZED KdV EQUATION:

The KdV equation for \(u \in C(\mathbb{R}), u_t = 6uu_x - 2u_{xxx}\) where \(u\) describes the amplitude of waterwaves in a narrow channel of finite depth has been studied over recent years.

The KdV equation can be written as

\[
u_t = \frac{\partial}{\partial x} (3u^2 - u_{xx}) = \frac{\partial}{\partial x} \left( \frac{\delta H}{\delta u(x)} \right)
\] (2.5.1)

where the functional \(H\) is

\[
H[u] = \int_{-\infty}^{\infty} \left( \frac{u_x^2}{2} + u^3 \right) dx
\] (2.5.2)

and the symbol \(\delta\) denotes the variational derivative. Expressing \(\frac{\partial}{\partial x} \frac{\delta H}{\delta u(x)}\) as

\[
- \int_{-\infty}^{\infty} \delta'(x - x') \frac{\delta H}{\delta u(x')} dx'
\] (2.5.3)

we note that equation (2.5.1) takes the form of (2.2.6). We interpret the coordinates of a function \(u\) (points in the phase space) as the set of its values at the points on \(x\)-axis,
so that \( x \) and \( x' \) are the suffixes in (2.2.6): \( \omega(x, x') = -8(x - x') \). That is, \( \omega \) is skew-symmetric and on functions, it is non-degenerate as the operator \( d/dx \) is reversible on such functions. Also, since \( \omega \) is not dependent on the point \( u \) in the phase space, the Poisson bracket

\[
\{S, R\} = \int_{-\infty}^{\infty} \frac{\partial S}{\partial u(x)} \frac{\partial R}{\partial u(x)} dx
\]  

(2.5.4)

satisfies the Jacobi identity. Thus, the KdV equation is a Hamiltonian equation. The functional \( \mathcal{H} \) (2.5.2) is a Hamiltonian function, the phase space consists of sufficiently smooth functions \( u(x) \) which decrease at infinity, and the Hamiltonian structure is defined by the Poisson bracket (2.5.4), i.e., by the skew-symmetric operator \( \frac{\partial}{\partial x} \) in \( L_2(\mathbb{R}) \).

2.5.5 Fact: P. Lax [34] discovered that the KdV equation is identical to the operator equation \( L = [A, L] \) where

\[
L = -\frac{d^2}{dx^2} u + \frac{d^4}{dx^4} \left( u \frac{d}{dx} + \frac{d}{dx} u \right).
\]

P. Lax [34] and Gardner [14] have shown that the known polynomial integrals \( I_n(u) = \int_{-\infty}^{\infty} P_n(u, \cdots, u^n) dx \) of the KdV equation (the \( I_n \) are expressed in terms of the spectrum of the operator \( L \)) all determine equations

\[
\dot{u} = \frac{d}{dx} \frac{\delta I_n}{\delta u(x)}
\]

admitting the Lax representation \( L = [A_n, L] \) where \( L = \frac{d^4}{dx^4} + u \) and the \( A_n \) are certain skew-symmetric operators of order \( 2n + 1 \),

\[
\begin{align*}
I_0 &= \int u^2 dx, \quad I_1 = \int \left( \frac{u^2}{2} + u^3 \right) dx, \quad I_2 = \int \left( \frac{u^2}{2} - \frac{5}{2} u^2 u_{xx} + \frac{5}{2} u^4 \right) dx, \\
A_0 &= \frac{d}{dx}, \quad A_1 = \frac{4d^2}{dx^3} - 3 \left( u \frac{d}{dx} + \frac{d}{dx} u \right), \quad A_2 = \frac{d^5}{dx^5} - \frac{5}{2} u \frac{d^3}{dx^3} - \frac{15}{4} u^2 \frac{d^2}{dx^2} + \frac{15}{8} u^2 - \frac{25}{8} u_{xx} \frac{d}{dx} + \frac{15}{8} \left( uu_x - \frac{u_{xxx}}{2} \right)
\end{align*}
\]

These equations are called "higher KdV equations".

2.5.6: By the above fact (2.5.5) there exists a denumerable set of local polynomial integrals of the KdV equation. The KdV equation is a completely integrable system, for, the substitution \( u(x) \rightarrow 'scattering data' \) is a transformation in the phase space
to "action-angle" type variables and each polynomial $I_j[u]$, which is an integral of the KdV, gives a completely integrable Hamiltonian system defined by the non-linear partial differential equation:

$$u_t = \frac{\partial}{\partial x} \frac{\delta I_j}{\delta u(x)} \quad (2.5.7)$$

Equations of type (2.5.7) for $j > 3$ are called higher order KdV equations. They can be integrated with the same substitution of variables and have the same integrals of motion as the KdV equation.

### 2.6 THE GEL’FAND-DIKII SYSTEM:

P. Lax [34] described the KdV equation in the form:

$$\frac{dL}{dt} = [B, L] \quad \text{where} \quad L = -\partial_x^2 + q(x, t), \quad B = -4\partial_x^{n-2}\partial(q\partial_x + \partial_x q)$$

Gel’fand-Dikii generalized the Lax form of the KdV equation with

$$L = (-i\partial_x)^n + \sum_{j=0}^{n-2} q_j(-i\partial_x)^j, \quad q_i \in C_0^\infty(\mathbb{R}), \quad i = 0, \ldots, n - 2,$$

then the Lax equations $\frac{dL}{dt} = [B, L]$ for appropriate choice of $B$, is a Hamiltonian system with an infinite sequence of involutive polynomial integrals, in analogy with $L = -\partial_x^2 + q(x, t)$. In the case of the generalized KdV equation or the Gel’fand-Dikii system, the Lax operator is a differential operator of the form

$$L_n = \partial^n + u_{n-2}\partial^{n-2} + \cdots + u_0, \quad \partial = \frac{d}{dx} \quad \text{or} \quad \left(\partial = \sqrt{-1}\frac{d}{dx}\right)$$

$$(D = \sqrt{-1}\frac{d}{dx})$$

(These systems are infinite dimensional Hamiltonian systems having infinitely many constants of motion in involution, which are integrals of local polynomial densities. These systems are also (formal) infinitesimal isospectral deformations of (formal) linear ordinary differential operators. A formal linear ordinary differential operator or linear differential
expression is a polynomial in \( \frac{d}{dx} \) (or in \( D = -\sqrt{-1}\frac{d}{dx} \)) with coefficients which are functions of \( x \), which are infinitely differentiable in some open interval on the real line).

We consider the coefficients \( u_i \)'s as symbols to which the operator \( d \) can be applied:
\[
\partial u_k = u'_k, \quad \partial u_k = u''_k, \ldots, \partial u_{k}^{(i-1)} = u_i^{(i)}, \quad u_k^{(0)} = u_k.
\]
Differential polynomials, i.e., expression of the form \( f = \sum a_{i_1 \cdots i_k} u_{k_1}^{(i_1)} \cdots u_{k_r}^{(i_r)} \) form a differential algebra \( \mathcal{A} \), i.e., an algebra with a given derivation \( d \). Here the coefficients \( a \) can be either complex or real numbers or smooth functions of \( x \).

The ring of pseudo-differential operators \( R \) consists of formal sums of the form \( X = \sum_{i=0}^\infty a_i \partial^i \), \( a_i \in \mathcal{A} \) with the commutation law,
\[
\partial^i f = \sum_{k=0}^\infty \binom{i}{k} f^{(k)} \partial^{i-k}, \quad \forall \ i,
\]
where \( f^{(k)} \) is the \( k \)-th derivative of the function \( f(x) \) and \( \binom{i}{k} = \frac{i!}{(i-k)!(k!)}. \) There is a decomposition \( R = R_+ + R_- \) where \( R_+ \) is the subring of differential operators \( \{ X = \sum_{i=0}^\infty a_i \partial^i \} \) and \( R_- \) that of integral or 'Volterra' operators \( R_- = \{ X = \sum_{i=0}^\infty a_i \partial^i \}. \)

We remark that the operators can also be written in the 'left' form \( \sum_i \partial^i a_i \) using the given commutation law.

For a pseudo-differential operator of the form \( X = \sum a_i \partial^i \) we define its residue \( (\text{Res} X) \) as the coefficient of the \( \partial^{-1} \) term, namely, \( \text{Res} X = a_{-1} \). The Trace is defined by
\[
\text{Tr} X = \int \text{Res} X \, dx \quad (2.6.1).
\]
Tr satisfies the property \( \text{Tr}(XY) = \text{Tr}(YX) \) \( (2.6.2) \) and it follows that \( \text{Tr}[X, Y] = 0 \) \( (2.6.3) \) for any two pseudo-differential operators \( X \) and \( Y \).

If \( X = \sum_{-\infty}^N a_i \partial^i \) and \( a_N = 1 \), then the following operators exist:
\[
X^{-1} = \sum_{-\infty}^{-N} f_{-} \partial^{-n} \quad \text{and} \quad X^{1/N} = \sum_{-\infty}^1 c_i \partial^i \text{where} \ b_{-N} = c_1 = 1 \quad \text{and hence} \quad X^{m/N} \text{also exists}.
\]
These operators commute with \( X \).

A formal isospectral deformation of \( L \) is a specification
\[
\hat{L} = \sum_{k=0}^{n-2} u_k D^k = [P, L] = P \circ L - L \circ P \quad (2.6.4)
\]
where \( P \) is a linear differential expression whose coefficients depend polynomially on \( u_0, \ldots, u_{n-2} \) and their derivatives, having the property that the commutator appearing
on the right hand side of (2.6.4) is of order \( n - 2 \) or less. Equation (2.6.4) can be regarded as a collection of \( (n - 1) \)-partial differential equations for the coefficients \( u_k \), where the dot is interpreted to mean differentiation with respect to a (time) parameter \( t \), for example for \( n - 2 \), choosing \( L = D^2 + u \), \( P = D^3 + \frac{3}{2}uD^2 + \frac{3}{4}Du \), equation (2.6.4) is equivalent to the KdV equation for the coefficient \( u \), \(-iu = \frac{1}{4}D^3u + \frac{3}{4}Du\). Let \( m \) be any positive integer. Then \( L^{m/n} \) is a \( u \)-symbol of order \( m \) whose homogeneous pieces have integral degree.

**2.6.2 Lemma:** \([L^{m/n}_+, L] \) where \((L^{m/n})_+ = L^{m/n}_+\) is the differential part of \( L^{m/n} \) is a differential operator of order \( < n - 2 \).

**Proof:** 
\[ [L^{m/n}_+, L] = 0, \text{ hence } [L^{m/n}_+, L] = -[L^{m/n}_-, L]. \] The right hand side is of order \( < -1 + n - 1 = n - 2 \).

Suppose \( u_k \) depends on the parameter \( t \). Then the equation \( L = [L^{m}_+, L] \) \((L = \partial L/\partial t)\) makes sense because \( L \) is an operator of order \( n - 2 \) as in the right hand side. The set of all these equations, for all \( m \), is called the \( n \)-th KdV hierarchy (for \( n = 2, m = 3 \), we obtain the KdV equation).

The \( k \)-th flow of the hierarchy is defined by the Lax equation,

\[
\partial_t^k L_n = ((L_n^{k/n})_+, L_n) = [L_n, (L_n^{k/n})_-], \quad \partial_t = \partial/\partial t_k \tag{2.6.6}
\]

From (2.6.6), it can also be written as

\[
0 = \partial_t^k L_n - ((L_n^{k/n})_+, L_n) = \sum_{i=0}^{n-1} L_n^{i/n} \left( \partial_t^k L_n^{1/n} - ((L_n^{k/n})_+, L_n^{1/n}) \right) L_n^{(n-i-1)/n},
\]

from which it follows that

\[
\partial_t^k L_n^{1/n} = ((L_n^{k/n})_+, L_n^{1/n}).
\]

Alternately, we can write \( \partial_t^k L_n^{1/n} = ((L_n^{k/n})_+, L_n^{1/n}) \tag{2.6.7} \) for any arbitrary integer \( l \).

From equation (2.6.6) and (2.6.7), we have that \( \partial_t \partial_t^k L_n = \partial_t \partial_t^k L_n \). In other words,
any two operators in the hierarchy commute. Therefore, we can solve all these equations simultaneously, obtaining \( L(t_1, t_2; \cdots) \).

Now taking the trace of equation (2.6.7) and using the relation (2.6.3), we obtain
\[
\partial_t \text{Tr}(L_{n}^{l/n}) = 0.
\]
If we define \( H_l = \frac{n}{l} \text{Tr}(L_{n}^{l/n}) \) \( l \) , these are conserved under any flow and these are the constants of motion (or first integrals) of all the equations of the \( n \)-th hierarchy.

We now describe the Lax pair associated with \( L^{m/n} \). We set \( L^{m/n} = P_m + N_m \) where
\[
(A_p \text{ is defined as follows: Let } R(\lambda) \text{ be the resolvent symbol for } L \text{ defined by } R(\lambda) = L - A) = \lambda. \text{ We define the symbol } L^s, \text{ for complex } s, \text{ by } L^s = \frac{1}{2\pi i} \oint \lambda^s R(\lambda) d\lambda \text{ where } 0 \text{ is the contour from } ReX = \frac{1}{2} \text{ in the semicircle } |A| = \frac{1}{2} \text{ in the counterclock wise direction to } ReX = -\frac{1}{2}. \text{ We obtain } L^s = \sum_{p=0}^{\infty} A_p(s) \text{ by evaluating the above integral where } \text{ord} A_p = n \text{ Res } s - p.
\]
Here \( L \) is defined by \( L(q, \xi) = \xi^n + \sum_{k=0}^{n-1} u_k \xi^k \) and \( A(x, \xi) = \sum_{i=0}^{\infty} A_l(x, \xi) \).

Thus \( P_m \) is a polynomial \( u \) symbol and \( \text{ord} N_m < -1 \). If \( m \) is not divisible by \( n \), then \( A_p(\frac{m}{n}) = 0 \) for \( p > m + 1 \). Since \([I, L^{m+n}] = 0\), we have \([P_m, I] = [I, N_m]\). Since the left hand side of this equation is polynomial \( u \) symbol, so is the right hand side. Also, since the right hand side has order \(< n - 2\), so does the left hand side. Thus \( L = [P_m, L] \) is a Lax equation for each positive integer \( m \). This is Gel’fand-Dikii’s construction of Lax pairs [15,16].

2.7 THE GEODESICS ON AN ELLIPSOID: (a) Let \( A \) be a positive definite symmetric \( n \times n \) matrix with distinct eigenvalues and \( x \in \mathbb{R}^n \) be a vector. Then the \((n - 1)\)-dimensional ellipsoid has equation
\[
< A^{-1}x, x > = 1.
\]
Also the differential equation of the geodesics is given by

$$\frac{d^2 x}{dt^2} = -\nu A^{-1} x$$

where $\nu = \frac{< A^{-1} y, y >}{|A^{-1} x|^2}$, $y = \frac{dx}{dt}$ (2.7.2)

Further the family of confocal quadrics related to the ellipsoid (2.7.1) is given by the equation

$$< (z - A)^{-1} x, x > + 1 = 0$$

(2.7.3)

Introduce the notation

$$Q_z(x, y) = < (z - A)^{-1} x, y >, \quad Q_z(x) = Q_z(x, x)$$

(2.7.4)

Note that $Q_0(x) + 1 = 0$ is the ellipsoid (2.7.1) and (2.7.3) can be written as $Q_z(x, x) I = 0$. Let $L(x, y)$ be an isospectral symmetric matrix of $A$ obtained by a process of rank 2 perturbation (which will be explained later) which depends on two vectors $x, y \in \mathbb{R}^n$. Define

$$\frac{|y|^2}{z} \frac{\det(zI - L)}{\det(zI - A)} =: \Phi_z(x, y)$$

(2.7.5)

which is a rational function of $z$ with poles at the eigenvalues $\alpha_1, \cdots, \alpha_n$ of $A$ and zeros at $\lambda_1, \cdots, \lambda_n$, the non-trivial eigenvalues of $L(x, y)$. In fact, one eigenvalue of $L$ is $\lambda_n = 0$ with corresponding eigenvector $y = \frac{dx}{dt}$ and these eigenvalues $\lambda_1, \cdots, \lambda_n$ are preserved by the geodesic flow (2.7.2) (here note that we denote by $x$ the position vector on the ellipsoid and its velocity by $y = \frac{dx}{dt}$). As a function of $x, y$, $\Phi_z$ is a quartic polynomial. The partial fraction expansion of $\Phi_z(x, y)$ corresponding to its poles $\alpha_1, \cdots, \alpha_n$ is

$$\Phi_z = \sum_{j=1}^{n} \frac{G_j(x, y)}{z - \alpha_j}$$

(2.7.6)

$G_j(x, y)$ ($j = 1, \cdots, n$) are quartic polynomials of $x$ and $y$ which axe the integrals of the flow (2.7.2). In fact only $n - 1$ of them are independent on the ellipsoid, since there is a relation $\Phi_0 = -\sum_{i=1}^{n} \alpha_i^{-1} G_j(x, y)$ among them. For a given parameter $z$ we denote
the quadric $Q_z(z) + 1 = 0$ in the confocal family of (2.7.1) (i.e., that particular member of (2.7.3)) by $U_z$.

Consider the eigenvalue equation $\Phi_z(x, y) = 0$ defined by (2.7.5). We have the following identities connecting $\Phi_z$ and $Q_z$ as

$$\Phi_z(x, y) = Q_z(y)(1 + Q_z(x)) - Q^2_z(x, y), \quad (2.7.7)$$

so that for a fixed $z$ and $x$, this represents a quadratic form in $y$. The equation $\Phi_z(x, y) = 0$ represents the quadratic cone of tangents to the quadric $U_z$, passing through the point $x$, after the point $x$ is translated to the origin. Also we have,

$$\Phi_z(x + sy, y) = \Phi_z(x, y),$$

so that $\Phi_z$ is constant along any line $x = x_0 + sy, \ y \neq 0$ (*). Hence we get that for a given line $x = x_0 + sy$, the roots $z = \lambda_1, \ldots, \lambda_{n-1}$, of the equation $\Phi_z(x_0, y) = 0$ are such that the above line (*) is tangent to the confocal quadrics $U_{\lambda_j} \ (j = 1, \ldots, n - 1)$ (†).

That is the equation

$$Q_z(y)(1 + Q_z(x)) - Q^2_z(x, y) = 0$$

is the equation of tangency of the confocal quadric family. We consider the Hamiltonian system

$$\frac{dy}{dx}$$

restricted to the surface $\Phi_z = 0$. Then the differential equation (2.7.2) can be expressed as

$$\frac{d^2(x + sy)}{dt^2} = k \nabla Q_x$$

at the point $x_0 + sy, \ k$ constant. That is, this differential equation governs the motion of the tangents to the hyperquadric $Q_z(x) + 1 = 0$, i.e., $U_z$ along the geodesics by (†). In otherwords, the geodesic flow is obtained by just following the motion of the point of
tangency, \( x_0 + sy \) by reducing the system by \( \sqrt{\lambda^2} \) by an integral process. If we put \( z = 0 \) in this, we get the geodesic flow on the ellipsoid \( Q_0(x) + 1 = 0 \).

2.7.8 Remarks: (1) Geometric preparation: First we note that, a given line in \( \mathbb{R}^n \) touches exactly \( n - 1 \) con focal quadrics. The set of all lines tangent to these \( n - 1 \) quadrics \( U_{\lambda_1}, U_{\lambda_2}, \cdots, U_{\lambda_{n-1}} \) is called a normal congruence. Then the spectrum of \( L(x, y) \) can be given the following geometrical interpretation: The “isospectral manifold” of matrices \( L(x, y) \) with a fixed distinct spectrum \( \lambda_1, \cdots, \lambda_{n-1} \) is identified with the normal congruence of common tangents to \( n - 1 \) confocal quadrics \( U_{\lambda_j} \) \( (j = 1, \cdots, n - 1) \).

(2) The eigenvalue \( \lambda_n = 0 \) corresponds to the eigenvector \( \phi_n = y = \frac{dy}{dt} \) and the other eigenvalues \( \phi_j \) (corresponding to \( \gamma \)) are the normals of \( U_{\lambda_j} \) at the point of contact of the line \( x = x_0 + sy \). Since \( L = L(x, y) \) is a symmetric matrix, these \( n \) vectors are pairwise orthogonal. Then under the geodesic flow (2.7.2) the orthogonal frame \( (\phi_j)_{j=1}^n \) will undergo a motion given by a skew- symmetric matrix \( B \) so that

\[
\phi_j = B\phi_j, \quad L = [B, L],
\]

which is the Lax representation of the geodesic flow where

\[
B = -(\alpha_i^{-1}\alpha_j^{-1}(x_i y_j - x_j y_i)).
\]

(3) Rank 2 perturbation of a given symmetric matrix \( A \) (Adler-Moser approach):
Let \( A \) be a fixed symmetric matrix and \( x, y, \xi, \eta \) be four \( n \)-vectors. We call \( A + x \otimes \xi + y \otimes \eta \) a rank 2 perturbation of \( A \). Take \( \xi = ax + by, \eta = cx + dy \), where \( a, b, c, d \) are reals with \( A = ad - 6c \neq 0 \). Then define

\[
L(x, y) = A + ax \otimes x + bx \otimes y + cy \otimes x + dy \otimes y
\]

which is a matrix depending on two \( n \)-vectors, \( x \) and \( y \). By the isospectral manifold of \( L(x, y) \), we mean the algebraic manifold \( \mathcal{M}(\lambda_1, \cdots, \lambda_n) \) consisting of those \( x, y \in \mathbb{R}^n \).
Chapter 2. Complete Integrability of some integrable systems (CIHS)

for which $L(x,y)$ has the fixed spectrum $\lambda_1, \cdots, \lambda_n$. Let $\omega = \sum_{j=1}^{n} dy_j \wedge dx_j$, be the symplectic 2-form on $\mathbb{R}^{2n}$. Then the eigenvalues of $L(x,y)$ of (2.7.11) are in involution with respect to $\omega$, i.e., $\{\lambda_j, \lambda_k\} = 0$ where the corresponding Poisson bracket is defined by $\{F, G\} = \sum_j (F_{x_j} G_{y_j} - F_{y_j} G_{x_j})$ the standard one. Again we can consider, even the symmetric functions of the eigen values $\lambda_j$ or even the more general function of $G_j^s$ in

$$
\Phi_j(x,y) = \sum_{j=1}^{n} \frac{G_j(x,y)}{z - \alpha_j} = 1 - \frac{\det(zI - L)}{\det(zI - A)},
$$

which are quartic polynomials in $x, y$. Hence, as in (a) above, we have $n$ quartic polynomials $G_j$ which are in involution.

(4) Let $// = \phi(G_1, \cdots, G_n)$ be any Hamiltonian function in these or even any Hamiltonian function depending on the spectrum of $L$ only. Then the corresponding Hamiltonian vector field $X_H$ is tangential to the isospectral manifold $\mathcal{M}_\lambda$ and

$$
X_H = \sum_{j=1}^{n} \frac{\partial H}{\partial G_j} X_{G_j},
$$

where we also have $[X_{G_j}, X_{G_k}] = -X_{\{G_j, G_k\}} = 0$ and hence all these vector fields commute. Thus $\mathcal{M}_\lambda$ is a Lagrange manifold and all these Hamiltonian systems are integrable. i.e., the corresponding vector field $X_H$ of this system admits $n$ integrals $G_j$ in involution for which $dG_j (j = 1, \cdots, n)$ are independent over a dense open subset.

(b) We close this section by giving the integrals for the geodesic flow on the ellipsoid. Let the setup be as above where we have, $\mathbb{R}^n$, $<, >, A$, a positive symmetric matrix with distinct eigenvalues. Assume $A = \text{diag}(\alpha_1, \cdots, \alpha_n)$ with $0 < \alpha_1 < \cdots < \alpha_n$. Then $< A^{-1} x, x > = 1$ defines an ellipsoid. The quadrics $\mathcal{U}_z$ confocal to this ellipsoid are given by the equation

$$
< (z - A)^{-1} x, x > + 1 = 0.
$$

(2.7.12)
Consider the bilinear form

\[ Q_z(x, y) = \langle (z - A)^{-1}x, y \rangle, \quad Q_z(x) = Q_z(x, x) \] \hspace{1cm} (2.7.13)

so that \( \mathcal{U}_0 \) is defined by

\[ Q_z(x) + 1 = 0 \] \hspace{1cm} (2.7.14)

and note that (2.7.12) is simply \( \mathcal{U}_0 \). Through any point \( x = (x_1, \cdots, x_n) \) with \( x_1^2 \cdot \cdots \cdot x_n^2 \neq 0 \) there pass exactly \( n \) confocal quadrics which intersect each other perpendicularly.

For any given point \( x_0 \in \mathbb{R}^n \) we ask for the cone of lines which are tangent to a quadric \( Q_z(x) + 1 = 0 \). The equation of this cone is given by

\[
\begin{vmatrix}
1 + Q(x) & 1 + Q(x, x_0) \\
1 + Q(x, x_0) & 1 + Q(x_0)
\end{vmatrix} = Q(x) - 2Q(x, x_0) + Q(x_0) + Q(x)Q(x_0) - Q^2(x, x_0) = 0.
\]

In otherwords, if we set \( y = x - x_0 \) this equation becomes

\[
\begin{vmatrix}
Q(y) & Q(x_0, y) \\
Q(x_0, y) & 1 + Q(x_0)
\end{vmatrix} = Q(y) + Q(x_0)Q(y) - Q^2(x_0, y) = 0,
\]

which for fixed \( x_0 \) describes a cone with vertex at the origin. Now we note that this equation agrees with \( \Phi_z(x_0, y) = 0 \) of the above paragraph (a) with \( a = 0, \ b = -c = 1, \ d = -1 \) (or \( a = 0, \ b = c = i, \ d = -1 \)). Hence we can geometrically understand this equation as the set of lines \( x = x_0 + sy \) tangent to \( \mathcal{U}_z \). In particular, this equation \( \Phi_z(x_0, y) = 0 \) describes the tangents to the ellipsoid \( \mathcal{U}_0 \). Then the Hamiltonian differential equations are

\[
\dot{x} = \frac{\partial}{\partial y} \Phi_0(x, y), \quad \dot{y} = -\frac{\partial}{\partial x} \Phi_0(x, y)
\] \hspace{1cm} (2.7.15)
which when restricted to $\Phi_0 = 0$ describes the motion of such tangent lines and the point of contact with $U_0$ moves along a geodesic while the point $x$ moves perpendicular to this tangent. In fact, if the line through $x$ in the direction $y \neq 0$ has the point of contact $x + sv = \xi$ with $U_0$, then we have

$$Q(x + sy, y) = 0, \quad \text{or} \quad s = -\frac{Q(x, y)}{Q(y)}.$$  

Then

$$\frac{d}{dt} \xi = \frac{d}{dt} (x + sy) = \dot{x} + sy + s \dot{y}$$

$$= \frac{2\Phi_0(x, y)}{Q(y)} A^{-1} y + \dot{s} y = \dot{s} y,$$

since $\Phi_0(x, y) = 0$, and

$$\frac{dy}{dt} = -2Q(y)A^{-1}x + 2Q(x, y)A^{-1}y = -2Q(y)A^{-1}\xi.$$  

Introduce $\tau$ by $d\tau/dt = \dot{s}$, then

$$\frac{d\xi}{d\tau} = y, \quad \frac{d^2\xi}{d\tau^2} = \frac{dy}{d\tau} = -\frac{2Q(y)}{s} A^{-1} \xi$$

and so the point of contact $\xi = \xi(t)$ moves on a geodesic. From the equations

$$\langle \dot{x}, y \rangle = 2\Phi_0(x, y) = 0, \quad \langle y, y \rangle = 0,$$

we get that $x$ is perpendicular to $y$, i.e., the direction of this line and that $\langle y, y \rangle$ is a constant. Thus (2.7.15) can be viewed as an extension of the geodesic flow on the ellipsoid to a flow in $T^*\mathbb{R}^n = \mathbb{R}^{2n}$. Putting the other way, the geodesic flow is obtained by constraining (2.7.15) to the symplectic manifold $Q_0(x, y) = 0, \quad |y|^2 = \text{constant} > 0,$ and to the energy manifold $\Phi_0(x, y)$. Then the relations $\Phi_0 = 0, Q_0(x, y) = 0$ are equivalent to

$$Q_0(x) = 1 = 0, \quad Q_0(x, y) = 0, \quad \text{if} \quad Q_0(y) \neq 0.$$
which is the tangent bundle description of the ellipsoid (cf. paragraph (a) above). This constrained flow takes place on this tangent bundle. To establish the geodesic flow as an integrable system, it suffices to show that the extended flow (2.7.15) is integrable. This follows as was done in above paragraph with

$$\Phi_0(x, y) = \sum_{j=1}^{n} \frac{G_j}{z - \alpha_j}$$

and hence these $G_1, \ldots, G_n$ are the integrals of motion which are in involution.

2.7.16 Remark: Note that the $n - 1$ roots of $\Phi_z(x, y)$ are the eigenvalues of $L$ and the $n$-th eigenvalue $\lambda_n = 0$ corresponds to the eigenvector $y$. The matrix $L$ undergoes isospectral deformation under the flow (2.7.15) in Lax form $\frac{d}{dt}L = [B, L]$ with an appropriate matrix $B$. More generally, define

$$H = \frac{1}{2} \sum_j \beta_j y_j^2 + \frac{1}{2} \sum_{i<j} \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j} (x_i y_j - x_j y_i)^2 = -\frac{1}{2} \sum_{j=1}^{n} \beta_j G_j$$

(2.7.17)

with arbitrary constants $\beta_1, \ldots, \beta_n$. Note that $\beta_j = 2\alpha_j^{-1}$ we get $H = \Phi_0$ the one considered above. Then for this function $H$, we get the Hamiltonian system $x = H_y, y = -H_x$ with $H$ given in (2.7.17) can be put in the matrix form $\frac{d}{dt}L = [B, L]$ (2.7.18) where $L$ is given by

$$L(x, y) = \left\{ I - \frac{y \otimes y}{\langle y, y \rangle} \right\} (A - x \otimes x) \left\{ I - \frac{y \otimes y}{\langle y, y \rangle} \right\}$$

and

$$B = -\left( \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j} (x_i y_j - x_j y_i) \right)$$

with zeros on the diagonal. This $L$ and $B$ can be put in tensor product notation which will be done in chapter 3 with details by introducing suitable notations.