CHAPTER 3

Classification of Partially Ordered Loops and Lattice ordered loops
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CLASSIFICATION OF PARTIAL ORDERED LOOPS
AND LATTICE ORDERED LOOPS

§3.1. INTRODUCTION:

Garrett Birkhoff (1942) firstly initiated the notion of lattice ordered groups. Then Bruck (1944) contributed various results in the theory of quasigroups. Zelinski (1948) described about ordered loops. The concept of non associative number theory was thoroughly studied by Evans (1957). Bruck (1963) explained about what is a loop? Various crucial properties of lattice ordered groups were established by Garrett Birkhoff in 1964 and 1967. Evans (1970) described about lattice ordered loops and quasigroups. Richard Hubert Bruck (1971) made a survey of binary systems. In the recent past Hala (1990) made a description on quasigroups and loops.

In this document we furnish definitions, examples and some properties of Partial ordered loops, lattice ordered loops and some relations between lattice ordered loops and distributive lattices in additive notation. In this manuscript mainly there are two topics, one is about Partially Ordered Loops and other is about Lattice Ordered Loops and the definitions, examples and properties are in additive notation. Here we provide some of the foremost properties:

1. \((S,+,\leq)\) is partially ordered loop iff (i) \(x\leq y \iff x+z\leq y+z\), (ii) \(x\leq y \iff z+x\leq z+y\)
2. In a partially ordered loop there are four equivalent conditions are satisfied.
3. In any lattice ordered loop there are six conditions and their duals are also satisfied.
4. Any lattice ordered loop is a distributive lattice.
5. Any lattice ordered loop is a modular lattice.
6. In any lattice ordered loop $L$, if $\forall x \land y = 0$ and $x \land z = 0$ then $x \land (y+z) = 0 \forall \ x, y, z \in L$.

7. Different properties of distributive and modular lattices in lattice ordered loops.

8. In a lattice ordered loop disjoint positive elements are permutable.

§3.2. PARTIAL ORDERED LOOPS AND LATTICE ORDERED LOOPS:

**Definition 3.2.1:** A system $(S, +, \leq)$, where $S$ is a nonempty set, $+$ is a binary operation on $S$, $\leq$ is a binary relation on $S$ satisfying

(i) $(S, +)$ is a loop

(ii) $(S, \leq)$ is a partially ordered set

(iii) The translations $x \mapsto a + x$ and $x \mapsto x + b$ are order automorphisms of $S$,

is called a partially ordered loop (briefly P.O.loop).

**Example 3.2.1:** $(\mathbb{Z}, +, \leq)$ is a partially ordered loop.

**Theorem 3.2.1:** In a partially ordered loop we have

(1) $x \leq y$ if and only if $x+z \leq y+z$

(2) $x \leq y$ if and only if $z+x \leq z+y$

Conversely if $(S, +, \leq)$, where $(S, +)$ is a loop and $\leq$ is a partial ordering on $S$, satisfying the conditions (1), (2) then it is a partially ordered loop.

**Proof:** Let $(S, +, \leq)$ be a partially ordered loop.

Since $x \mapsto x + z$ and $x \mapsto z + x$ are order automorphisms

We have (1) and (2)

Converse is clear.
**Definition 3.2.2:** A lattice ordered loop is a partially ordered loop in which the partial order is a lattice order.

**Theorem 3.2.2:** In any partially ordered loop

(i) \( x \leq y \) if and only if \( x - z \leq y - z \)

(ii) \( x \leq y \) if and only if \( -z + x \leq -z + y \)

(iii) \( x \leq y \) if and only if \( z - x \geq z - y \)

(iv) \( x \leq y \) if and only if \( -x + z \geq -y + z \)

**Proof:**

(i) \( x \leq y \)

Then \( (x - z) + z \leq (y - z) + z \)

\[ \iff x - z \leq y - z \] (by condition (1) of the above theorem 3.2.1)

(ii) \( x \leq y \)

i.e. \( z + (-z + x) \leq z + (-z + y) \)

\[ \iff -z + x \leq -z + y. \] (by condition (2) of the above theorem 3.2.1)

(iii) \( x \leq y \)

\[ \Rightarrow (z - y) + y = z = (z - x) + x \]

\[ \leq (z - x) + y \] (by condition (2) of theorem 3.2.1)

\[ \Rightarrow z - y \leq z - x \] (by condition (1) of the above theorem 3.2.1)

Conversely suppose

\( z - y \leq z - x \)

\[ \Rightarrow (z - y) + y \leq (z - x) + y \] (by condition (1) of the theorem 3.2.1)
i.e. \( z \leq (z-x) + y \)

i.e. \((z-x) + x \leq (z-x) + y\)

\[ \Rightarrow x \leq y \text{ (by condition (1) of theorem 3.2.1)} \]

Hence \( x \leq y \) if and only if \( z-x \geq z-y \).

(iv) \( x \leq y \)

\[ \Rightarrow y + (-y+z) = z = x + (-x+z) \]

\[ \leq y + (-x+z) \text{ (by condition (1) of theorem 3.2.1)} \]

\[ \Rightarrow -y+z \leq -x+z \text{ (by condition (2) of theorem 3.2.1)} \]

Conversely, \((-y+z) \leq (-x+z)\)

\[ \Rightarrow y + (-y+z) \leq y + (-x+z) \text{ (by condition (2) of theorem 3.2.1)} \]

\[ \Rightarrow z \leq y + (-x+z) \]

\[ \Rightarrow x + (-x+z) \leq y + (-x+z) \]

\[ \Rightarrow x \leq y \text{ (by condition (1) of the theorem 3.2.1)} \]

Therefore \( x \leq y \) if and only if \(-x+z \geq -y+z\).

Hence the proof.

**Note 3.2.1:** Since \((p+z) - z = p\) and \(-z + (z+p) = p\), the mappings \( x \mapsto x-z \) and \( x \mapsto -z+x \) are on to and hence order automorphisms. (By (i) and (ii) of the theorem 3.2.2)

**Note 3.2.2:** Since \( p + (-p+z) = z \) and \((z-p) + p = z\), the mapping \( x \mapsto z-x \), \( x \mapsto -x+z \) are on to respectively, and are dual order automorphisms.

Now we recall the following (without proof)
**Theorem 3.2.3:** (G.szasz) Let \((P, \leq)\) and \((Q, \leq)\) be lattices.

Let \(\theta : P \rightarrow Q\) be an order isomorphism. Then we have the following.

“For any \(\{a_\alpha\}_{\alpha \in \Delta}\) in \(P\) for which \(\lor a_\alpha\) exists. Then \(\lor_\alpha \theta(\{a_\alpha\})\) exists and \(\theta(\lor_\alpha a_\alpha) = \lor_\alpha \theta(\{a_\alpha\})\). Similarly for meets”.

**Theorem 3.2.4:** (G.szasz) Let \((P, \leq)\) and \((Q, \leq)\) be lattices.

Let \(\theta\) be dual isomorphism. Then we have the following.

“For any \(\{a_\alpha\}_{\alpha \in \Delta}\) for which \(\lor a_\alpha\) exists then \(\lor_\alpha \theta(\{a_\alpha\})\) exists and \(\theta(\lor_\alpha a_\alpha) = \lor_\alpha \theta(\{a_\alpha\})\). Similarly for joins”.

**Theorem 3.2.5:** In any lattice ordered loop

1. \((x \lor y) + z = (x + z) \lor (y + z)\); \((x \land y) + z = (x + z) \land (y + z)\)

2. \(z + (x \lor y) = (z + x) \lor (z + y); z + (x \land y) = (z + x) \land (z + y)\)

3. \((x \lor y) - z = (x - z) \lor (y - z); (x \land y) - z = (x - z) \land (y - z)\)

4. \(-z + (x \lor y) = (-z + x) \lor (-z + y); -z + (x \land y) = (-z + x) \land (-z + y)\)

5. \(z - (x \lor y) = (z - x) \land (z - y); z - (x \land y) = (z - x) \lor (z - y)\)

6. \((-x \lor y) + z = (-x + z) \lor (-y + z); (-x \land y) + z = (-x + z) \lor (-y + z)\) for all \(x, y, z\).

Proof:

Let \((S, +, \leq)\) be a lattice ordered loop.

1. By definition, the mapping \(\theta : S \rightarrow S\) defined by \(\theta(x) = x + z\) is an order automorphism.

\[\therefore \theta(x \lor y) = \theta(x) \lor \theta(y)\]

i. e. \((x \lor y) + z = (x + z) \lor (y + z)\)
and $\theta (x \land y) = \theta(x) \land \theta(y)$

i. e. $(x \land y) + z = (x+z) \land (y+z)$

(2) By def, the mapping $\theta : S \rightarrow S$ defined by

$\theta (x) = z+x$ is an order automorphism.

$\therefore \theta(x\lor y) = \theta(x) \lor \theta(y)$

i. e. $z + (x\lor y) = (z+x) \lor (z+y)$

and $\theta (x\land y) = \theta(x) \land \theta(y)$

i. e. $z+(x\land y) = (z+x) \land (z+y)$

(3) By Note (3.2.1), the mapping $\theta : S \rightarrow S$ defined by

$\theta (x) = x-z$ is an order automorphism.

$\therefore \theta(x\lor y) = \theta(x) \lor \theta(y)$

i. e. $(x \land y)-z = (x-z) \lor (y-z)$

and $\theta (x\land y) = = \theta(x) \land \theta(y)$

i.e. $(x \land y) - z = (x-z) \land (y-z)$

(4) By Note (3.2.2), the mapping $\theta : S \rightarrow S$ defined by

$\theta(x) = -z+x$ is an order automorphism.

$\therefore \theta(x\lor y) = \theta(x) \lor \theta(y)$

i.e. $-z+(x\lor y) = (-z+x) \lor (-z+y)$

and $\theta (x\land y) = \theta(x) \land \theta(y)$

i.e. $-z+(x\land y) = (-z+x) \land (-z+y)$
(5) By Note (3.2.2), the mapping \( \theta:S \rightarrow S \) defined by \( \theta(x) = z - x \) is a dual order automorphism.

\[ \therefore \theta(x \lor y) = \theta(x) \land \theta(y) \]

i.e. \( z - (x \lor y) = (z - x) \land (z - y) \)

_and_ \( \theta(x \land y) = \theta(x) \lor \theta(y) \)

i.e. \( z - (x \land y) = (z - x) \lor (z - y) \)

(6) By Note (3.2.2), the mapping \( \theta:S \rightarrow S \) defined by \( \theta(x) = -x + z \) is a dual order automorphism.

\[ \therefore \theta(x \lor y) = \theta(x) \land \theta(y) \]

i.e. \( -(x \lor y) + z = (-x + z) \land (-y + z) \)

_and_ \( \theta(x \land y) = \theta(x) \lor \theta(y) \)

\( -(x \land y) + z = (-x + z) \lor (-y + z) \)

Hence the proof.

**Definition 3.2.3:** A lattice \((L, \lor, \land)\) is said to be distributive if, for any \(x, y, z \in L\),

\[ x \lor (y \land z) = (x \lor y) \land (x \lor z) \quad \text{or} \quad x \land (y \lor z) = (x \land y) \lor (x \land z) \]

**Example 3.2.2:** For any set \(X\), the lattice \((P(X), \cup, \cap)\) is a distributive lattice.

**Solution:** since \(A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\),

\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \forall A, B, C \in P(X) \] and hence \((P(X), \cup, \cap)\) is a distributive lattice.

**Example 3.2.3:** Every chain in a partially ordered loop is distributive.

**Solution:** Let \(L\) be a chain in a partially ordered loop.

To show that \(L\) is distributive.
Let a, b, c ∈ L, then any two of these are comparable.

Case (i): Suppose b ≤ c ⇒ b∧c=b and b∨c=c.

Also b ≤ c ⇒ a∧b ≤ a∧c.

Now a∧(b∨c) = a∧c and (a∧b) ∨ (a ∧ c) = a∧c

Therefore L is distributive.

Case (ii): Suppose c ≤ b ⇒ c ∨ b = b & a ∧ c ≤ a ∧ b

Now a∧(b∨c) = a∧b and (a∧b) ∨ (a ∧ c) = a∧b

∴ a∧(b∨c) = (a∧b) ∨ (a ∧ c)

∴ L is distributive.

Hence every chain in a partial ordered loop is a distributive.

**Definition 3.2.4:** A lattice (L, ∨, ∧) is called modular if, for all x, y, z ∈ L, x ≤ z implies x ∨ (y ∧ z) = (x ∨ y) ∧ z (Modular identity)

**Lemma 3.2.1:** In a lattice ordered loop L,

\[-[x-(x∧y)] + (x∨y) = y, \text{ for all } x, y \in L\]

**Proof:** Let L be a lattice ordered loop. Let x, y ∈ L

Now \[x-(x ∧ y)] + y\]

\[= [(x-x) ∨ (x-y)] + y\]

\[= [0 ∨ (x-y)] + y\]

\[= (0+y) ∨ [(x-y) + y]\]

\[= y∧x\]

Hence y = \[-[x-(x ∧ y)] + (x ∨ y), \text{ for all } x, y \in L\]
**Definition 3.2.5:** In any lattice ordered loop L, two positive elements $x, y \in L$ are said to be orthogonal or disjoint if $x \wedge y = 0$.

**Theorem 3.2.6:** In any lattice ordered loop L, if $x \wedge y = 0$ and $x \wedge z = 0$ then $x \wedge (y+z) = 0 \forall x, y, z \in L$

**Proof:** Let L be a lattice ordered loop. Let $x, y, z \in L \text{ and } x, y, z \geq 0$ and so $x \wedge (y+z) \geq 0 \rightarrow (1)$

Now $0 = \{x+(x \wedge z)\} \wedge \{y+(x \wedge z)\}$

$= (x+x) \wedge (x+z) \wedge (y+x) \wedge (y+z)$ (by use of the distributivity of addition over meet)

Hence $0 \geq x \wedge (y+z) \rightarrow (2)$

Therefore from $(1)$ and $(2)$, we conclude that $x \wedge (y+z) = 0, \forall x, y, z \in L$.

**Note 3.2.3:** Similarly we can also prove that if $x \vee y = 0$ and $x \vee z = 0$ then $x \vee (y+z) = 0, \forall x, y, z \in L$, where L is a lattice ordered loop.

**Proof:** By taking the dual of the theorem (3.2.6)

**Note 3.2.4:** From the above we conclude that the class of loops as a variety.

**Definition 3.2.6:** Two relations $\theta$ and $\theta^1$ on S are permutable if and only if $\theta \theta^1 = \theta^1 \theta$

**Definition 3.2.7:** In any partial ordered loop, an element is called positive when $x \geq 0$.

**Theorem 3.2.7:** In a lattice ordered loop L, disjoint positive elements are permutable.

i.e. if $x \wedge y = 0$ then $x + y = y + x$, for all $x, y \in L$

**Proof:** Let L be a lattice ordered loop. We know that in a lattice ordered loop $x-(x \wedge y)+y = y \forall x$, for all $x, y \in L$ (by lemma (3.2.1))
$$\implies x - 0 + y = y \lor x \text{ (since } x \land y = 0)$$

$$\implies x + y = y \lor x.$$  

Similarly $y + x = x \lor y$

Therefore $x + y = y + x$ (since $x \lor y = y \lor x$), for all $x, y \in L$.

We recall the following (without proof)

**Theorem 3.2.8:** (Birkhoff) A lattice $(L, \lor, \land)$ is distributive if and only if, for any $a, x, y \in L$, $a \land x = a \land y$ and $a \lor x = a \lor y \implies x = y$. (This property can be called as cancellation rule)

**Lemma 3.2.2:** Any lattice ordered loop is a distributive lattice.

**Proof:** Let $L$ be a lattice ordered loop.

Let $a \lor x = a \lor y$ and $a \land x = a \land y$ for all $a, x, y \in L$ 

Now $y = -[a-(a \land y)] + (a \lor y)$ (by lemma 3.2.1)

$= -[a-(a \land x)] + (a \lor x)$

$= x$ (by lemma 3.2.1)

So by the Birkhoff theorem (3.2.8), $L$ is a distributive lattice.

Hence any lattice ordered loop is a distributive lattice.

**Lemma 3.2.3:** Any lattice ordered loop is a modular lattice.

**Proof:** we know that every distributive lattice is modular.

Since any lattice ordered loop is distributive (by lemma (3.2.2)), and hence any lattice ordered loop is modular lattice.

**Note 3.2.5:** By lemma 3.2.1 we have

$[x-(x \land y)] + y = x \lor y \implies (1)$
and hence putting  \( y=0 \)

\[ x = (x \lor 0) + (x \land 0) \quad \rightarrow \quad (2) \]

Also from (1) if \( x \land y = 0 \) then \( x+y = x \lor y \quad \rightarrow \quad (3) \)

§3.3. CONCLUSION:

This research work make possible that \((S, +, \leq)\) is partially ordered loop
iff (i) \( x \leq y \Leftrightarrow x+z \leq y+z \), (ii) \( x \leq y \Leftrightarrow z+x \leq z+y \). It is observed that in a partially
ordered loop there are four equivalent conditions. In any lattice ordered loop
there are six conditions and their duals are also satisfied. In any lattice ordered
loop \( L \), if \( x \land y = o \) and \( x \land z = 0 \) then \( x \land (y+z) = 0 \) \( \forall \) \( x, y, z \in L \). In a lattice ordered
loop disjoint positive elements are permutable. Further any lattice ordered loop is
a distributive lattice and hence any lattice ordered loop is modular.