CHAPTER – 5

Characteristics of the Atoms in Lattice Ordered Loops
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CHARACTERISTICS OF THE ATOMS IN LATTICE ORDERED LOOPS

§5.1. INTRODUCTION:

Garrett Birkhoff (1942) firstly initiated the notion of lattice ordered groups. Then Bruck (1944) contributed various results in the theory of quasigroups. Zelinski (1948) described about ordered loops. The concept of non associative number theory was thoroughly studied by Evans (1957). Bruck (1963) explained about what is a loop? Various crucial properties of lattice ordered groups were established by Garrett Birkhoff in 1964 and 1967. Evans (1970) described about lattice ordered loops and quasigroups. Richard Hubert Bruck (1971) made a survey of binary systems. In the recent past Hala (1990) made a description on quasigroups and loops.

In this manuscript, we assume that L is a lattice ordered loop. Further, we furnish definitions, examples and some properties of atoms in lattice ordered loops. We initiate the concepts of positive and negative atoms, dual atom, atomic lattice, meet irreducible element, join irreducible element, descending chain condition and ascending chain condition, right Archimedean when compared with groups, in additive notation. In this manuscript mainly there are two topics, one is about atoms in lattice ordered loops and the other is about Archimedean property. The properties and the definitions, examples are in additive notation.

§5.2. ATOMS IN LATTICE ORDERED LOOPS:

Definition5.2.1: A system \((S, +, \leq)\), where S is a nonempty set, \(+\) is a binary operation on S and \(\leq\) is a binary relation on S satisfying

(i) \((S, +)\) is a loop

(ii) \((S, \leq)\) is a partially ordered set
(iii) The translations \( x \mapsto a + x \) and \( x \mapsto x + b \) are ordered automorphisms of \( S \), is called a partially ordered loop (briefly P.O. loop).

**Example 5.2.1:** \((\mathbb{Z}, +, \leq)\) is a partially ordered loop.

**Definition 5.2.2:** A lattice ordered loop \( L \) is a partially ordered loop in which the partial order is a lattice order.

**Definition 5.2.3:** Let \( L \) be a lattice ordered loop. An element \( x \) of \( L \) is called positive element if \( x \geq 0 \).

**Definition 5.2.4:** In a lattice ordered loop \( L \), a positive element which covers 0 is called an atom.

**Note 5.2.1:** An atom is never minimal

**Proof:** Let \( L \) be a lattice ordered loop and \( p \) be an atom of \( L \).

By definition of an atom \( p \) covers some minimal element of \( L \) and hence \( p \) is not minimal.

Hence an atom is never minimal.

**Note 5.2.2:** An atom is an element that is minimal element among the nonzero elements.

**Proof:** By definition of atom and by Note (5.2.1), an atom is an element that is minimal element among the nonzero elements.

**Definition 5.2.5:** Let \( L \) has a bottom element 0. An element \( x \) of \( L \) is an atom if \( 0 < x \) and there is no element \( y \) of \( L \) such that \( 0 < y < x \).

**Examples:** 5.2.2 In the power set \( P(X) \) of a nonempty set \( X \),

for any \( a \in X, \{a\} \) is an atom.

Because in \((P(X), \subseteq)\) is a poset in a lattice ordered loop \( L \) with unique minimal element \( \emptyset \).
Thus all singleton sub sets \{a\} of \(P(X)\) are atoms in \(L\).

**Example 5.2.3:** Let \(N\) be the set of all natural numbers. If \((N, \leq)\) is a partly ordered set in a lattice ordered loop \(L\), then 2 is the only atom because it is covered by the minimal element 1 in \(N\) and hence by def of atom 2 is the only atom in \(N\) by the usual relation \(\leq\).

**Example 5.2.4:** If \((N, |)\) is a partly ordered set in a lattice ordered loop \(L\) then every prime number is an atom.

**Solution:** If \((N,|)\) is a partly ordered set in a lattice ordered loop \(L\).

Here “|” is a relation called “divides”.

So \(N\) is the set of natural numbers with the relation “divides”.

As prime has only divisors 1 and itself, so \(1|p\), for all primes \(p\) in \(N\).

Here 1 is covered by all primes in \(N\).

So every prime is an atom in \((N,|)\).

**Definition 5.2.6:** In a lattice ordered loop \(L\), dually an element ‘\(m\)’ of a lattice bounded above is said to be a dual atom if \(m \prec 1\). That is \(m\) is covered by 1 or 1 covers \(m\).

**Example 5.2.5:** In \(P(X)\), for any \(a \in X\), \(X - \{a\}\) is a dual atom.

**Definition 5.2.7:** A lattice ordered loop \(L\) bounded below is said to be an atomic lattice if for every \(a \neq 0\) of \(L\) there exists an atom \(p\) of \(L\) such that \(p \leq a\).

**Definition 5.2.8:** Let \(L\) be a lattice ordered loop. Then \(L\) is said to satisfy the minimum condition or descending chain condition or simply d.c.c. if for every descending sequence \(a_1 \geq a_2 \geq \ldots\), of elements of \(L\) there exists a positive integer \(n\) such that \(a_n = a_{n+1} = a_{n+2} = \ldots\).

**Definition 5.2.9:** Let \(L\) be a lattice ordered loop. \(L\) is said to satisfy the maximum condition or ascending chain condition or simply a.c.c. if for every
ascending sequence $a_1 \leq a_2 \leq \ldots$, of elements of $L$ there exists a positive integer $n$ such that $a_n = a_{n+1} = a_{n+2} = \ldots$.

**Example 5.2.6:** Let $N$ be the set of natural numbers and $\leq$ is the usual ordering then $(N, \leq)$ is a partly ordered set in a lattice ordered loop $L$ then it satisfies the minimum condition but does not satisfy the maximum condition.

**Example 5.2.7:** Let $R$ be the set of real numbers and $\leq$ is the usual ordering then $(R, \leq)$ is a partly ordered set in a lattice ordered loop $L$ then it satisfies neither maximum nor minimum conditions.

**Example 5.2.8:** If $X$ is an infinite set then its power set $P(X)$ satisfies neither the maximum nor the minimum condition.

**Theorem 5.2.1:** Let $L$ be a lattice ordered loop, which is bounded below. If it satisfies the minimum condition, then it is atomic.

**Proof:** Let $L$ be a lattice ordered loop and bounded below satisfying the minimum condition.

Let $o \neq x \in L$. Consider $A = \{a \in L / 0 < a \leq x\}$.

Since $L$ satisfies the minimum condition, $A$ has a minimal element say $p$.

Then $p$ is an atom and $p \leq x$.

Therefore for every $0 \neq x \in L$ there is an atom $p$ such that $p \leq x$.

Hence $L$ is an atomic lattice.

**Theorem 5.2.2:** Let $L$ be a lattice ordered loop. An element $p$ of $L$ is an atom if and only if for each element $x$ of $L$ either $p \leq x$ or $x \wedge p = 0$.

**Proof:** Let $L$ be a lattice ordered loop and $p \in L$ is an atom.

Let $x \in L$. Then we have $0 \leq x \wedge p \leq p$.

Since $p$ is an atom we have either $x \wedge p = p$ or $0 = x \wedge p$. 
This implies either \( p \leq x \) or \( x \land p = 0 \).

Conversely suppose that for each element \( x \) of \( L \).

Let \( 0 \leq x \leq p \).

Then by our assumption \( x \land p = 0 \) or \( x = p \).

Then either \( x = 0 \) (since \( x \land p = x \)) or \( x = p \).

Therefore \( P \) is an atom.

**Definition 5.2.10:** An element ‘\( a \)’ of a lattice ordered loop \( L \) is said to be meet irreducible if \( a = a_1 \land a_2 \) \( (a_1, a_2 \in L) \) then either \( a = a_1 \) or \( a = a_2 \). That is it cannot be decomposed by elements greater than ‘\( a \)’.

**Definition 5.2.11:** An element ‘\( a \)’ of a lattice ordered loop \( L \) is said to be join irreducible if \( a = a_1 \lor a_2 \) \( (a_1, a_2 \in L) \) then either \( a = a_1 \) or \( a = a_2 \). That is it cannot be decomposed by elements less than \( a \).

**Example 5.2.9:** Find join and meet irreducible elements of the lattice ordered loop given below.

\[ \text{L}_1 \]

\[ \text{L}_2 \]
**Solution:** Join irreducible elements of $L_1 = \{0, a, b, c, e\}$

Meet irreducible elements of $L_1 = \{c, e, f, g, 1\}$

Join irreducible elements of $L_2 = \{0, a, c, e, f, h\}$

Meet irreducible elements of $L_2 = \{b, e, f, g, i, 1\}$

**Note 5:** Clearly the least element and every atom of a lattice bounded below is join irreducible while the greatest element and every dual atom of a lattice bounded above is meet irreducible by the above definitions (5.2.10), (5.2.11).

**Note 5.2.4:** Every element of a chain is meet irreducible as well as join irreducible.

**Result 5.2.1:** In a lattice ordered loop $L$, $L$ is a chain if and only if every one of its elements is meet irreducible.

**Proof:** Suppose $L$ is a chain and $a \in L$.

If $a = b \land c$ then either $a = b$ or $c$.

Since $b$ and $c$ are comparable we have $b \land c = b$ or $c$.

Hence $a$ is meet irreducible.

Conversely suppose that every element of $L$ is meet irreducible and let $a, b \in L$.

Now, by assumption $a \land b$ is meet irreducible and $a \land b = a \land b$.

This implies, $a \land b = a$ or $a \land b = b$.

Hence $a$ and $b$ are comparable.

Thus $L$ is a chain.

**Theorem 5.2.3:** In a lattice ordered loop $L$, every atom is join irreducible.

**Proof:** Let $L$ be a lattice ordered loop and $p \in L$ is an atom.
Claim: $p$ is join irreducible.

Let $a_1, a_2 \in L$ such that $p = a_1 \lor a_2$

$\Rightarrow a_1 \leq p$ and $a_2 \leq p$

$\Rightarrow 0 \leq a_1 \leq p$ and $0 \leq a_2 \leq p$.

This implies, $0 = a_1$ or $a_1 = p$ and $0 = a_2$ or $a_2 = p$ (since $p$ is atom).

This leads to $p = 0 \lor a_2$ or $a_1 = p$ and $p = a_1 \lor 0$ or $a_2 = p$.

Then $p = a_2$ or $a_1 = p$ and $p = a_1$ or $a_2 = p$.

Thus we have $a_1, a_2 \in L$ with $p = a_1 \lor a_2$.

Hence $p = a_1$ or $p = a_2$.

Therefore $p$ is join irreducible

Hence every atom in a lattice ordered loop $L$ is join irreducible.

Theorem 5.2.4: In a lattice ordered loop $L$, every dual atom is meet irreducible.

Proof: Let $L$ be a lattice ordered loop and $p \in L$ is dual atom in $L$.

Claim: $p$ is meet irreducible.

Let $a_1, a_2 \in L$ such that $p = a_1 \land a_2$.

Then $p \leq a_1$ and $p \leq a_2$.

This implies $p \leq a_1 \leq 1$ and $p \leq a_2 \leq 1$.

So $p = a_1$ or $a_1 = 1$ and $p = a_2$ or $a_2 = 1$.

Then $p = a_1$ or $p = 1 \land a_2$ and $p = a_2$ or $p = a_1 \land 1$.

This leads to $p = a_1$ or $p = a_2$ and $p = a_2$ or $p = a_1$. 

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Thus we have $a_1, a_2 \in L$ with $p = a_1 \land a_2$.

Hence, $p = a_1$ or $p = a_2$.

Therefore $p$ is meet irreducible.

Hence in a lattice ordered loop $L$, every dual atom is meet irreducible.

**Definition 5.2.12:** Let $L$ is a lattice ordered loop with descending chain condition (d.c.c.) on its positive elements, which we call positive atoms.

**Note 5.2.5:** Any two positive atoms in a lattice ordered loop $L$ are orthogonal. That is, $x \land y = 0$, for all $x, y$ in $L$.

**Theorem 5.2.5:** In a lattice ordered loop $L$, if $x, y, z$ are positive atoms such that $x \land y = 0$ and $x \land z = 0$ then $x \land (y + z) = 0$.

**Proof:** Let $L$ be a lattice ordered loop and $x, y, z$ are positive atoms such that $x \land y = 0$ and $x \land z = 0$.

Let $x, y, z \geq 0$ and so $x \land (y + z) \geq 0 \rightarrow (1)$

Now $0 = \{x + (x \land z)\} \land \{y + (x \land z)\} = (x + x) \land (x + z) \land (x \land (y + z)) \land (y + z)$ (by use of the distributives of addition over meet).

Hence $0 \geq x \land (y + z) \rightarrow (2)$

Therefore from (1) and (2), we conclude that $x \land (y + z) = 0$.

**Note 5.2.6:** Similarly we can also prove that in a lattice ordered loop $L$, if $x, y, z$ are positive atoms such that $x \lor y = 0$ and $x \lor z = 0$ then $x \lor (y + z) = 0$

**Proof:** By taking the dual of the theorem.

**Definition 5.2.13:** Let ‘$a$’ be a positive atom. We will denote the unique element $u$ such that $u + a = a + u = 0$ by $\bar{a}$, and call it a negative atom.
**Theorem 5.2.6:** In a Lattice ordered loop $L$, if $a$ is a positive atom and $a + a > x > 0$, then $x = a$.

Proof: Let $L$ be a lattice ordered loop, since $x > 0$, by the descending chain condition, $x$ contains some atom.

If $x \geq a$, then $a + a > x > 0$ and hence $a > x - a \geq 0$.

Hence $x - a = 0$ and $x = a$.

If $x \geq b$, where $b$ is an atom different from $a$,

then $a + a > x$ implies $(a + a) \wedge b \geq x \wedge b$.

That is, $0 \geq b$, a contradiction.

Hence in a lattice ordered loop $L$, if $a$ is a positive atom and $a + a > x > 0$, then $x = a$.

**Theorem 5.2.7:** In a Lattice ordered loop $L$, if $a, b$ are positive atoms and $a + b > x > 0$, then $x = a$ or $x = b$.

Proof: Same as theorem (5.2.6)

**Theorem 5.2.8:** In a Lattice ordered loop $L$, if $a$ is a positive atom, $0 - a = -a + 0$.

(That is, the left and right inverses of $a$ are equal.)

**Proof:** Let $L$ be a lattice ordered loop, let $u + a = 0$.

Now $a + u$ covers $u$ and so $(a + u) + a$ covers $u + a$.

Hence, $(a + u) + a > 0$.

Also, $a + a = (a + (u + a)) + a > (a + u) + a$.

By theorem (5.2.7), $(a + u) + a = a$ and so $a + u = 0$.

Now if $a$ is a positive atom we will denote the unique element $u$
such that \( u + a = a + u = 0 \) by \( \alpha \), and call it a negative atom.

Negative atoms are covered by 0, since \( \alpha < x < 0 \) implies \( 0 = \alpha + a < x + a < a \).

Also any element \( u \) covered by 0 is a negative atom, since \( 0 - u \) will cover 0 and so is a positive atom \( a \).

Hence \( u = -(0 - u) + 0 = \bar{a} \).

Thus, the negative atoms are precisely the maximal negative elements.

**Note 5.2.7:** Clearly any nonempty set of negative elements in \( L \) has a maximal element and the negative elements satisfy the ascending chain condition (a.c.c).

**Theorem 5.2.9:** In a Lattice ordered loop \( L \), if \( a, b \) are positive atoms in \( L \) and \( a > x > \alpha \), then \( x = 0 \) or \( x = a + \bar{b} \).

**Proof:** Let \( L \) is a lattice ordered loop and we note that if \( a = b \) then \( a > x > \alpha \) and so \( a + a > x + a > 0 \).

Hence by theorem (5.2.6), \( x + a = a \) and \( x = 0 \).

Assume that \( a \neq b \). If \( a > x > b \), then \( a + b > x + b > 0 \).

Hence \( x + b = a \) or \( x + b = b \).

If \( x + b = b \) then \( x = 0 \).

This leaves the case \( x + b = a \).

Now \( a > (a + \bar{b}) > \bar{b} \) and so \( a + b > (a + \bar{b}) + b > 0 \).

By theorem (5.2.7) \( (a + \bar{b}) + b = a \) or \( (a + \bar{b}) + b = b \).

But \( (a + \bar{b}) + b \) implies \( a + \bar{b} = 0, a = b \).

Hence we must have \( (a + \bar{b}) + b = a \).

Since \( x + b = a \), by cancellation we have \( x = a + \bar{b} \).
Hence in a Lattice ordered loop L, if a, b are positive atoms in L and \( a > x > b \), then \( x = 0 \) or \( x = a + b \).

**Example 5.2.10:** Let A be a set and \( P = 2^A \) its power set. \( P \) is a partial ordered set ordered by \( \subseteq \) with a unique minimal element \( \phi \). Thus, all singleton subsets \( \{a\} \) of A are atoms in \( P \).

**Example 5.2.11:** \( \mathbb{Z}^+ \) is partially ordered set in a lattice ordered loop L and if we define \( a \leq b \) to mean that \( a \mid b \). Then \( \mid \) is a minimal element and any prime number \( P \) is an atom in L.

We recall the following (without proof)

**Theorem 5.2.10:** (Birkhoff): A lattice \((L, \lor, \land)\) is distributive if and only if, for any \( a, x, y, \in L \), \( a \land x = a \land y \) and \( a \lor x = a \lor y \implies x = y \). (This property can be called as cancellation rule).

**Lemma 5.2.1:** In a lattice ordered loop L, \(-[x - (x \land y)] + (x \lor y) = y\) for all \( x, y, z \in L \).

**Proof:** Let L be a lattice ordered loop and \( x, y, z \in L \).

Now \( [x-(x \land y)] + y = [(x-x) \lor (x-y)] + y = [0 \lor (x-y)] + y \)

\[ = (0+y) \lor [(x-y) + y] = y \lor x \]

Hence \( y = -[x-(x \land y)] + (x \lor y) \), for all \( x, y, z \in L \).

**Lemma 5.2.2:** Any lattice ordered loop is a distributive lattice.

**Proof:** Let L be a lattice ordered loop.

Let \( a \lor x = a \lor y \) and \( a \land x = a \land y \) in a lattice ordered loop L.

Now \( y = -[a-(a \land y)] + (a \lor y) \) (by lemma 5.2.1)

\[ = -[a-(a \land x)] + (a \lor x) = x \) (by lemma 5.2.1)
So by the Birkhoff theorem (5.2.8), L is a distributive lattice.

Hence any lattice ordered loop is a distributive lattice.

**Definition 5.2.14:** An element a of A is called a right Archimedean of A, if a ≥ 0 and if, for every element b, there is a positive integer n such that na ≥ b.

**Note 5.2.8:** If all positive elements of A are right Archimedean, it is called to be right Archimedean.

**Theorem 5.2.11:** Let L be a Lattice ordered loop. Then L is Archimedean, if and only if na ≤ b, n = 1, 2, … implies a ≤ 0.

In other words, in an ordered loop L, the concepts of the integrally closed and Archimedean are equivalent.

**Proof:** Let L be a lattice ordered loop and Archimedean.

If na ≤ b, n = 1, 2, …, and a > 0, then n₀a ≥ b for some n₀, which is a contradiction to na ≤ b.

By definition of ordered loop, a must be non-positive.

That is a ≤ 0.

Conversely suppose if ordered loop is integrally closed and let ‘a’ be a positive element.

Suppose na ≤ b (n = 1, 2…), then element a must be non-negative, which contradicts a > 0.

Hence for some natural number n₀, n₀a > b.

Hence L is Archimedean.
**Theorem 5.2.12:** Let $L$ be a lattice ordered loop. If $L$ has an atom and if its atom is Archimedean, then it is a cyclic group.

**Proof:** Given that $L$ be a lattice ordered loop.

Let $p$ be atom of $L$, then we have the identities:

\[ p + np = np + p \text{ (} n = 1, 2\ldots\text{)} \text{ and} \]

\[ p + (a + b) = (p + a) + b = (a + b) + p, \text{ for } a, b \text{ of } L \rightarrow (1) \]

In terminology of the theory of loops, $p$ is in the center of $L$ since $p$ is atom, there exists no element $c$ such that $a + b < c < p + (a + b)$, $a + b < c < (p + a) + b$ and $a + b < c < (a + b) + p$.

This shows the equality (1).

Thus the set \{np / n = 0, ±1, 2\ldots\} is a cyclic group.

Since $p$ is Archimedean, there is a positive integer such that $0 < a < np$ for any fixed positive element $a$.

Let $n$ be such a minimal positive integer.

Then we have $(n-1)p \leq a < np$.

Thus $0 \leq a - (n - 1)p < p$.

Since $p$ is atom, $a = (n-1)p$.

Therefore, every element $a$ of $L$ is written in the form $a = np$ (n = 0, ±1, 2\ldots)

By the above theorem (5.2.11), $L$ must be cyclic group.

**Corollary 5.2.1:** Let $L$ be an Archimedean ordered loop. If $L$ has an atom, it is isomorphic to the set of all integers.
**Proof:** Suppose L is an Archimedean ordered loop.

If L has atom then by the above theorem every element a of L is written in the form $a = np$, where n is an integer, and hence it is isomorphic to the set of integers.

Hence if Archimedean ordered loop has atom then it is isomorphic to the set of all integers.

§5.3. **CONCLUSION:**

This study makes it possible the following results: A lattice ordered loop, if it is bounded below satisfies the minimum condition then it is atomic. An equivalent condition established for an element of a lattice ordered loop to be an atom. Every element of a chain is meet irreducible as well as join irreducible. In a lattice ordered loop, every atom is join irreducible where as every dual atom is meet irreducible. Any lattice ordered loop is a distributive lattice. Established an equivalent condition for an ordered loop to be Archimedean. If an ordered loop has atom and if its atom is Archimedean, then it is a cyclic group.