CHAPTER - I

INTRODUCTION AND SUMMARY
1.1 Introduction.

This Thesis deals with some aspects of non-Convex programming and analysis. In the first part of the thesis, the weak, strong and converse duality results for a pair of symmetric dual nonlinear programming problems for semi-locally Pseudo-Convex/semilocally Pseudo Concave functions with an additional feasibility condition are derived. The same result for semilocally strongly Pseudo Convex/semilocally strongly Pseudo Concave functions is also discussed.

In the second part of the thesis the relationship between the known and the new non Convex functions which are more general than the known ones are established.

In the third part of the thesis Convex fuzzy mappings and Convex functionals are discussed.

1.2. Non Convex programming and analysis.

Convexity plays a key role in mathematical programming. Many significant results in mathematical programming have been derived under convexity assumptions. As most
of the real world problems are non convex in nature, the continuing interest in convex programming and analysis is to extend the idea for more general non convex problems.

Solution of non convex programming problems are analytically challenging and practically useful. Though it is analytically hard to solve the general non-convex programming problems, yet the present thrust of research for their solutions is essential in two aspects. At first, a systematic attempt is being done by many authors (for example, see Rockafellar [1970], Martos [1975], and Bazaraa and Shetty [1979] for detail) to extend the domain of application of convex programming and convex analysis results to non-convex situations which are close to their convex counterparts in some respect. Secondly algorithms are being developed for general non-convex programming problems using the established results for convex situations. The main motivation behind such algorithms is that at least they can give a local (if not global) optimum solution to general non-convex problems. Here Convexity assumptions serve as the basis upon which algorithms and axioms for more general non convex situations can be developed. Convexity thus plays the same role as that of linearity in the study of dynamic systems.

In this thesis attempts are made to extend the application domain of some familiar results of Convex programming and analysis to more general situations which are only close to
their Convex counterparts in some respect. Since this thesis deals with various types of non-convex (but close to convex in some sense) problems, it is necessary to summarize their basic definitions. Table 1.1 and Table 1.2 presented at the end of this Chapter give the definitions of some known Convex and generalized Convex functions and some newly defined non-convex functions respectively.

In order to put the contributions of this thesis in its correct perspective, the rest of this chapter contains a concise review of the literature which are only relevant to this thesis. Finally in section 1.3 a brief outline of this thesis is presented.

1.3. Symmetric duality in Non-linear programming:

Duality principle relates to constrained minimization and maximization problems. (One of which is primal and the other is dual) According to this principle, the existence of a solution to one of these problems ensures a solution to other and the extrema of the two problems are equal.

Duality also plays a key role in non-linear programming as it plays in case of linear programming. Several authors have formulated different nonlinear dual problems: some of the important formulations are:
(1) Duality via conjugate functions
   (Fenchel [1949], Rockafellar [1966, 1968, 1969, 1970],
   Whinston [1967]).

(2) Duality via Lagrangian multipliers
   (Geoffrion [1970])

(3) Duality of minimax type
   (Stoer and Witzgall [1970])

(4) Duality of symmetric type
   (Dantzig et al [1968], Bazaraa and Goode [1973])

A pair of symmetric dual nonlinear programming problems
has the following general formulation:

\[(P_0)\) (Primal) : \text{Min } f(x,y) = K(x,y) - y^t \nabla_y K(x,y)\]
   \text{such that } (x,y) \in C_1 \times C_2, \nabla_y K(x,y) \in C^* \]

\[(D_0)\) (Dual) : \text{Max } g(x,y) = K(x,y) - x^t \nabla_x K(x,y)\]
   \text{such that } (x,y) \in C_1 \times C_2, - \nabla_x K(x,y) \in C_i^* \]

where \(K(x,y)\) is real valued twice differentiable function
in \(R^n \times R^m\), \(C_1\) and \(C_2\) are closed convex cones in \(R^n\) and \(R^m\)
respectively, \(C_i^*\) \((i = 1,2)\) is the polar of \(C_i\), \(\nabla_x K(x,y)\)
and \(\nabla_y K(x,y)\) are gradient vectors of \(K(x,y)\) with respect
to \(x\) and \(y\) respectively. For convex/concave function together
with some regularity condition Bazaraa and Goode [1973] have established weak forward and converse duality theorems for the problems \((P_0)\) - \((D_0)\) above. The same results have been obtained by Dantzig et al [1965] for non-negative orthant in place of closed Convex cones.

Mishra et al [1984] studied the above type of duality for more general Pseudo Convex/Pseudo Concave functions with an additional feasibility condition and obtained weak, forward and converse duality results. They also derived the duality results for strongly Pseudo Convex/Strongly Pseudo Concave function where the feasibility condition was not required.

Here first of all the weak, forward and converse duality theorems for a pair of non linear programming problems for semilocally Pseudo Convex/semilocally Pseudo Concave functions under feasibility condition are established. Secondly, the above theorems for semilocally Invex are presented. The detail of the results are presented in Chapter II.

1.4. Generalized convexity and a pair of non linear mixed integer programming problems:

Let us consider the following pair of non-linear programs.
\[(P_0) \quad \begin{align*} 
\text{Max } & \min_{x^1, x^2, y} \left\{ f = K(x, y) - (y^2)^t \nabla y^2 K(x, y) \right\} \\
\text{such that } & x^1 \in U, (x^2, y) \in C_1 \times T, \text{ and} \\
\quad & \nabla y^2 K(x, y) \in C_2^* 
\end{align*} \]

\[(D_0) \quad \begin{align*} 
\text{Min } & \max_{y^1, x, y^2} \left\{ g = K(x, y) - (x^2)^t \nabla x^2 K(x, y) \right\} \\
\text{such that } & y^1 \in V, (x, y^2) \in S \times C_2, \text{ and} \\
\quad & -\nabla x^2 K(x, y) \in C_1^* 
\end{align*} \]

U and V are any two arbitrary sets of integers in $\mathbb{R}^{n_1}$ $(0 \leq n_1 \leq n)$ and $\mathbb{R}^{m_1}$ $(0 \leq m_1 \leq m)$ respectively. $C_1$ and $C_2$ are closed Convex Cones with vertices at the origin and with non empty interiors. $C_1^*$ and $C_2^*$ are the polars of $C_1$ and $C_2$ respectively. $K(x, y)$ is real valued twice differentiable function defined on an open set in $\mathbb{R}^n \times \mathbb{R}^m$ containing $S \times T$ where $S = U \times C_1$ and $T = V \times C_2$. $\nabla x^2 K(\bar{x}, \bar{y})$ and $\nabla y^2 K(\bar{x}, \bar{y})$ denote the gradient vector of $K$ with respect to $x^2$ and $y^2$ respectively at the point $(\bar{x}, \bar{y})$.

This problem was studied by Balas [1970] and Mishra and Das [1980]. While Balas [1970], considered Concave/Convex function and non-negative orthant as the cone, Mishra and Das [1980] have generalised this to any arbitrary cone.
Mishra et al. [1984] have derived the weak, forward and converse duality results for Pseudo Convex/Pseudo Concave functions with a feasibility condition. They have also derived the same results for strongly Pseudo Convex/strongly Pseudo Concave function where the feasibility condition is not required.

Here weak, forward and converse duality theorems for a pair of non-linear mixed integer programming problems are established by considering the function to be semilocally Pseudo Convex and a class of functions called semilocally Invex, which generalizes the class of semilocally strongly Pseudo Convex functions. The detail of this is done in Chapter III.

1.5. Generalization of Convex and related functions:

The concept of convexity was generalized to semilocally convexity by Bǎnging [1977]. In a similar manner Kaul and Kaur [1982] introduced the concepts of semilocally Pseudo Convex and semilocally Quasi Convex functions. Logarithmic Convexity was discussed by Klinger and Mongasarian [1968]. Another new concept, called Harmonic convexity was introduced by Das [1975].

It is natural to think that the concepts like semilocally logarithmic and semilocally Harmonic Convexities can be
introduced and this will fill up a gap in the existing literature. The purpose here is to introduce such concepts. Some other concepts called Harmonic Quasi Convex and Harmonic Pseudo Convex have also been introduced here. The relationships with various known and new concepts are discussed. Details of these are given in Chapter IV.

1.6. Generalized Convexity and related concepts:

Here several new kinds of generalized Convex functions are defined. They are Logarithmic Convex like, Harmonic Convex like, Quasi Convex like, Convex type, Logarithmic Convex type, Harmonic Convex type, Quasi Convex type. Some theorems are established which give relationship and implications among known and new kinds of generalized Convexities. Also some results are established which deal with certain new types of Convex functions over subconvex sets. Further some counter examples have been constructed which show that the concepts introduced in this chapter are more general than those existing in the literature. These are discussed in detail in Chapter V.

1.7. Convex fuzzy mapping:

In 1965, Prof. L.A. Zadeh laid the foundation of fuzzy set theory. Fuzzy set theory is the concepts and techniques which lay a form of mathematical precision to
human thought processes which, in many ways, are imprecise and ambiguous by the standards of classical mathematics. The fuzzy field has bloomed into a many faceted field of inquiry, drawing on and contributing to a wide spectrum of areas ranging from para-mathematics to human perception and judgement. Fuzzy sets have been suggested for handling the imprecised real world problems by using truth values ranging between the usual 'true' and 'false'.

There are classes of objects encountered in the real physical world which do not have precisely defined criteria of membership. For example consider the small natural numbers or all the students with brilliant academic career or all tall men etc. All the classes defined above do not constitute sets in the usual mathematical sense. Yet, the fact remains that such imprecisely defined 'classes' play an important role in human thinking particularly in the domain of pattern recognition, communication of information and abstraction.

L.A. Zadeh in 1965 gave a preliminary concept for these classes of ambiguities. He introduced the fuzzy set which is defined as a class with continuum of grades of membership. The behaviour of fuzzy sets is similar in many sense with that of the ordinary sets but are more general than the later. It has much wider scope of applicability, particularly in the fields of pattern classification and information processing. It provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class of membership.
rather than presence of random variables. Thus a set can be called a fuzzy set but not vice-versa.

The concept of fuzzy sets and fuzzy set operations are as follows.

Let $X$ be a set and $L$ a lattice. In particular $L$ could be the closed interval $[0, 1]$. A fuzzy set $A$ in $X$ is characterised by membership function $\mu_A : X \rightarrow L$ which associates with each point $x \in X$ its 'grade or degree of membership' $\mu_A(x) \in L$. Some definitions are quoted here which will be needed in the sequel.

Let $A$ and $B$ be fuzzy sets in $X$. Then

$$A = B, \text{ if and only if } \mu_A(x) = \mu_B(x) \text{ for all } x \in X,$$

$$A \subseteq B, \text{ if and only if } \mu_A(x) \leq \mu_B(x) \text{ for all } x \in X,$$

$$C = A \cup B, \text{ if and only if } \mu_C(x) = \max \left\{ \mu_A(x), \mu_B(x) \right\}, \text{ for all } x \in X.$$

$$D = A \cap B, \text{ if and only if } \mu_D(x) = \min \left\{ \mu_A(x), \mu_B(x) \right\}, \text{ for all } x \in X.$$

More generally, if $L$ is a complete lattice, then for a family of fuzzy sets
A = \bigcup_{j \in J} A_j : j \in J \} the union \quad C = \bigcup_{j \in J} A_j\) and the intersection \quad D = \bigcap_{j \in J} A_j\) are defined by

\mu_C(x) = \sup_{j \in J} \mu_{A_j}(x), \quad \mu_D(x) = \inf_{j \in J} \mu_{A_j}(x), \quad x \in X.

\text{K}_c denotes the fuzzy set in } X \text{ with membership function } \mu_{K_c}(x) = C \text{ for all } x \in X. \text{ The fuzzy sets } K_1 \text{ and } K_0 \text{ respectively correspond to the set } X \text{ and the empty set } \emptyset.

The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh [1965] and subsequently several authors including Zadeh have discussed various aspects of the theory and applications of fuzzy sets. The concept of Convex fuzzy sets has been discussed in Katsaras and Lin [1977], Lowen [1980] and Nanda [1986]. Here the concept of Convex fuzzy mapping, Logarithmic Convex fuzzy mapping and Quasi Convex fuzzy mapping are discussed. Different results relating to these functions are derived in Chapter VI.

1.8. Convex functionals and related concepts:

Real-valued Convex functionals defined on Convex subsets of real Banach spaces have been extensively studied in the literature. Let } X \text{ be any linear space over the field of real numbers } \mathbb{R}. \text{ A mapping from } X \text{ into } \mathbb{R}\)
is called a real functional. The concept of Convexity for real functionals is known in the literature (see, for example, Luenberger [1970], Rockafellar [1970], Bose and Joshi [1984]). Here some non-Convex which are close to Convex functionals are defined and the behaviour of such types of functionals when they are Gateaux or Fréchet differentable are discussed. To be more precise the concepts like Quasi-, Pseudo-, Logarithmic and Harmonic Convexities for such functionals have been introduced and discussed. The details of these are given in Chapter VII.

1.9. A brief outline of the thesis:

For the purpose of suitable presentation the rest of the thesis is divided into seven chapters. A chapter-wise summary is given below.

In Chapter II, a pair of symmetric non-linear programming problems are solved. First, the weak duality, forward duality and converse duality theorems for a pair of non-linear programming problems for semilocally Pseudo Convex/semilocally Pseudo Concave functions are established under the feasibility condition. Then the weak duality, forward duality and converse duality theorems for semilocally Invex are established.
In Chapter III a pair of mini-max and maxi-mini nonlinear mixed integer programming problems are formulated, which are nonsymmetric in duality sense, but in particular case they reduce to the symmetric form. First of all the problem is solved for semilocally Pseudo Convex functions with an additional feasibility condition. Secondly it is solved for semilocally Invex, which generalizes the class of semilocally strongly Pseudo Convex functions.

In Chapter IV the concepts like semilocally logarithmic and semilocally Harmonic Convexities are introduced. Other concepts like Harmonic Quasi Convex and Harmonic Pseudo Convex are also introduced. The relationships with various known and new concepts are discussed and some counter examples are given to have the complete proof of theorems.

In Chapter V several new kinds of generalized Convex functions are defined. The relationships and implications among known and new kinds of generalized Convexities are established. Some counter examples have been constructed which show that the concepts introduced in this chapter are more general than those existing in the literature.
In Chapter VI the concepts of Convex, Logarithmic Convex and Quasi Convex for fuzzy mappings are introduced and the related theorems are also discussed.

In Chapter VII, Quasi-, Pseudo-, Logarithmic and Harmonic Convexities and some related concepts for real functionals defined on subsets of a real normed linear space are introduced and discussed.

Chapter VIII gives a brief conclusion and certain problems for future research.
Table No. 1

<table>
<thead>
<tr>
<th>Name of the functions</th>
<th>Abbreviation</th>
<th>Definition of the function</th>
<th>References</th>
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<tbody>
<tr>
<td>Convex</td>
<td>C</td>
<td>f is convex if ( f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2) ) for all ( \lambda \in [0,1] ) and all ( x_1, x_2 ) in its domain. If ( f ) is differentiable, an alternate definition is given by ( f(x_1) - f(x_2) \geq (x_1-x_2)^t \nabla f(x_2) )</td>
<td>Rockafellar [1970] Mond [1983]</td>
</tr>
<tr>
<td>Strictly Convex</td>
<td>StC</td>
<td>If strict inequality holds in the above definition for each ( x_1, x_2 ) in the domain, then ( f ) is strictly Convex.</td>
<td></td>
</tr>
<tr>
<td>Logarithmic Convex</td>
<td>LC</td>
<td>A positive function ( f ) defined on a set ( S ) in ( \mathbb{R}^n ) is Logarithmic Convex, if for each ( x_1, x_2 \in S ) and ( 0 &lt; \lambda &lt; 1 ), ( f(\lambda x_1 + (1-\lambda)x_2) \leq (f(x_1))^\lambda (f(x_2))^{1-\lambda} ).</td>
<td>Klinger and Mangasarian [1968]</td>
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<tbody>
<tr>
<td>Strictly Logarithmic Convex</td>
<td>St LC</td>
<td>If strict inequality holds in the above definition for each $x_1, x_2$ in the domain then $f$ is strictly LC.</td>
<td>Ponstein 1967</td>
</tr>
<tr>
<td>Harmonic Convex</td>
<td>HC</td>
<td>A positive function $f$ defined on a Convex set $S \subseteq \mathbb{R}^n$ is HC on $S$, if for each $x_1, x_2 \in S$ and $0 &lt; \lambda &lt; 1$</td>
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<tr>
<td></td>
<td></td>
<td>$f(\lambda x_1 + (1-\lambda)x_2) \leq \frac{1}{\frac{\lambda}{f(x_1)} + \frac{1-\lambda}{f(x_2)}}$</td>
<td>Das 1975</td>
</tr>
<tr>
<td>Strictly Harmonic Convex</td>
<td>StHC</td>
<td>If strict inequality holds in the above definition for each $x_1, x_2$ in the domain then $f$ is strictly HC.</td>
<td>Ponstein 1967</td>
</tr>
<tr>
<td>Quasi Convex</td>
<td>QC</td>
<td>$f$ defined on a convex set $S \subseteq \mathbb{R}^n$ is QC on $S$, if for each $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1,$</td>
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<td></td>
<td></td>
<td>$f(\lambda x_1 + (1-\lambda)x_2) \leq \max (f(x_1), f(x_2))$</td>
<td>Martos 1967, Cottle and Ferland 1972</td>
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<td>Name of the functions</td>
<td>Abbreviation</td>
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<tr>
<td>Strictly Quasi Convex</td>
<td>StQC</td>
<td>If strict inequality holds in the above definition then f is strictly QC on S.</td>
<td>Simons [1978]</td>
</tr>
<tr>
<td>Convex like</td>
<td>Cl</td>
<td>$f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is Convex like if for all $y \in \mathbb{R}^m$ and all $x_1, x_2 \in \mathbb{R}^n$, there exists an $x_3 \in \mathbb{R}^n$ such that $f(x_3, y) \leq \frac{f(x_1, y) + f(x_2, y)}{2}$. If the strict inequality holds good then f is called strictly Convex like.</td>
<td></td>
</tr>
<tr>
<td>Semilocally Convex</td>
<td>S1C</td>
<td>$S \subseteq \mathbb{R}^n$ is locally star-shaped at $\bar{x} \in S$, if corresponding to $\bar{x}$ and each $x \in S$ there exists a maximum positive number $a(\bar{x}, x) &lt; 1$ such that $(1-\lambda)\bar{x} + \lambda x \in S$, $0 &lt; \lambda &lt; a(\bar{x}, x)$. A real valued function $f$ defined on a set $S \subseteq \mathbb{R}^n$ is S1C at $\bar{x} \in S$, if $S$ is locally star shaped at $\bar{x}$ and corresponding to $\bar{x}$ and each $x \in S$ there exists a positive number $d(\bar{x}, x) \leq a(\bar{x}, x)$ such that $f((1-\lambda)\bar{x} + \lambda x) \leq (1-\lambda)f(\bar{x}) + \lambda f(x)$, $0 &lt; \lambda &lt; d(\bar{x}, x)$</td>
<td></td>
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Kaul and Kaur [1982]
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<th>Name of the Abbreviation</th>
<th>Definition of the function</th>
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<td><strong>functions</strong></td>
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<td>Mond [1983]</td>
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<tr>
<td>Kaul and Kaur [1982]</td>
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If \( d(\bar{x}, x) = a(\bar{x}, x) = 1 \) for each \( x \in S \), then \( f \) is said to be Convex at \( \bar{x} \). If \( f \) is SLQC for each \( \bar{x} \in S \), then \( f \) is SLQC on \( S \).

\( f \) is said to be semilocally concave on \( S \) if \( -f \) is SLQC on \( S \).

Semilocally Quasi Convex

A real valued function \( f \) defined on a set \( S \subseteq \mathbb{R}^n \) is SLQC at \( \bar{x} \in S \), if \( S \) is locally star shaped at \( \bar{x} \) and corresponding to \( \bar{x} \) and each \( x \in S \) there exists a positive number \( d(\bar{x}, x) \leq a(\bar{x}, x) \) such that

\[
\left\{ \begin{array}{l}
    f(x) \leq f(\bar{x}) \\
    0 \leq \lambda \leq a(\bar{x},x)
\end{array} \right\} \implies f((1-\lambda)\bar{x}+\lambda x) \leq f(\bar{x})
\]

If \( d(\bar{x},x) = a(\bar{x},x) = 1 \) for each \( x \in S \), then \( f \) is Quasi Convex at \( \bar{x} \in S \). If \( f \) is SLQC at each \( \bar{x} \in S \), then \( f \) is SLQC on \( S \).

Invex

A function \( f \) that satisfies

\[
f(y)-f(x) \geq h^t(x,y)\nabla f(x)
\]

for all \( x, y \) in the domain, and for some vector function \( h(x,y) \) is known as Invex with respect to \( h(x,y) \).

Mond [1983]

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<tbody>
<tr>
<td>Pseudo Invex</td>
<td>PI</td>
<td>$f$ is said to be Pseudo Invex if for some vector function $h(x,y)$ and for all $(x,y)$ in the domain $h^t(x,y) \nabla f(x) \geq 0 \Rightarrow f(y) - f(x) \geq 0$.</td>
<td>Mond [1983]</td>
</tr>
<tr>
<td>Pseudo-Convex</td>
<td>PC</td>
<td>A differentiable function $f$ is Pseudo Convex if for all $x,y$ in the domain $(y-x)^t \nabla f(x) \geq 0 \Rightarrow f(y) - f(x) \geq 0$.</td>
<td>Mangasarian [1965]</td>
</tr>
<tr>
<td>Strongly Pseudo Convex</td>
<td>SPC</td>
<td>If $f$ is a scalar valued differentiable function on a convex set $S \subseteq \mathbb{R}^n$ and $K(x,y)$ is an arbitrary positive scalar function satisfying $K(x,y)(f(y) - f(x)) \geq (y-x)^t \nabla f(x)$ then $f$ is strongly Pseudo Convex with respect to $K(x,y)$.</td>
<td>Mond [1983]</td>
</tr>
</tbody>
</table>
If the strict inequality holds in the above definition, then 

\[ f: R^n \times R^m \rightarrow R \] is Logarithmic Convex like, if for all \( y \in R^m \) and all \( x_1, x_2 \in R^n \), there exists an \( x_3 \in R^n \) such that

\[
f(x_3, y) \leq \left( \frac{1}{\sqrt{\varepsilon}} \right)^2 \left( \frac{1}{\sqrt{\varepsilon}} \right)^2 .
\]

\[ f(x_3, y) \leq \left( \frac{f(x_1, y)}{\sqrt{\varepsilon}} \right)^2 \frac{1}{\sqrt{\varepsilon}} .
\]

If the strict inequality holds in the above definition, then 

\[ f: R^n \times R^m \rightarrow R \] is Harmonic Convex like if for all \( y \in R^m \), and all \( x_1, x_2 \in R^n \), there exists an \( x_3 \in R^n \) such that,

\[
f(x_3, y) \leq \frac{2f(x_1, y)f(x_2, y)}{f(x_1, y) + f(x_2, y)}
\]

If the strict inequality holds in the above definition, then 

\[ f: R^n \times R^m \rightarrow R \] is Strictly Logarithmic Convex like, if for all \( y \in R^m \) and all \( x_1, x_2 \in R^n \), there exists an \( x_3 \in R^n \) such that

\[
f(x_3, y) < \left( \frac{1}{\sqrt{\varepsilon}} \right)^2 \left( \frac{1}{\sqrt{\varepsilon}} \right)^2 .
\]

If the strict inequality holds in the above definition, then 

\[ f: R^n \times R^m \rightarrow R \] is Strictly Harmonic Convex like, if for all \( y \in R^m \), and all \( x_1, x_2 \in R^n \), there exists an \( x_3 \in R^n \) such that,

\[
f(x_3, y) < \frac{2f(x_1, y)f(x_2, y)}{f(x_1, y) + f(x_2, y)}
\]
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<tr>
<td>Quasi Convex like</td>
<td>QCL</td>
<td>$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is Quasi Convex like if for all $y \in \mathbb{R}^m$, and all $x_1, x_2 \in \mathbb{R}^n$, there exists an $x_3 \in \mathbb{R}^n$ such that, $f(x_3, y) \leq \max(f(x_1, y), f(x_2, y))$.</td>
</tr>
<tr>
<td>Strictly Quasi Convex like</td>
<td>StQCL</td>
<td>If the strict inequality holds good in the above definition then $f$ is strictly QCL.</td>
</tr>
<tr>
<td>Convex type</td>
<td>Ct</td>
<td>$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is Convex type, if for all $y \in \mathbb{R}^m$ and all $x_1, x_2 \in \mathbb{R}^n$, there exists an $x_3 \in \mathbb{R}^n$ such that $x_3 \leq \frac{x_1 + x_2}{2}$ and $f(x_3, y) \leq \frac{f(x_1, y) + f(x_2, y)}{2}$.</td>
</tr>
<tr>
<td>Strictly Convex type</td>
<td>StCt</td>
<td>$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is strictly Convex type, if for all $y \in \mathbb{R}^m$ and all $x_1, x_2 \in \mathbb{R}^n$, there exists an $x_3 \in \mathbb{R}^n$ such that $x_3 \leq \frac{x_1 + x_2}{2}$, and $f(x_3, y) &lt; \frac{f(x_1, y) + f(x_2, y)}{2}$.</td>
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<tbody>
<tr>
<td>Logarithmic Convex type</td>
<td>Lct</td>
<td>$f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is Logarithmic Convex type, if for all $y \in \mathbb{R}^m$ and all $x_1, x_2 \in \mathbb{R}^n$, there exists an $x_3 \in \mathbb{R}^n$ such that $x_3 \leq \frac{x_1 + x_2}{2}$ and $f(x_3, y) \leq \left(\frac{1}{2} f(x_1, y)\right) \left(\frac{1}{2} f(x_2, y)\right)$</td>
</tr>
<tr>
<td>Strictly Logarithmic Convex type</td>
<td>StLct</td>
<td>$f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is said to be strictly Logarithmic Convex type, if for all $y \in \mathbb{R}^m$ and all $x_1, x_2 \in \mathbb{R}^n$, there exists an $x_3 \in \mathbb{R}^n$ such that $x_3 \leq \frac{x_1 + x_2}{2}$ and $f(x_3, y) &lt; \left(\frac{1}{2} f(x_1, y)\right) \left(\frac{1}{2} f(x_2, y)\right)$</td>
</tr>
<tr>
<td>Harmonic Convex type</td>
<td>Hct</td>
<td>$f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is said to be Harmonic Convex type, if for all $y \in \mathbb{R}^m$ and all $x_1, x_2 \in \mathbb{R}^n$, there exists an $x_3 \in \mathbb{R}^n$ such that $x_3 \leq \frac{x_1 + x_2}{2}$ and $f(x_3, y) \leq \frac{2f(x_1, y)f(x_2, y)}{f(x_1, y) + f(x_2, y)}$</td>
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<tr>
<td>Name of the Abbreviation</td>
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</table>
| Strictly Harmonic Convex type | StHct | \( f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) is said to be strictly Harmonic Convex type, if for all \( y \in \mathbb{R}^m \) and all \( x_1, x_2 \in \mathbb{R}^n \), there exists an \( x_3 \in \mathbb{R}^n \) such that \[
    x_3 \leq \frac{x_1 + x_2}{2} \quad \text{and} \quad f(x_3,y) < \frac{2f(x_1,y) + f(x_2,y)}{f(x_1,y) + f(x_2,y)}.
\] |
| Quasi Convex type | Qct | \( f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) is said to be Quasi Convex type, if for all \( y \in \mathbb{R}^m \) and all \( x_1, x_2 \in \mathbb{R}^n \), there exists an \( x_3 \in \mathbb{R}^n \) such that \[
    x_3 \leq \frac{x_1 + x_2}{2}, \quad \text{and} \quad f(x_3,y) \leq \max(f(x_1,y), f(x_2,y)).
\] |
| Strictly Quasi Convex type | StQct | \( f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) is said to be strictly Quasi Convex type, if for all \( y \in \mathbb{R}^m \) and all \( x_1, x_2 \in \mathbb{R}^n \), there exists an \( x_3 \in \mathbb{R}^n \) such that \[
    x_3 \leq \frac{x_1 + x_2}{2}, \quad \text{and} \quad f(x_3,y) < \max(f(x_1,y), f(x_2,y)).
\] |
A real valued function $f$ defined on $S \subseteq \mathbb{R}^n$ is semilocally Logarithmic Convex at $x \in S$ if $S$ is locally star shaped at $x$ and corresponding to $x$ and each $x \in S$ there exists a positive number $d(x,x) < a(x,x)$ such that

$$f((1-X)x+Xx) < (f(x))^1 - X^*(f(x))^X, \text{ for } 0 < X < d(x,x).$$

A differentiable function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be Harmonic Pseudo Convex if and only if for all $x_1, x_2 \in S$

$$\frac{f(x_2) - f(x_1)}{f(x_2)} \leq 0 \Rightarrow \frac{\nabla f(x_1) \cdot (x_2 - x_1)}{f(x_1)} \leq 0.$$

A differentiable function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be Harmonic Quasi Convex if and only if for all $x_1, x_2 \in S$

$$\frac{\nabla f(x_1) \cdot (x_2 - x_1)}{f(x_1)} \geq 0 \Rightarrow \frac{f(x_2) - f(x_1)}{f(x_2)} \geq 0.$$

A real valued function $f$ defined on $S \subseteq \mathbb{R}^n$ is semilocally Logarithmic Convex at $\bar{x} \in S$ if $S$ is locally star shaped at $\bar{x}$ and corresponding to $\bar{x}$ and each $x \in S$ there exists a positive number $d(\bar{x}, x) \leq a(\bar{x}, x)$ such that

$$f((1-\lambda)\bar{x} + \lambda x) \leq (f(\bar{x}))^{1-\lambda}(f(x))^\lambda, \text{ for } 0 < \lambda < d(\bar{x}, x).$$

If $f$ is S1LC for each $\bar{x} \in S$, then $f$ is S1LC on $S$.  

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<tr>
<td>Semilocally Harmonic Convex</td>
<td>S1HC</td>
<td>A real valued function $f$ defined on a set $S \subseteq \mathbb{R}^n$ is semilocally Harmonic Convex at $\vec{x} \in S$, if $S$ is locally star-shaped at $\vec{x}$ and corresponding to $\vec{x}$ and each $x \in S$, there exists a positive number $d(\vec{x}, x) \leq a(\vec{x}, x)$ such that $f((1-\lambda)\vec{x}+\lambda x) \leq \frac{1}{1-\lambda} + \frac{\lambda}{f(\vec{x}) + f(x)}$ for $0 &lt; \lambda &lt; d(\vec{x}, x)$. If $f$ is S1HC for all $x \in S$, then $f$ is S1HC on $S$.</td>
</tr>
<tr>
<td>Semilocally Invex</td>
<td>S1I</td>
<td>A real valued function $f$ defined on a set $S \subseteq \mathbb{R}^n$ is semilocally Invex at $\vec{x}$ if for each $x \in S$ the right differential $(df)^+(\vec{x}, x-\vec{x})$ of $f$ at $\vec{x}$ in the direction of $x-\vec{x}$ exists and $f(y) - f(x) \geq h^{t}(x,y)(df)^+(\vec{x}, x-\vec{x})$ for all $x,y$ in the domain and for some vector function $h(x,y)$.</td>
</tr>
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| Semilocally Pseudo Convex | S1PC | \( f : C_1 \times C_2 \rightarrow \mathbb{R} \) is semilocally Pseudo-Convex at \((\bar{x}, y) \in C_1 \times C_2\), if \[(df)^+_{x} ((\bar{x}, y), (x-\bar{x}, y)) \geq 0 \Rightarrow f(x, y) \geq f(\bar{x}, y)\] \( f \) will be semilocally Pseudo Convex/sesilocally Pseudo Concave on \( C_1 \times C_2 \), if and only if \( f(\cdot, y) \) is semilocally Pseudo Convex on \( C_1 \) for each \( y \in C_2 \) and \( f(x, \cdot) \) is semilocally Pseudo Concave on \( C_2 \) for each given \( x \in C_1 \).

| Semilocally Strongly Pseudo Convex | S1SPC | \( f : C_1 \times C_2 \rightarrow \mathbb{R} \) is semilocally strongly Pseudo Convex at \((u, v)\), if the right differential \((df)^+_{x} ((\bar{x}, v), (x-\bar{x}, v))\) of \( f \) in the direction of \((x-\bar{x})\) exists for a given \( y = v \) and \( \Psi(x, y) \) is an arbitrary positive scalar function such that \[\Psi(x, y)(f(x, v) - f(u, v)) \geq (x-u)^t(df)^+_{u} (u, v)\].

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<td>Convex fuzzy mapping</td>
<td></td>
<td>Let $K \subset V$ and $F : K \rightarrow L(R)$. $F$ is said to be convex fuzzy mapping at $\bar{x} \in K$, if</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x \in K, \lambda \in [0,1], (1-\lambda)\bar{x} + \lambda x \in K$</td>
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<tr>
<td></td>
<td></td>
<td>$\Rightarrow F((1-\lambda)\bar{x} + \lambda x) \leq (1-\lambda)F(\bar{x}) + \lambda F(x)$.</td>
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<tr>
<td></td>
<td></td>
<td>Here $V$ denotes the vector space over the field $R$. $L(R)$ denotes the set of all fuzzy numbers which are upper semicontinuous and have compact support.</td>
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<td></td>
<td></td>
<td>If $K$ is a convex set, then $F$ is said to be Convex on $K$, if</td>
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<tr>
<td></td>
<td></td>
<td>$F((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)F(x_1) + \lambda F(x_2)$</td>
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<tr>
<td></td>
<td></td>
<td>for $0 \leq \lambda \leq 1$ and all $x_1, x_2 \in K$. $F$ is said to be Concave fuzzy mapping at $\bar{x} \in K$ (on a Convex set $K$) if $-F$ is Convex at $\bar{x} \in K$.</td>
</tr>
<tr>
<td>Logarithmic Convex fuzzy mapping</td>
<td></td>
<td>Let $K \subset V$ and $F : K \rightarrow L(R)$. $F$ is said to be</td>
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<tr>
<td></td>
<td></td>
<td>Logarithmic Convex fuzzy mapping at $\bar{x} \in K$ if</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x \in K, \lambda \in (0,1), (1-\lambda)\bar{x} + \lambda x \in K$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Rightarrow F((1-\lambda)\bar{x} + \lambda x) \leq (F(\bar{x}))^{1-\lambda}(F(x))^\lambda$</td>
</tr>
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</table>
If $K$ is a convex set, then $F$ is said to be Logarithmic Convex on $K$, if

$$F((1-\lambda)x_1 + \lambda x_2) \leq (F(x_1))^{1-\lambda}(F(x_2))^{\lambda}$$

for $\lambda \in (0,1)$ and $x_1, x_2 \in K$.

$F$ is said to be Logarithmic Concave on $K$ if $-F$ is Logarithmic Convex on $K$.

Let $K \subseteq V$ and $F : K \to L(R)$. $F$ is said to be Quasi Convex fuzzy mapping at $\bar{x} \in K$, if

$$x \in K, \lambda \in (0,1), (1-\lambda)\bar{x} + \lambda x \in K$$

$$\Rightarrow F((1-\lambda)\bar{x} + \lambda x) \leq \max(F(\bar{x}), F(x))$$

Here $V$ denotes a vector space over the field $R$. $L(R)$ denotes the set of all fuzzy numbers which are upper semicontinuous and have compact support.

$F$ is said to be Quasi-Convex fuzzy mapping on $K$, if

$$F((1-\lambda)x_1 + \lambda x_2) \leq \max(F(x_1), F(x_2))$$

for all $\lambda \in (0,1)$ and all $x_1, x_2 \in K, (1-\lambda)x_1 + \lambda x_2 \in K$. 
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<td>Generalized Concavity</td>
<td></td>
<td>F is said to be Quasi Concave fuzzy mapping at $\bar{x} \in K$ if $-F$ is Quasi Convex at $\bar{x} \in K$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>If $f$ is generalized Convex in $\bar{x} \in K$ in its domain, then $-f$ is generalized Concave there.</td>
</tr>
<tr>
<td>Strict generalized Convexity</td>
<td></td>
<td>$f$ will be strictly generalized Convex in its domain if strict inequality holds for distinct points there.</td>
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