INTRODUCTION AND SUMMARY

The breadth of integrable systems is that it ranges over mechanics, differential equations, global analysis, algebraic geometry and Lie theory and these are just some of their mathematical aspects, ignoring the vast intersection with physics. The subject accommodates a whole range of points of view from very "pure" to very "applied".

Integrable systems first appeared as mechanical systems for which the equations of motion could be solved by quadratures, i.e., by a sequence of operations which included only algebraic operations, integration and application of the inverse function theorem. Apart from some non-trivial examples which were constructed before, the first main result (due to Liouville, but essentially an application of a result due to Hamilton) was that if a mechanical system with \( n \) degrees of freedom of the form

\[
\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (i = 1, \cdots, n)
\]

(\( H \) any function in the co-ordinates \( q_i, p_i \)) has \( n \) independent integrals in involution then it can be solved by quadratures. Two functions \( f \) and \( g \) are said to be in involution if their Poisson bracket

\[
\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}
\]

vanishes, \( \{f, g\} = 0 \) and \( f \) is called an integral of the system if \( f \) and \( H \) are in involution.

Mechanical systems which satisfy the conditions of Liouville's theorem are called Liouville integrable or integrable in the sense of Liouville. A quite short - but important - list of (non-trivial) examples of Liouville integrable systems were found during the last century: a few integrable tops (the Euler top, the Lagrange top, the Kowalevski top, free motion of a particle on an ellipsoid (Jacobi), motion of a rigid body in an ideal fluid (Kirchhoff and Steklov case), motion in the field of a central potential (Newton) and few others. Both finding these systems (i.e., showing that enough integrals exist, which was
clone by constructing them) and solving them explicitly by quadratures required a lot of ingenuity and often quite long calculations. In the more complicated cases the solution was written down in terms of two-dimensional theta functions by a non-trivial use of the rich analytical properties of these functions. In turn it motivated the research in theta functions and Abelian varieties, which originated in the beginning of that century in the works of Riemann and Abel.

Traditionally integrable systems are considered as differential geometric objects. The phase space is a smooth (or analytical) manifold, equipped with a symplectic structure and the functions in involution are smooth (or analytic) functions.

In the above classical definition of integrability in the sense of Liouville the existence of a sufficient number of integrals in involution among themselves and with a given function (the Hamiltonian) is demanded, sufficient meaning equal to the degrees of freedom of the system. For integrable Hamiltonian systems, it is better to consider the algebra of functions generated by the integrals and if the integrals are in involution, then this whole algebra is involutive. Giving only the function (Hamiltonian) does not suffice to determine the whole algebra, which confirms that the integrable system should consist of an algebra and not of a single function. Having a sufficient number of functions in involution corresponds to this algebra having maximal dimension. The algebra should be complete in the sense that every function which is in involution with all elements of the algebra actually belongs to the algebra.

Thus a completely integrable system is a Hamiltonian system that admits the maximum possible number of first integrals, i.e., if the notion of integrability means the existence of integrals of motion, then complete integrability means that these integrals exist in sufficient number. For a system of $n$ first order autonomous ordinary differential equations, sufficient means $n - 1$ time-independent (where the system can be reduced to a single quadrature) invariants or $n$ time-dependent ones (in which case the solutions can be
obtained by solving an algebraic problem). The study of completely integrable systems proceeds in three stages: (i) identification of the symplectic structure which gives the system its Hamiltonian character; (ii) identification of first integrals (or constants of motion or action variables); (iii) identification of a complementary set of variables, called angle variables and computation of their evolution under the various Hamiltonian flows associated to the first integrals if possible in terms of elementary functions. The symplectic manifolds on which the systems are defined are orbits of the co-adjoint action of a Lie group $G$ on the dual of its Lie algebra, with the natural symplectic structure. Complete integrability is strongly related to either Lie algebra or algebraic curve theory. For instance, in chapter 4, it is shown that both KdV equation and the Toda system can be viewed as Hamiltonian systems on the co-adjoint orbit of a Lie group with Kostant-Kirillov structure and the complete integrability of these systems can be traced to a single abstract Lie algebra theorem. The main thrust of the method is to associate with all the above Hamiltonian systems a Lax matrix differential equation which contains a parameter, i.e., equations of the form $A = [A, B] = AB - BA$, (†) where $A$ and $B$ are matrices whose entries depend on the phase space variable and are polynomials in the indeterminate $h$ and $h^{-1}$. Then the curve in $(h, z)$ space $X : \det(A - z) = Q(z, h) = 0$ is formed whose coefficients are functions of the phase space. From (†), the curve $X : Q(z, h) = 0$ (of genus $g$) is time independent. (cf.section 3.3), i.e., its co-efficients are integrals of the motion (†). Then we linearise (†) and its associated flows on the $\text{Jac}(X)$ of $X$. (cf.section 3.4)

Chapter 1 is concerned with certain partial differential equations and as such partial differential equations is a multi-faceted subject, created to describe the mechanical behaviour of objects such as vibrating strings and blowing winds, it has developed into a body of material that interacts with many branches of mathematics such as differential
equations, complex analysis, harmonic analysis and a very important factor in the de-
scription and elucidation of problems in Mathematical Physics. This branch of partial
differential equations in Mathematical Physics was studied by great mathematicians like
Riemann, Jacobi, Weierstrass and Poincare etc.

For the last two decades certain special non-linear equations of Mathematical Physics like
the Strum-Liouville equation, equations of non-linear string, the Korteweg-deVries(KdV)
equation, the Kadomtsev-Petviashvili(KP) equation for dispersive waves in shallow wa-
ter theory were extensively investigated by the Soviet school[23,25,37] and the Courant
School [40].

Several interesting connections were discovered in these works. For example, there are
interlinks between Classical Mechanics, in particular infinite- dimensional integrable sys-
tems[37], spectral theory of differential operators[17,28,40] and algebraic geometry, in
particular, geometry of complex tori [6,22,30,45,61].

In chapter 1, we discuss the solutions of the KdV and KP equations which are periodic
and conditionally periodic and these solutions live on the Jacobian variety of a compact
Riemann surface. Section 1.1 is concerned with basic definitions and results from com-
plex analytic theory of a compact Riemann surface, divisors, abelian variety, Jacobian
variety and the Abel map which gives the necessary and sufficient condition for a divisor
of degree zero to be a principal divisor. We have also defined the Riemann-theta function
and studied its properties. In section 1.2, we give a brief account of the KdV equation,
the KP equation and the Boussinesq equation of which the KdV equation arose in the
nineteenth century in connection with the theory of waves in shallow water and the KP
equation which is a more general form of the KdV equation. In section 1.3, we intro-
duce and study the Akhiezer functions and their relation with the KdV equations and
Riemann-theta functions. We have proved the theorem (Theorem: 1.3.20) in which the
solutions of the KP equation are expressed in terms of the Riemann-theta function.
Section 1.4 is like an inverse problem of section 1.3 where we construct the operators $L_1, L_2$ using the data (of an algebraic curve (CRS), Akhiezer function and special divisors). We have proved that for each Akhiezer function $\Psi$, $\exists$ unique pair of differential operators $L_1$ and $L_2$ of form (1.4.1) such that $L_1 \Psi = \frac{\partial}{\partial t} \Psi, L_2 \Psi = \frac{\partial}{\partial y} \Psi$ (Theorem 1.4.5).

We have also discussed some special cases in the above two sections. In section 1.5, we give some remarks on the interconnection of the KP (KdV) equations theory with other topics in dynamical systems and deeper areas of algebraic geometry. In summary we proved here a common transcendental solution for the partial differential equations we considered (Theorem 1.3.20 and Theorem 1.4.5) in this chapter.

Subsequent chapters, starting with chapter 2, are concerned with the complete integrability of some integrable (Hamiltonian) systems (CIHS). Here we have introduced systems such as (a) The Toda Lattice (b) the Lagrange top (c) the geodesic motion on an ellipsoid in the finite dimensional case and (d) the KdV and the generalised KdV equations and (e) the Gel’fand-Dikii system in the infinite dimensional case, where we have given the description of these systems in the classical sense. The main purpose of chapter 2 is to introduce the above systems and describe them from classical view point and all these systems will be discussed in subsequent chapters from several other angles. In section 2.1, we have given an introduction to the various aspects of Poisson brackets where the purpose is to lay down the differential geometric basics for the use of Poisson brackets in the subsequent parts of the thesis, where the emphasis is on the integrable (Hamiltonian) systems. Section 2.2 deals with various forms of Hamiltonian structures via Poisson structures or sympletic structures used in the general theory of integrable Hamiltonian systems and also the natural sympletic structures on the co-tangent bundle and on the dual of a Lie algebra are given. In section 2.3, we have discussed the example of the Toda lattice, invented by M.Toda around 1968 and the Poisson bracket on it and the corresponding Hamiltonian structure are given.
Introduction and Summary

Section 2.4 deals with the example of the three dimensional rigid body motion about a fixed point under the influence of gravity which has the Lagrange top as a special case. In section 2.5, we have discussed the KdV equations and the generalized KdV equations for which we have given the Hamiltonian structure and the Poisson bracket associated with it. Section 2.6 deals with the Gel’fand-Dikii systems which are generalizations of KdV equations to the $n$-th order partial differential operators. The higher order flows of these KdV hierarchy are also given (cf. 2.6.6). These have remarkable properties such as that they are infinite-dimensional Hamiltonian systems, having infinitely many constants of motion in involution, which are also integrals of local polynomial densities.

Finally in section 2.7 of chapter 2 we have introduced the differential equation for geodesics on an ellipsoid and also the confocal quadric family of this ellipsoid. First we considered an ellipsoid defined by a positive definite symmetric matrix $A$ of order $n \times n$ and then passing to its isospectral matrix $L(x, y)$ got a special rational function $\Phi_z(x, y)$ whose partial fraction expansion gives the $n$ first integrals of the geodesic flow (cf. 2.7.6). The eigenvalues of $L$ are related to the common tangent cone of a certain family of confocal quadrics. The method of rank 2 perturbation of matrix $A$ to get $L(x, y)$ is explained and the Hamiltonian form and Lax form of geodesics equations are given. Then the Hamiltonian equations of geodesics for ellipsoid in $\mathbb{IR}^n$ were extended to the cotangent bundle $T^*\mathbb{IR}^n = \mathbb{IR}^{2n}$ and using the standard symplectic form on it, the integrals of motion of the geodesic were obtained and generalized (cf part(b) of (2.7)).

Chapter 3 is concerned with Lax representation, cohomological interpretation and linearization of flows of some completely integrable Hamiltonian systems (CIHS) given in chapter 2. In section 3.1, we discuss the Lax equations of the KdV and the KP systems from the viewpoint of isospectral deformations of ordinary differential operators with formal power series coefficients. In otherwords, following the work of Gel’fand and Dikii closely the general isospectral deformation theory of general differential operators in the
larger set of pseudo-differential operators was developed first and then applied to special partial differential equations or operators arising out of them. The geometric way to understand the KP equation

\[
\frac{3}{4} u_{yy} - (u_t - \frac{1}{4} u_{xx} - 3uu_x)_x = 0
\]  

is to introduce a large system of non-linear equations which contain the KP equation (1) as the first equation. This system of equations, called the KP system or KP hierarchy are the equations that describe the universal deformations of ordinary differential equations (cf. Remark 3.1.17). In remark (3.1.16), if we set \( P = d^2 + 2u \), then we get the KdV equation

\[ u_t - \frac{1}{4} u_{xxx} - 3uu_x = 0 \]

In section 3.2, we discuss the Lax equations with a parameter associated with some of the completely integrable Hamiltonian systems discussed in chapter 2. The Euler equations of motion for the \( n \)-dimensional free rigid body about a fixed point as a Hamiltonian system on adjoint orbits of \( so(n) \) are given in this section, where we have proved the theorem (cf. Theorem 3.2.5). In this case if we put \( n = 3 \), then we get the usual Euler equations (cf. Remark 3.2.7). For the three dimensional heavy rigid body, it is noted that the Euler - Poisson equations can be written in Lax form [48] with variables formal matrix polynomials if and only if the equations describe the Lagrange top (two of the principal moments of inertia are equal and the center of mass is on the axis of symmetry of the body) or the heavy symmetric top (all three moments of inertia are equal) (cf. Theorem 3.2.10). We also discuss the examples of the Toda lattice (3.2.12) and the geodesic motion on an ellipsoid (3.2.13) in this section.

In section 3.3, we discuss the spectral curves associated with the Lax equations with a parameter \( \xi \), for the examples discussed in section 3.2 for which we have given some preparation from algebraic geometry ([6], [20],[26]). That is, suppose we have a Lax equation
with a parameter $\xi$, $\frac{d}{dt} A_\xi = [A_\xi, B_\xi]$ where $A$ and $B$ are complex or real matrices which have entries in the ring of real or complex Laurent polynomials in the variable $\xi$, which is called the spectral parameter. This Lax equation has many first integrals since the matrix $A$ stays in the same conjugacy class, its eigenvalues will be ‘constants of motion’. In other words, the co-efficients of the characteristic polynomial of $A$ are first integrals. Then the characteristic polynomial is a polynomial in two variables $Q(\xi, \eta) = \det(\eta I - A_\xi)$ which is a complex curve $C$ of the equation $Q(\xi, \eta) = 0$ and which is called the spectral curve. It describes the eigenvalues, the spectrum of $A_\xi$. The co-efficients in the equation of $C$ are the first integrals and so there are many spectral curves, one for each value of the set of integrals and $Q(\xi, \eta) = 0$ describes a family of curves corresponding to the curve $C$. For a general point $p = (\xi, \eta) \in C$, if we assume $\dim \ker \| \eta I - A(\xi, t) \| = 1$, then there is a uniquely determined vector $\nu(p, t) \in V$ (upto non-zero scalars) satisfying $A(\xi, t) \nu(p, t) = \eta \nu(p, t)$ (cf. 3.4.2) and the correspondence $p \rightarrow \mathbb{C} \nu(p, t)$ $C V$ determines a family of holomorphic mappings depending holomorphically on $t, f_t : C \rightarrow \mathbb{P}V$, which are called the eigenvector mappings associated to the given Lax equation (cf. section 3.4). For the rigid body motion, the Lagrange top, the Toda lattice and the geodesics on ellipsoid the corresponding spectral curves are given along with their genus.

Section 3.4 deals with the cohomological interpretation of Lax equations and the linearization of flows. The aim is, given a Lax equation with a parameter, we associate to it an algebraic curve $C$ (its spectral curve) together with a dynamical system or flow $\{L_t\}$ on its Jacobian $J(C)$. Then give the necessary and sufficient condition on $B$ in the Lax pair $(A, B)$ that the flow $t \rightarrow L_t$ be linear on $J(C)$. The answer to this lies in Theorem (3.4.13) and Theorem (3.4.20). The eigenvalues of $A(\xi, t)$ are fixed as time evolves and the eigenvectors of $A(\xi, t)$ will change with $t$. This leads to the eigenvector mappings $f_t : C \rightarrow \mathbb{P}V$ and we set $L_t = f_t^*(\mathcal{O}_{\mathbb{P}V}(1)) \in \text{Pic}^d(C)$ where $\text{Pic}^d(C)$ is the set of line bundles of degree $d$ on $C$. Choosing $L_0 \in \text{Pic}^d(C)$, we have the map
L → L ⊗ L_0^{-1} where under an isomorphism $\text{Pic}^d(C) = J(C)$ and the canonical isomorphism $T_L(Pic^d(C)) \cong H^1(\partial_C)$. Then the condition on $B$ becomes that the acceleration vector $\frac{d^2 L_t}{dt^2}$ be a multiple of $\frac{dL_t}{dt}$, i.e., $\frac{d^2 L_t}{dt^2} = \mu_t \frac{dL_t}{dt}$. The Lax equation is invariant under a substitution $B \rightarrow B + P(A, \xi)$, where $P(x, \xi) \in \mathbb{C}[x, \xi]$ and this suggests that it has invariant cohomology meaning. The necessary and sufficient condition for $\{L_t\} \subset Pic^d(C)$ be linear is given in Corollary (3.4.21) and this is applied to the rigid body motion or Lagrange top; Toda lattice and geodesics on ellipsoid to conclude linearity of flows by showing the vanishing of the residue of $B$ (cf. Examples 3.4.30-3.4.33).

In chapter 4, we discuss the geometric Adler - Kostant - Symes (AKS) principle and its application to some integrable systems such as, the Toda system, the Lagrange top, the geodesic motion on an ellipsoid, the KdV and the generalized KdV equations, and the Gel'fand-Dikii system. In section 4.2, we discuss some preliminaries from symplectic manifold theory and define the Kostant-Kirillov-Souriau orbit symplectic structure (cf. Definition 4.2.6). The KdV and the Toda systems and others as well are completely integrable Hamiltonian systems whose equations of motion are expressible in terms of the Lax isospectral equations. The splitting of a Lie algebra into a vector space direct sum of Lie algebras is responsible for the complete integrability of the above systems and the Lax isospectral equations associated with these systems.

Of the different approaches to the study of integrable systems, our approach here is that, a given non-linear system is written as a Hamiltonian dynamical system with respect to some Hamiltonian structure on the underlying space. (For finite dimensional manifolds, the term 'Poisson structure' is usually preferred, that of 'Hamiltonian structure' being more frequently applied to the infinite dimensional case). For finite dimensional Hamiltonian systems on a symplectic manifold of dimension $2n$, integrability in the sense of Liouville (1855) and Arnold (1974) is defined by the requirement that $3n$ conserved quantities that are functionally independent on a dense open set and in involution, i.e.,
whose pairwise Poisson brackets vanish and the geometric methods can be applied in various ways (cf. section 4.3).

In section 4.4, we discuss the above-mentioned examples of which for the Toda lattice, the relevant group $G$ is the group of lower triangular matrices with nonzero diagonal elements, as contained in $SL(n, \mathbb{R})$. We identify the dual algebra of $(\mathcal{L}, \mathcal{L}^*)$, with the upper triangular matrices through the trace form. The orbit Hamiltonian phase space $\theta_A$ is of the form $\theta_A = \{[U^{-1}AU]_+/U \in G, A \in \mathcal{L}^*\}$, where $[B]_+$ denotes the matrix formed from $B$ by setting its lower triangular entries equal to zero and $A$ is subject to certain conditions (cf. 4.4.1). For the case of the generalized KdV equations, the relevant group $G$ is the formal pseudo-differential symbols of negative type translated by the identity element 1, whose dual $\mathcal{L}^*$ is identified, through a trace form, with the differential symbols of non-negative type, which are identified with formal differential operators. The algebra in which everything takes place is the algebra of formal pseudo-differential operators. The Hamiltonian orbit space is of the same form as $\theta_A$ (cf. 4.4.15).

We have also discussed the examples of the Lagrange top using the Kac-Moody Lie algebra (cf. 4.4.3) and the geodesic motion on ellipsoid based on the polynomials in the indeterminate $\hbar$, $\hbar^{-1}$ with co-efficients in a Lie algebra (cf. 4.4.7). Thus in summary, in chapter 4 the geometric principle of Adler-Kostant-Symes is formulated in complete generality and is proved (cf. Theorem 4.3.1) and then applied to various systems of both finite and infinite dimensions and the corresponding set up and data for various systems is tabulated at the end of chapter 4.

In many problems of physical interest involving partial as well as ordinary differential equations, it is possible to find quantities that are invariants (which are equivalent to first integrals of equations of motion). This chapter 5, in which we study the motion of a non-linear string, that is concerned with the study of the Boussinesq equation and state its integrability in relation with a recursive scheme of Lenard and obtain various
integrals of motion, gives a unified method for finding invariants for a class of equations that includes crystal lattices (the Toda system) and water waves (Boussinesq and KdV equations). The approach here is that at the center of the theory of integrable systems lies the notion of a Lax pair, describing the isospectral deformation of a linear operator. A Lax pair \((L, M)\) is such that the time evolution of the Lax operator \(L = [L, A]\), is equivalent to the given non-linear system. The study of the associated linear problem \(L\psi = \lambda\psi\) can then be carried out by various methods.

Following the above approach, we have discussed the examples of Toda lattice and showed that its continuum limit the Boussinesq equation in partial differential equations. We have also discussed a common method for the construction of Lenard relations which along with Gel'fand - Dikii type operator trace formulae, yield an explicit recursive construction of the heirarchy of the integrable systems associated with each of the above systems. In section 5.1, we discuss a general method of construction to determine the Lenard relations (cf. 5.1.10-5.1.13) which can be applied to other systems (the Toda and Boussinesq systems) and in particular, we have discussed this here for the KdV equations (cf. relation 5.1.12). In section 5.2, we discuss the Boussinesq equation and derive the recursive relations of Lenard which is given by \(A\nabla J = J\nabla\lambda\) for appropriate \(A, J\) and \(L\psi = \lambda\psi\) (cf. Theorem 5.2.20). Section 5.3 is concerned with the Hamiltonian structure of the Boussinesq equation where we have constructed the Hamiltonians \(H_0, H_1, H_2, \ldots\), etc., using the trace functional approach of Adler[5]. Section 5.4 deals with the construction of an operator valued function which yields the infinitesimal generators of the Lax type isospectral deformations associated with the examples discussed in the previous section. Section 5.5 deals with the subhamiltonian system where the flows associated with the Boussinesq equation are restricted to the manifold \(r = 0\). The subhamiltonian system is an integrable system in its own right specifically as \(\phi = A_1\nabla_n E_n, n = 1,2,\ldots\), (cf. Theorem 5.5.6) where \(E_n\) satisfy a certain recursion scheme (cf.relation 5.1.12).
Finally following closely the works of B.A. Kupershmidt and Yu.I. Manin ([31], [32]) we discuss in this chapter 6, the 1-dimensional and 2-dimensional equations for long waves moving in long channel along a free surface with rigid bottom and various generalizations of these and also the associated Benney's equations for the moment functions of horizontal velocity component. We interpret them as completely integrable Hamiltonian systems by giving the Hamiltonian structure and its first integrals by using the AKS principle. Several variations of Benney's equations are also given as integrable systems including the super symmetric case. In the supersymmetric case, we have realized the super Poisson bracket as a non-trivial part of a commutator $[,]$ in a ring of formal Laurent (differential) series with coefficients in an algebra.

In section 6.1, we discuss the mathematical aspects of long non-linear wave propagation on a free surface and we have obtained some special solutions (cf. 6.1.16). The moment equations and the conservation laws for long non-linear waves are also discussed in this section (cf. 6.1.21). Then we define the moment function for the horizontal velocity function $u(x, y, t)$ of long waves by $A_n(x, t) = \int_0^t u(x, y, t)^n dy$ (cf. Definition 6.1.30). The conservation laws can be written in the form $\frac{\partial P_n}{\partial t} + \frac{\partial Q_n}{\partial x} = 0$ ($n = 1, 2, 3$) (cf. 6.1.33), where $P_n$ and $Q_n$ are polynomials in $A_0, A_1, \cdots, A_n$. The equation (6.1.34) for the Benney's equations of long waves satisfied by the moment functions are also described. We also discuss a recursive method of Benney for constructing an infinite number of conservation laws for long waves (cf. Theorem 6.1.40) following the technique of generating function.

Section 6.2 deals with the Lax representation of Benney's equation and application of the geometric AKS principle to the Benney's system. The Benney's system admits a Lax representation given by $L_t = [L, P]$ where $L$ and $P$ are given by

$$L = (1 + \Phi_\xi) \frac{\partial}{\partial x} - \Phi_x \frac{\partial}{\partial \xi},$$
The Benney's system satisfies the AKS principle, that is, they fit into a general scheme of constructing Hamiltonian systems with Lax representation and involutive conservation laws and that the relevant Hamiltonian structure could be identified with the canonical symplectic form on the orbits of the co-adjoint representation of a convenient Lie algebra. Then we have the Benney's system as the quasiclassical limit of the KP system or the generalized KdV equations. That is in the limit $\epsilon \to 0$, $[a, b]_\epsilon = \lim_{\epsilon \to 0} [a, b]_\epsilon$ and the Lie algebra $G_\epsilon$ of Benney system is the quasiclassical limit of the Lie algebra $G_1$ of the KdV algebra equation (cf. 6.2.14).

Section 6.3 discusses the Hamiltonian structure of Benney's system from analytical point of view. Here we have given the construction of an infinite number of polynomial conserved densities $H_i \in A_i + \mathbb{Z}[A_0, \cdots, A_{i-1}], i \in \mathbb{Z}_+$ starting with $H_0, H_1, \cdots$ (cf. Theorem 6.3.3). Here the conservation laws are obtained from a non-linear integral equation involving a parameter. We also give a method of construction for higher Benney equations having an infinite set (cf. 6.3.2) of polynomial conserved densities. In particular, the higher flow equation is given by $A_{i,t} = A_{i+2,x} + A_0 A_{i,x} + (i+1)A_i A_{i,x} + i A_{i-1} A_{1,x}, i \in \mathbb{Z}_+$ (cf. Theorem 6.3.9). We have also discussed the Hamiltonian flows with Hamiltonian structure

$$B_{ij} = i A_{i+j+1,0} \partial_x \partial_j A_{i+j-1}, i, j \in \mathbb{Z}_+, \partial = \frac{\partial}{\partial x}$$

so that the flow $\# m$ can be written as

$$A_{i,t} = \sum_j B_{ij} \left( \frac{\partial \bar{H}}{\partial A_j} \right), \bar{H} = \frac{1}{m} H_m, m \in \mathbb{N}$$

(cf. Theorem 6.3.11). The flows have a common Poisson structure given by $L_t = \{P_+, L\} = \{L, P_-\}(\ast)$, where $L = \xi + \sum_{i=0}^{\infty} A_i \xi^{-(i+1)}$.

In section 6.4, we discuss the supersymmetric Benney's system. That is to understand,
what happens with the flows (6.3.15) when the plane $T^*\mathbb{R}^1 = \mathbb{R}^2$ is extended into the super plane $\mathbb{R}^{2N}$ equipped with the super Poisson bracket

$$\{F, G\} = F_\xi G_x - F_x G_\xi + \frac{1}{2\xi} \sum_{r=1}^{N} D_r(F) D_r(G),$$

(cf. 6.4.1). Then we have defined the super flows (cf. 6.4.2) and the supersymmetric Benney hierarchy is a semi-integrable system meaning that the extended flows do not commute between themselves but nevertheless, they have a common infinite set of polynomial conserved densities. That is, the new hierarchy of flows is not Hamiltonian, despite having a common infinite set of conserved densities (cf. Theorem 6.4.7). In section 6.2, we noted that the Benney's system is a quasiclassical limit of the KP or the KdV system. On the contrary, in the super symmetric Benney's system, the super Poisson bracket is realized as the first non-trivial term of the commutator followed by other symmetric expressions (cf. 6.4.13) and the relation between the super symmetric Benney's system and the super KP hierarchy needs to be explored. Also analogous formula to (6.4.13) are computed for lower dimensional superplanes.

In section 6.5, the Hamiltonian structure of Benney's long wave equations is discussed from a different view point than that was discussed in section 6.3. Here the algebraic approach is followed. The Hamiltonian structure here is defined in a different way by using a special Lie product and the first integrals of motion are obtained. Also, Benney's differential algebra and the operator $B$ is defined in section (6.5.5). In otherwords, here we understand the Hamiltonian structure of Benney's system (6.3) as the formal analogue of the Kirillov structure on the orbits of the co-adjoint representation of Lie algebras(cf.6.5.15). Moreover the Poisson bracket defined here is invariant one rather than the one defined in section (6.3).

Now some comments are in order. This study embodies the results obtained from our
attempt to understand the integrable Hamiltonian systems or partial differential equations by differential and algebraic geometric methods as well as by differential analytic methods and Lie methods. For example chapters 1 and 3 contain several results where algebraic geometric techniques were effectively used. Chapter 4 contains results obtained by us of analytic and Lie theoretic flavour. Chapter 2 simply explains various ways of understanding Poisson and Hamiltonian structures on systems. They were explained clearly from analytic point of view and also various Hamiltonian systems which we studied throughout to apply some unifying principles formulated were described from an elementary standpoint in Chapter 2. The Lie theoretic AKS principle after proving was applied for the systems of chapter 2 and these results obtained are summarized in a tabular form in chapter 4. In chapter 5 we studied Boussinesq equation by analytic methods namely Lenard relations and determining certain invariant coefficients whose integral densities give the conservations. Then we have reinterpreted the Hamiltonian structure of Mckean of the Boussinesq equation in terms of the above method. Further this Lenard scheme is also applied to other systems because Boussinesq equation is the continuous limit of discrete Toda system which we studied throughout.

Finally chapter 6 gives the study of Benney's equations in a very systematic and also a thorough understanding of this topic. The classical flavour of this topic is given first. Then Lax type of understanding of this was given and the infinite family of conserved polynomial densities were computed by analytic methods. Then the Poisson structure and then the Hamiltonian structure of Benney's system were given. The AKS principle setup for Benney system was derived by us. Then the super Benney system was shown to be a semi-integrable system. Then we answered by explicitly relating the super Poisson Bracket and the commutator product on a differential ring Lie algebra a question of Kupershmidt - Manin. Also we computed the above relation for lower dimensional super plane. This section contains many new results. Finally in the last section we have
given an invariant definition of Poisson bracket on a Benney differential algebra which we defined and made into a Lie algebra and related this to Kupershmidt-Manin Poisson bracket via an integral representation. In a sense this chapter is the heart of this thesis as it gives a complete understanding of the Benney system to date.

Throughout the thesis several new proofs were given and new interpretations and better understanding of known facts were inserted throughout. For example in chapter 5, our way of reinterpreting McKean’s conserved quantities for Boussinesq is a correct procedure. Nevertheless it must be mentioned except for chapters 1 and 3 the rest of the chapters have computational flavour rather than usual mathematical theorem like results and this is justified by the fact after all we are studying properties of Hamiltonian systems arising out of physical contexts from classical mechanics, applied mathematics, theoretical physics etc. We have given reasonably good collection of references used even though it may not be complete as these topics were studied by theoretical physicists also. We have made every effort and taken care to minimize the typing mistakes and since this thesis contains lots of computational formulae, still some typing mistakes may exist which escaped our attention. Finally we have followed the usual indexing pattern of Proposition X.Y.Z means in chapter X, in section Y, the proposition Z.