CHAPTER-5

HYDROMAGNETIC FLOW AND HEAT TRANSFER BETWEEN TWO HORIZONTAL PLATES BOTH THE PLATES BEING STRETCHING SHEETS
CHAPTER - 5

5.1 INTRODUCTION

Rott [1] studied the non-similar solution corresponding to the viscous flow at a stagnation point on a wall moving with constant velocity.

Danberg and Fansler [2] have studied the viscous flow in which the free stream velocity is constant and the wall is being stretched with a velocity proportional to \( x \) (\( x \) being the distance along the wall). Srivastava [3] has discussed the flow of second order fluids with heat transfer between two plates, one moving and the other at rest. The momentum, heat and mass transfer in a viscous flow past a stretching sheet has been studied by many authors [4-7]. The boundary layer flow generated by a continuous moving solid surface in an otherwise quiescent fluid has been studied by Sakiadis [8-10] and experimentally verified by Tsou et al. [11].

This problem, being significantly relevant to the continuous extrusion processes used in manufacturing of sheets and fibres in the glass and polymer industries, has attracted the attention of several research workers [12-14]. These studies are based on the assumption that the moving sheet is
inextensible. However, Mc Cormack and Crane [15] have pointed out that situations may arise in polymer industry in which one has to deal with a stretching plastic sheet.

Since the physical properties of the ambient fluid effectively influence the boundary layer characteristics, the study of non-Newtonian fluid flow over a moving sheet has gained considerable importance. Srivastava and Sharma [16] have studied the problem in a Newtonian electrically conducting fluid in the presence of a transverse magnetic field. Borkakoti and Bharali [17] have discussed the heat transfer on the MHD flow between two co-axial non-conducting porous discs when one rotates and the other is at rest [18]. Chakrabarti and Gupta [19] have investigated the motion of an electrically conducting fluid past a horizontal plate in presence of a magnetic field, the motion being caused solely by the stretching of the plate.

Recently, Borkakoti and Bharali [20] have investigated hydromagnetic flow and heat transfer between two horizontal plates, the lower plate being a stretching sheet.

The study to be reported here in , considers the stretching effect of both the plates on the flow of viscous incompressible fluid in the presence of transverse magnetic field and uniform injection at the upper plate.
This problem has various applications to polymer technology (where one deals with stretching plastic sheets) and metallurgy where hydromagnetic techniques have recently been used. To be more specific, it may be pointed out that many metallurgical processes involve the cooling of continuous strips of filaments by drawing them through a quiescent fluid and that in the process of drawing, these strips are sometimes stretched. Mention may be made of drawing, annealing and tinning of copper wires. In all these cases the properties of the final product depend to a great extent on the rate of cooling. By drawing such strips in an electrically conducting fluid subject to a magnetic field, the rate of cooling can be controlled and final products of desired characteristics might be achieved. Another interesting application of hydromagnetics to metallurgy lies in the purification of molten metals from non-metallic inclusions by the application of magnetic field [21].

5.2 FLOW ANALYSIS

The flow of an incompressible viscous electrically conducting fluid (with electrical conductivity \( \sigma \)) between two horizontal parallel non-conducting plates, is considered in the presence of a transverse magnetic field \( B_0 \), imposed along \( y \)-axis. A cartesian coordinate system is used where the \( y \)-axis is perpendicular to the plates located at \( y = +h \) and \( y = -h \). The lower and the upper plates are stretched by introducing
two equal and opposite forces so that the positions of the points \((0, h)\) and \((0, -h)\) remain unchanged. The fluid is injected through the upper porous plate with constant velocity \(v_0\). The induced magnetic field is neglected, which is justified for small magnetic Reynolds number \([22]\). It is also assumed that the external electric field is zero and the electric field due to polarization of charges is negligible \([19]\). Under the above assumptions, the equations governing the steady flow are

\[
\begin{align*}
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho} u, \quad (5.2.1) \\
\frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}, \quad (5.2.2) \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \quad (5.2.3)
\end{align*}
\]

where \(u, v\) are the fluid velocity components along the \(x\) and \(y\) directions. All variables are assumed to be independent of \(Z\).

The boundary conditions are:

\[
\begin{align*}
u = cx, \quad v = 0 \text{ at } y = -h; \\
u = cx, \quad v = -v_0 \text{ at } y = +h \quad (5.2.4)
\end{align*}
\]

We assume

\[
u = cx f'(\eta), \quad v = -c h f(\eta), \quad \eta = y/h \quad (5.2.5)
\]

where the prime denotes differentiation with respect to \(\eta\) and \(u = cx\) represents the velocity of both the plates with \(c > 0\).
Substituting (5.2.5) into equations (5.2.1) and (5.2.2), we have

\[
-\frac{1}{\rho} \frac{\partial p}{\partial x} = c^2 x \left[ f'^2 - ff'' - \frac{1}{R} f''' + \frac{M^2}{R} f' \right], \\
-\frac{1}{\rho h} \frac{\partial p}{\partial \eta} = c^2 h \left[ ff' + \frac{1}{R} f'' \right],
\]

where

\[ R = \frac{ch^2}{v}, \quad M = \frac{\alpha}{\rho v} \frac{1}{2} B_0 h \]

Differentiating (5.2.7) with respect to \( x \), we have

\[ \frac{\partial^2 p}{\partial x \partial \eta} = 0, \text{ which suggests that} \]

\[ f''' - R (f'^2 - ff'') - M^2 f' = A \]

where \( A \) is a constant. For small values of \( R \), a regular perturbation scheme can be developed by expanding \( f \) and \( A \) in ascending powers of \( R \) as

\[ f = \sum_{n=0}^{\infty} R^n f_n, \quad A = \sum_{n=0}^{\infty} R^n A_n \]

The coefficients of higher powers of \( R \) (\( n \geq 2 \)) have negligible contribution; thus we neglect them. By this method we can obtain, theoretically, the approximate solution.
Substituting (5.2.10) in (5.2.9) and equating like powers of R, we have

\[ f_0''' - M^2 f_0' = A_0, \quad (5.2.11a) \]
\[ f_1''' - M^2 f_1' = A_1 + \left( f_0^{(2)} - f_0 f_0'' \right), \quad (5.2.11b) \]

e tc. The corresponding boundary conditions are

\[ f_0 = \lambda, \quad f_0' = 1, \quad f_n = f_n' = 0 \text{ for all } n > 0 \text{ at } \eta = +1, \quad (5.2.12) \]
\[ f_0 = 0, \quad f_0' = 1, \quad f_n = f_n' = 0 \text{ for all } n > 0 \text{ at } \eta = -1 \]

where \( \lambda = (v_0/\text{ch}) \).

Equations (5.2.11) are solved subject to the boundary conditions (5.2.12). The expressions for \( f_0 \) and \( f_1 \) are

\[ f_0 = P_1 \eta + P_2 \sinh M \eta + \frac{\lambda}{2} \quad (5.2.13) \]
\[ f_1 = A_3 + A_5 \eta + \left[ \frac{A_1}{2M} \sinh hM + \frac{3P_1P_2}{2M} \eta - \frac{P_1P_2}{4} \right] \cosh M \eta \]
\[ + \left( \frac{P_1P_2}{4M} - \frac{P_2}{4} \right) \eta - \frac{3P_1P_2 \sinh M}{2(M \cosh M - \sinh M)} \right] \sinh M \eta, \quad (5.2.14) \]

where

\[ P_1 = \frac{(\lambda M \cosh M - 2 \sinh M)}{2(M \cosh M - \sinh M)}, \]
\[ P_2 = \frac{(2 - \lambda)}{2(M \cosh M - \sinh M)}, \]
\[ A_1 = \left( \frac{P_1 P_2 + P_2 \lambda M}{2} \right) \cosh M + \left( \frac{P_1 P_2 M - P_1 P_2 + P_2 \lambda}{2M} \right) \sinh M, \]

\[ A_2 = \left( \frac{P_1 P_2 + P_2 \lambda M}{4} \right) \cosh M - \left( \frac{P_1 P_2 - P_2 \lambda}{2M} \frac{P_1 P_2 M}{4} \right) \sinh M, \]

\[ A_3 = A_2 - \frac{A_1 (\cosh M + M \sinh M)}{2M \sinh M}, \]

\[ A_4 = \frac{A_1}{2} A_2 - \left( \frac{P_1 P_2 - P_2 \lambda}{4M} \right) \sinh M + \frac{3P_1 P_2 \sinh^2 M}{2 (M \cosh M - \sinh M)}, \]

\[ A_5 = A_4 - \left( \frac{3P_1 P_2 - P_1 P_2}{2M} \right) \cosh M. \]

### 5.3 Heat Transfer

The energy equation is

\[ \rho C_p \left[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right] = \left[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right] + \phi + \frac{\mathcal{J}^2}{\sigma} \quad (5.3.1) \]

where

\[ \phi = 2\rho v \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right] \]

is the viscous dissipation term and the last term is due to ohmic dissipation.

We consider here two different cases:

(i) When the plates are maintained at different temperatures;

(ii) When the lower plate is adiabatic and the upper plate is maintained at a constant temperature.
Case 1: When the plates are at different temperatures.

The boundary conditions are

\[ T = T_1 \text{ at } y = +h, \]
\[ T = T_0 \text{ at } y = -h, \text{ (} T_0 < T_1 \text{)} \]

(5.3.2)

Using the expression for temperature \( T \)

\[ T = T_0 + (C^2 h^2 / RC_p) \left[ \phi(\eta) + \frac{x^2}{h^2} \psi(\eta) \right], \]

(5.3.3)

Equation (5.3.1) reduces to (comparing coefficients of and the terms independent of \( x \))

\[ \psi'' = P_y R [2f' \psi - f \psi'] - (f'' r^2 + M^2 f' r^2) \]

(5.3.4a)

\[ \phi'' + 2 \psi = -P_y R [4f' r^2 + f \phi'], \]

(5.3.4b)

where \( P_y = (pvC_p / k) \)

We expand \( \phi \) and \( \psi \) in powers of \( R \) as

\[ \phi = \sum_{n=0}^{\infty} R^n \phi_n, \quad \psi = \sum_{n=0}^{\infty} R^n \psi_n \]

(5.3.5)

Substituting (5.3.5) in eqn (5.3.4) and comparing coefficients of like powers of \( R \) and considering upto second order approximations, we have
\[ \psi_0'' = 0, \quad (5.3.6a) \]
\[ \phi_0'' + 2\psi_0 = 0 \quad (5.3.6b) \]
\[ \psi_1'' = p_y \left[ 2f_0'\psi_0 - f_0\psi_0' - f_0'f_0'' - M^2f_0'' \right] \quad (5.3.6c) \]
\[ \phi_1'' + 2\psi_1 = -p_y \left[ 4f_0'^2 + f_0\phi_0' \right] \quad (5.3.6d) \]

The boundary conditions (5.3.2) reduce to

\[ \phi_0 = s, \quad \psi_0 = 0, \quad \phi_1 = 0 = \psi_1 \quad \text{for all } n > 0 \text{ at } \eta = \pm 1, \]
\[ \phi_n = \psi_n = 0 \quad \text{for all } n > 0 \text{ at } \eta = -1, \quad (5.3.7) \]

where \( s = (T_1 - T_0)RC_\rho/c^2h^2 \)

Eqsns (5.3.6) are solved subject to the boundary conditions (5.3.7):

\[ \psi_0 = 0, \quad (5.3.8a) \]
\[ \phi_0 = \frac{S}{2}(1 + \eta), \quad (5.3.8b) \]
\[ \psi_1 = M^2p_y \left[ A_0 - \frac{1}{4}p_2^2 \cosh 2M\eta - \frac{2p_1p_2}{M} \cosh M\eta - \frac{1}{2}p_1^2\eta^2 \right] \quad (5.3.8c) \]
\[ \phi_1 = p_y \left[ A_0 - (2p_1^2 + p_2^2 + A_0)\eta^2 - \frac{3}{8}p_2^2 \cosh 2M\eta \right. \]
\[ \left. - \frac{4p_1p_2}{M} \cosh M\eta + \frac{1}{12}p_1^2M^2\eta^4 \right] \quad (5.3.8d) \]

where

\[ A_0 = \frac{p_2^2}{2} + \frac{2p_1p_2}{M} \cosh M + \frac{1}{4}p_2^2 \cosh 2M, \]
\[ A_1 = \frac{1}{2}p_1^2M^2 + 2p_1p_2M \cosh M + \frac{1}{4}p_2^2M^2 \cosh 2M, \]
\[ A_4 = 2p_1^2 + (p_2^2 + \frac{5p_1^2}{12})M^2 + \left( \frac{4p_1p_2}{M} + 2p_1p_2M \right) \cosh M \]
\[ + \left( \frac{3p_2^2}{8} + \frac{p_2^2M^2}{4} \right) \cosh 2M \]
The non-dimensional expression for temperature is obtained from (5.3.3) as

\[
\frac{(T-T_0)}{(T_1-T_0)} = \frac{[\phi_0 + R\phi_1 + \frac{x^2}{h^2}(\psi_0 + R\psi_1)]}{E} = \{(1+\eta)/2\} + EX^2\psi_1
\]

(5.3.9)

where \( E = R/s \) and \( X = x/h \).

For a moderate distance from \( y \)-axis, the second term is negligible in comparison with other terms (1-3). Hence (5.3.9) reduces to

\[
\frac{(T-T_0)}{(T_1-T_0)} = \{(1+\eta)/2\} + EX^2\psi_1
\]

(5.3.10)

Case 2: When the lower plate is adiabatic:

In this case, the boundary conditions are

\[
T = T_1 \quad \text{at } y = + h \\
\frac{\partial T}{\partial y} = 0 \quad \text{at } y = - h
\]

(5.3.11)

Instead of (5.3.3), we use the expression for temperature

\[
T = \left( C^2 h^2 / R C_p \right) \left[ \phi(\eta) + \frac{x^2}{h^2} \psi(\eta) \right]
\]

(5.3.12)

Proceeding as in case 1, we can derive equation (5.3.6):

The boundary conditions become

\[
\phi_0 = s_1, \psi_0 = 0, \quad \phi_n = \psi_n = 0 \quad \text{for all } n > 0 \text{ at } \eta = +1,
\]

\[
\phi_n' = \psi_n' = 0 \quad \text{for all } n > 0 \text{ at } \eta = -1,
\]

(5.3.13)

where \( s_1 = (T_1 R C_p / C^2 h^2) \)
The expressions for $\psi_0$ and $\psi_1$ are

$$\psi_0 = 0 \quad \tag{5.3.14a}$$

$$\psi_1 = s_1 \quad \tag{5.3.14b}$$

$$\Psi_1 = M^2 P_y \left[ A_{10} - A_9 \eta - \frac{1}{2} P_1^2 \eta^2 - \frac{2 P_1 P_2}{M} \cosh M \eta \right]$$

$$- \frac{1}{4} P_2^2 \cosh 2M \eta \right] \quad \tag{5.3.14c}$$

$$\Phi_1 = P_y \left[ A_{13} - A_{12} \eta - A_{11} \eta^2 - \frac{4 P_1 P_2}{M} \cosh M \eta \right]$$

$$- \frac{3}{8} P_2^2 \cosh 2M \eta + \frac{1}{12} P_1^2 M^2 \eta^4 \right] \quad \tag{5.3.14d}$$

where

$$A_9 = P_1^2 + 2P_1 P_2 \sinh M + \frac{1}{2} P_2^2 M \sinh 2M,$$

$$A_{10} = A_9 + \frac{1}{2} P_1^2 + \frac{2 P_1 P_2}{M} \cosh M + \frac{1}{4} P_2^2 \cosh 2M,$$

$$A_{11} = 2P_1^2 + \frac{1}{2} P_1^2 M^2 + P_2^2 M^2 + 2P_1 P_2 M \cosh M$$

$$+ \frac{1}{4} P_2^2 M^2 \cosh 2M,$$

$$A_{12} = 2A_{11} - \frac{1}{3} P_1^2 M^2 + 4P_1 P_2 \sinh M + \frac{3}{4} P_2^2 M \sinh 2M,$$

$$A_{13} = 3A_{11} - \frac{5}{12} P_1^2 M^2 + \frac{3}{4} P_2^2 M \sinh 2M + 4P_1 P_2 \sinh M$$

$$+ \frac{3}{8} P_2^2 \cosh 2M + \frac{4 P_1 P_2}{M} \cosh M$$

The non-dimensional expression for temperature is obtained from (5.3.12) as
\[ \frac{T}{T_1} = \frac{[\phi_0 + R \phi_1 + \frac{x^2}{h^2} (\psi_0 + R \psi_1)]}{S_1} \]
\[ = 1 + \frac{R}{S_1} \phi_1 + \frac{x^2}{h^2} \frac{R}{S_1} \psi_1 \]
\[ = 1 + E \phi_1 + E X^2 \psi_1 \]

where \( E = \frac{R}{S_1} \) and \( X = \frac{X}{h} \)

At a moderate distance from y-axis, the second term is negligible in comparison with other terms. Hence (5.3.15) reduces to the non-dimensional expression for temperature as
\[ \frac{T}{T_1} = 1 + E X^2 \psi_1 \] (5.3.16)

5.4 DISCUSSIONS AND RESULTS

In course of discussion, effect of Lorentz force (M), suction parameter (\( \lambda \)), and Reynolds number (R) have been studied on the primary flow velocity (\( f' \)), transverse velocity (f) and temperature field in two different situations. i.e.,
(i) When plates are at different temperatures.
(ii) When the lower plate is adiabatic and the upper plate is maintained at a constant temperature.

Fig. 5.2 shows that velocity field is almost symmetric about the centre of the channel (\( \eta = 0 \)) in case of both the plates are being stretched at the same rate but it is not the case with the stretching of the lower plate only [20]. When \( \lambda \) is constant and small (\( \lambda = 1 \)), the effect of the Lorentz force on
the primary flow is to decrease near the lower plate and to increase near the upper plate. This result is in good agreement with [20].

When $M$ is constant, the effect of $\lambda$ on $f'$ is maximum at the centre of the channel. This observation is same for both the cases, i.e. single sheet being stretched and both the sheets being stretched. But an increase of 14% in the value of $f'$ at the centre is marked in case of the single sheet being stretched.

Comparative study of the two curves II and III for $\lambda=1$ and $\lambda=3$ respectively reveals that the suction parameter radically changes the primary flow velocity. The primary velocity for $\lambda = 3$ is five times that of the primary velocity for $\lambda = 1$ at the centre of the channel. From the curves III and IV, it is seen that an increase in the value of $R$ ($R < 1$) increases $f'$ in the lower half of the channel whereas the opposite effect is observed in the upper half.

Fig.5.3, shows that the Lorentz force decreases the transverse velocity, i.e. $f$, with the increase of the magnetic field strength when $\lambda$ is constant but the value of $f$ increases with the increase of $\lambda$ when $M$ is constant. It is also seen from the curves III and IV that an increase in the value of $R$ increases $f$ at all points and the transverse velocity increases with increase of $\eta$, channel width, when $M$ is constant.
Case 1: When the plates are at different temperatures:

Fig. 5.4 shows the temp distribution for $M=1,3$ and $\lambda=1,3$ when $P_{j}EX^{2}=5,10$ and 20. It is observed that the temperature of the fluid at all points increases with the increase of both magnetic parameter $M$ and suction parameter $\lambda$ but the suction parameter increases the temperature more than the magnetic parameter. This above observations are also true in case of the lower plate being only a stretching sheet [20]. The only peculiarity with the temperature profiles of the present study (i.e both the plates being stretching sheets) is the symmetricity of the profiles about the centre of the channel.

Case 2: Temperature distributions:

(adiabatic lower plate)

Fig. 5.5 shows the temperature distribution for $M = 1,3$ and $\lambda = 1,3$ when $P_{j}EX^{2} = 5, 10$ and 20. In this case it is also seen that an increasing values of $M$ and $\lambda$ result in increasing temperature at any point in the fluid. Moreover, the increase in temperature is more significant near the adiabatic wall.

Comparing the temperature distribution of the single plate being a stretching sheet with that of the present study it is found that stretching of both the plates is to decrease the temperature of the fluid considerably.
5.4 CONCLUSION

The injection parameter $\lambda (=v_0/ch)$ increases with the increase of the injection velocity and/or the decrease of the stretching velocity of both the plates. If the stretching velocity is kept constant and the injection velocity at the upper plate is increased, the primary flow velocity $f'$ increases and the maximum velocity occurs at the central plane of the channel because the injection velocity has no effect on the fluid motion at the lower plate which is evident from the boundary conditions. The similar observations are also derived if the injection velocity is kept constant and the stretching velocity of both the plates decreases. Further, it is to be noted that $f$, the transverse velocity which vanishes at the lower plate, increases near the upper plate with increasing $\lambda$.

The imposed magnetic field is perpendicular to the primary fluid velocity $u$ and parallel to the injection velocity $v_0$. Since the Lorentz force acts on the primary fluid flow in the opposite direction (equation (5.2.1)), the increase of the magnetic field increases the velocity near the plates but decreases velocity near the mid-plane of the channel (I,II).

As the fluid velocity increases with $\lambda$, the fluid temperature increases with $\lambda$ also. Similarly, the electric current generated in the fluid increases the strength of the magnetic field, and this causes the increase of temperature of the fluid.
Fig. 5-1. GEOMETRY OF THE PROBLEM.
Fig. 5.2. VARIATION OF $f' (\eta)$ WITH THE PARAMETERS M, $\lambda$, R.
Fig. 5.3. VARIATION OF $f(\eta)$ WITH THE PARAMETER $M, \lambda, R$. 

<table>
<thead>
<tr>
<th>CURVE</th>
<th>$M$</th>
<th>$\lambda$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1.0</td>
<td>1.0</td>
<td>0.20</td>
</tr>
<tr>
<td>II</td>
<td>3.0</td>
<td>1.0</td>
<td>0.20</td>
</tr>
<tr>
<td>III</td>
<td>3.0</td>
<td>3.0</td>
<td>0.20</td>
</tr>
<tr>
<td>IV</td>
<td>3.0</td>
<td>3.0</td>
<td>0.05</td>
</tr>
<tr>
<td>V</td>
<td>1.0</td>
<td>3.0</td>
<td>0.20</td>
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Fig. 5.4. TEMPERATURE DISTRIBUTIONS
(PLATES AT DIFFERENT TEMPERATURES).

<table>
<thead>
<tr>
<th>CURVE</th>
<th>M</th>
<th>( \lambda )</th>
<th>( PyE X^2 )</th>
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<tbody>
<tr>
<td>I</td>
<td>3.0</td>
<td>1.0</td>
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<tr>
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<td>III</td>
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<td>3.0</td>
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<tr>
<td>V</td>
<td>3.0</td>
<td>1.0</td>
<td>10.0</td>
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Fig. 5.5. TEMPERATURE DISTRIBUTIONS
(ADIABATIC LOWER PLATE).
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