CHAPTER 1

INTRODUCTION

Decision making is the process of identifying and choosing alternatives based on the values and preferences of the decision maker. It is the process of sufficiently reducing uncertainty and doubt about alternatives to allow a reasonable choice to be made from among them. Decision making based solely on a single criterion appears insufficient as soon as the decision-making process deals with the complex organizational environment. So, one must acknowledge the presence of several criteria that lead to the development of multi criteria decision making.

Optimization is a kind of the decision making, in which decisions have to be taken to optimize one or more objectives under some prescribed set of circumstances. These problems may be a single or multi objective and are to be optimized (maximized or minimized) under a specified set of constraints. The constraints usually are in the form of inequalities or equalities. Such problems which often arise as a result of mathematical modeling of many real life situations are called optimization problems.

1.1 Single Objective Optimization Problem

In many real life situations optimization problems are modeled and solved as single-objective optimization problems in a deterministic and crisp environment. The general form of single-objective optimization problem is:

\[
\begin{align*}
\text{Min.}(\text{Max.}) & \ f(X), & X = (x_1, x_2, \ldots, x_n) & \quad \ldots(1.1.1) \\
\text{sub.to} & \ & g_i(X) \leq 0, & i = 1, 2, 3, \ldots k \\
& & h_i(X) \leq 0, & i = 1, 2, 3, \ldots l \\
& & \phi_i(X) \leq 0, & i = 1, 2, 3, \ldots m \\
\end{align*}
\]

Where \( f, g_i, h, \) & \( \phi \) are real valued functions defined on \( \mathbb{R}^n \). \( X = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) is called decision vector and \( x_1, x_2, \ldots, x_n \) are called decision or unknown variables.
In case all the functions are linear then the above problem is called linear programming problem, otherwise it is called non-linear programming problem.

1.2 Transportation Problem

It is a special type of linear programming problem which arises in many practical applications. In the beginning it was formulate for determining the optimal shipping pattern, so it is called transportation problem. The conventional and very well known transportation problem consists in transporting various amount of single homogeneous commodity from m origins \(i=1, 2, \ldots, m\) to \(n\) destination \(j=1, 2, \ldots, n\). The origins are production facilities with respective capacities \(a_1, a_2, \ldots, a_m\) and the destinations are warehouses with required levels of demand \(b_1, b_2, \ldots, b_n\) for the transport of a unit of the given product from the \(i^{th}\) source to the \(j^{th}\) destination a cost \(c_{ij}\) is given for which, without loss of generality, we can assume \(c_{ij} \geq 0 \forall i, j\). Hence, one must determine the amounts \(x_{ij}\) to be transported from all the origins \(i=1, 2, \ldots, m\) to all the destinations \(j=1, 2, \ldots, n\) is such a way that total cost is minimized. This problem can be suitably modeled as a linear programming problem. Thus the conventional transportation problem can be mathematically expressed as:

\[
\text{Min} \ Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} \quad \text{...(1.2.1)}
\]

Subject to

\[
\sum_{j=1}^{n} x_{ij} = a_i, \quad i=1, 2, \ldots, m
\]

\[
\sum_{i=1}^{m} x_{ij} = b_j, \quad j=1, 2, \ldots, n
\]

\(x_{ij} \geq 0, \quad i=1, 2, \ldots, m, \quad j=1, 2, \ldots, n\)

\[
\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j
\]
(balanced condition).

**Definition 1.1**

A set of non-negative allocation $x_{ij}$ is said to be feasible solution of fuzzy linear programming problem if and only if it satisfies the row and column restriction of the problem.

**Definition 1.2**

If there are $m$ equations in $(m + n)$ variables then to solve these equations put any $n$ variables equal to zero and find the solution of $m$ equations in $m$ variables. If the obtained solution is unique then it is called basic solution, otherwise, it is called non-basic solution. The zero valued variables are called non-basic variables and the remaining variables are called basic variables.

**Definition 1.3**

A feasible solution to a $m$-origin and $n$-destination problem is said to be basic feasible solution if the number of positive allocations are $(m + n -1)$.

**Definition 1.4**

If the number of allocations in a basic feasible solutions are less than $(m + n -1)$, it is called degenerate basic feasible solution (otherwise non-degenerate).

**Definition 1.5**

A feasible solution (not necessarily basic) is said to be optimal solution if it minimizes the total transportation cost.

**1.3 Multi Objective Optimization Problems**

Many real life optimization problems are multi objective in nature and are to be optimized simultaneously subject to a common set of constraints. The most general mathematical model of a multi objective optimization problem is:
\[ \text{Min.}(\text{Max.}) \ F(X) = [f_1(x), f_2(x), \ldots, f_m(x)] , \quad X = (x_1, x_2, \ldots, x_n) \ \ldots(1.3.1) \]

subject to
\[ g_i(X) \leq 0, \quad i = 1, 2, 3, \ldots, k \]
\[ h_i(X) \leq 0, \quad i = 1, 2, 3, \ldots, l \]
\[ \phi_i(X) \leq 0, \quad i = 1, 2, 3, \ldots, m \]

Where \( f_1, f_2, \ldots, f_m \) are the objective functions. Variables \( x_1, x_2, \ldots, x_n \) are called decision variables and \( X \) is called decision vector. This problem is also called multi objective programming problem.

### 1.4 Multi Objective Transportation Problem (MOTP)

In real life situations, the transportation problems are not single objective. The transportation problems which are characterized by multiple objective functions are considered here. A special type of linear programming problem in which constraints are of equality type and all the objectives are conflicting with each other, are called MOTP. Similar to a typical transportation problem, in a MOTP problem a product is to be transported from \( m \) sources to \( n \) destinations and their capacities are \( a_1, a_2, \ldots, a_m \) and \( b_1, b_2, \ldots, b_n \) respectively. In addition, there is a penalty \( c_{ij} \) associated with transporting a unit of product from \( i^{th} \) source to \( j^{th} \) destination. This penalty may be cost or delivery time or safety of delivery or etc. A variable \( x_{ij} \) represents the unknown quantity to be shipped from \( i^{th} \) source to \( j^{th} \) destination. A mathematical model of MOTP with \( r \) objectives, \( m \) sources and \( n \) destinations can be written as:

\[
\text{Min.} z_k = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}, \quad k= 1, 2, \ldots, K \quad \ldots(1.4.1)
\]

subject to
\[
\sum_{j=1}^{n} x_{ij} = a_i, \quad i=1, 2, \ldots, m
\]
\[
\sum_{i=1}^{m} x_{ij} = b_j, \quad j=1, 2, \ldots, n
\]
The subscript on $Z_k$ and superscript on $c_{ij}^k$ denote the k-th penalty criterion, $a_i > 0$ for all $i$, $b_j > 0$ for all $j$, $c_{ij}^k \geq 0$ for all $(i, j)$, and $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ (balanced condition).

1.5 Different Types of Multi Objective Transportation Problem

Since different assumptions about the way of satisfying the constraints and some lack of precision in the statement of the problem can be supposed, several versions of the problem can arise.

1.5.1 MOTP with equality constraints

MOTP with equality constraint is of the form:

$$\min z_k = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^k x_{ij}, \quad k = 1, 2, \ldots, K$$

Subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \ldots, n$$

Here the problem is feasible $\iff$ $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ holds.

1.5.2 MOTP with inequality constraints

MOTP with inequalities both in supply and demand constraints can be presented as:
\[
Min. z_k = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^k x_{ij}, \quad k = 1, 2, \ldots, K \quad \ldots 1.5.2.1
\]

Subject to

\[
\sum_{j=1}^{n} x_{ij} \leq a_i, \quad i = 1, 2, \ldots, m
\]

\[
\sum_{i=1}^{m} x_{ij} \geq b_j, \quad j = 1, 2, \ldots, n
\]

\[
x_{ij} \geq 0, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n
\]

Here the problem is feasible \(\iff\) \(\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j\) holds.

**1.5.3 MOTP with mixed constraints**

Transportation problem constraints, that is, with equality supply constraints and inequality demand constraints is as follows:

\[
Min. z_k = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^k x_{ij}, \quad k = 1, 2, \ldots, K \quad \ldots 1.5.3.1
\]

Subject to

\[
\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \ldots, m
\]

\[
\sum_{i=1}^{m} x_{ij} \geq b_j, \quad j = 1, 2, \ldots, n
\]

\[
x_{ij} \geq 0, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n
\]

Here the problem is feasible \(\iff\) \(\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j\) holds.
(b) Transportation problems with inequality in supply constraints and equality in demand constraints can be presented as:

\[ \text{Min.} z_k = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^k x_{ij}, \quad k=1, 2, \ldots, K \quad \ldots \text{1.5.3.2} \]

Subject to

\[ \sum_{j=1}^{n} x_{ij} \leq a_i, \quad i=1, 2, \ldots, m \]

\[ \sum_{i=1}^{m} x_{ij} = b_j, \quad j=1, 2, \ldots, n \]

\[ x_{ij} \geq 0, \quad i=1, 2, \ldots, m, \quad j=1, 2, \ldots, n \]

Here the problem is feasible \( \iff \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \) holds.

### 1.6 Different Approaches to Solve MOTP

The linear MOTP is a special type of linear programming problem in which constraints are of equality type and the objectives are conflicting with each other. The existing solution procedure of this problem can be divided into two categories.

First category consists those that are generating all the sets of efficient solution and the second category represents the procedures that are seeking the best compromise solution among the set of efficient solution. From a practical point of view the knowledge of the set of efficient solutions is not always necessary. In such a case, a procedure is needed to determine a compromise solution. As a result, different approaches are developed in the context of MOTP to find the compromise solution.

The various approaches to solve MOTP are:

- Fuzzy programming technique
- Fuzzy goal programming approach
• Interactive procedure
• Spanning tree based genetic algorithm
• Geometric programming approach
• Genetic algorithm
• Interactive fuzzy multi objective linear programming

Each and every approach has its own drawbacks as well as its own strengths, and any rational choice as to which should be used is almost dependent on at least two vital considerations:

1. The type and size of the problem
2. The characteristics of the ultimate decision maker(s)

But in this thesis we studied different fuzzy programming technique to find an optimal compromise solution of a transportation problem with several objectives and for both linear and non-linear membership functions. And finally in last chapter some new and simple methods proposed to find initial solution of multi objective transportation problem.

**Definition 1.6**

A feasible solution \( \hat{x} = \{\hat{x}_{ij}\} \in X \) is said to be a non-dominated solution of MOTP if there is no other feasible solution \( x = \{x_{ij}\} \in X \) such that

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} x_{ij} \leq \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} \hat{x}_{ij} \quad \text{for all } k \text{ and}
\]

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} x_{ij} < \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} \hat{x}_{ij} \quad \text{for at least one } k.
\]

**Definition 1.7**

An optimal compromise solution of the MOTP is a solution \( \hat{x} = \{\hat{x}_{ij}\} \in X \) which is preferred by the decision maker to all other solutions, taking into consideration all criteria contained in the multi-objective function. Hence, an optimal compromise solution has to be a non-dominated solution according to the definition of non-dominated solution.
1.7 Fuzzy Set Theory & Decision Making.

1.7.1 Fuzzy sets and membership

The idea proposed by Lotfi Zadeh suggested that set membership is the key to decision making when faced with uncertainty. In fact, Zadeh made the following statement in his seminal paper of 1965:

The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing. Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables.

As an example, we can easily assess whether someone is over 6 feet tall. In a binary sense, the person either is or is not, based on the accuracy, or imprecision, of our measuring device. For example, if “tall” is a set defined as heights equal to or greater than 6 feet, a computer would not recognize an individual of height 5’11.999” as being a member of the set “tall.” But how do we assess the uncertainty in the following question: Is the person nearly 6 feet tall? The uncertainty in this case is due to the vagueness or ambiguity of the adjective nearly. A 5’11” person could clearly be a member of the set of “nearly 6 feet tall” people. In the first situation, the uncertainty of whether a person, whose height is unknown, is 6 feet or not is binary; the person either is or is not, and we can produce a probability assessment of that prospect based on height data from many people. But the uncertainty of whether a person is nearly 6 feet is nonrandom. The degree to which the person approaches a height of 6 feet is fuzzy. In reality, “tallness” is a matter of degree and is relative. Among peoples of the some tribal area a height for a male of 6 feet is considered short. So, 6 feet can be tall in one context and short in another. In the real (fuzzy) world, the set of tall people can overlap with the set of not-tall people, impossibility when one follows the precepts of classical binary logic.
This notion of set membership, then, is central to the representation of objects within a
universe by sets defined on the universe. Classical sets contain objects that satisfy precise
properties of membership: fuzzy sets contain objects that satisfy imprecise properties of
membership, that is, membership of an object in a fuzzy set can be approximate. For
example, the set of heights from 5 to 7 feet is precise (crisp); the set of heights in the
region around 6 feet is imprecise, or fuzzy. To elaborate, suppose we have an exhaustive
collection of individual elements (singletons) x, which make up a universe of information
(discourse), X. Further, various combinations of these individual elements make up sets,
say A, on the universe. For crisp sets, an element x in the universe X is either a member
of some crisp set A or not. This binary issue of membership can be represented
mathematically with the indicator function.

\[
\chi_A(x) = \begin{cases} 
1 & x \in A \\
0 & x \notin A 
\end{cases}
\]

Where the symbol \( \chi_A(x) \) gives the indication of an unambiguous membership of element
x in set A, and the symbols \( \in \) and \( \notin \) denote contained in and not contained in,
respectively. For our example of the universe of heights of people, suppose set A is the
crisp set of all people with \( 5 \leq x \leq 7 \) feet, shown in Figure 1.7.1.1. A particular
individual, \( x_1 \), has a height of 6.0 feet. The membership of this individual in crisp set A is
equal to 1, or full membership, given symbolically as \( \chi_A(x_1) = 1 \). Another individual, say
\( x_2 \) has a height of 4.99 feet. The membership of this individual in set A is equal to 0, or
no membership, hence \( \chi_A(x_2) = 0 \) also seen in Figure 1.7.1.1. In these cases the
membership in a set is binary, either an element is a member of a set or it is not.
Zadeh extended the notion of binary membership to accommodate various “degrees of membership” on the real continuous interval \([0, 1]\) where the endpoints of 0 and 1 conform to no membership and full membership, respectively, just as the indicator function does for crisp sets, but where the infinite number of values in between the endpoints can represent various degrees of membership for an element \(x\) in some set on the universe. The sets on the universe \(X\) that can accommodate “degrees of membership” were termed by Zadeh as fuzzy sets. Continuing further on the example on heights, consider a set \(H\) consisting of heights near 6 feet. Since the property near 6 feet is fuzzy, there is no unique membership function for \(H\). Rather, the analyst must decide what the membership function, denoted \(\mu_H\), should look like. Plausible properties of this function might be (1) normality \(\mu_H(6) = 1\), (2) monotonicity (the closer \(H\) is to 6, the closer \(\mu_H\) is to 1), and (3) symmetry (numbers equidistant from 6 should have the same value of \(\mu_H\)) (Bezdek, 1993). Such a membership function is illustrated in Figure 1.7.1.2. A key difference between crisp and fuzzy sets is their membership function; a crisp set has a unique membership function, whereas a fuzzy set can have an infinite number of membership functions to represent it. For fuzzy sets, the uniqueness is sacrificed, but flexibility is gained because the membership function can be adjusted to maximize the utility for a particular application.
James Bezdek provided one of the most lucid comparisons between crisp and fuzzy sets (Bezdek, 1993). It bears repeating here. Crisp sets of real objects are equivalent to, and isomorphically described by, a unique membership function, such as $\chi_A$ in Figure 1.7.1.1. But there is no set-theoretic equivalent of “real objects” corresponding to $\chi_A$. Fuzzy sets are always functions, which map a universe of objects, say $X$ onto the unit interval $[0,1]$; that is, the fuzzy set $H$ is the function $\mu_H$ that carries $X$ into $[0,1]$. Hence, every function that maps $X$ onto $[0,1]$ is a fuzzy set. Although this statement is true in a formal mathematical sense, many functions that quality on the basis of this definition cannot be suitable fuzzy sets. But, they become fuzzy sets when, and only when, they match some intuitively plausible semantic description of imprecise properties of the objects in $X$.

The membership function embodies the mathematical representation of membership in a set, and the notation used throughout this thesis for a fuzzy set is a set symbol with a tilde notation say $\tilde{A}$ where the functional mapping is given as

$$\mu_A(x) \in [0,1]$$

And the symbol $\mu_A(x)$ is the degree of membership of element $x$ in fuzzy set $A$. Therefore, $\mu_A(x)$ is a value on the unit interval that measures the degree to which element $x$ belongs to fuzzy set $A$; equivalently $\mu_A(x) =$ degree to which $x \in A$. 
**Definition 1.8**

A crisp set or a classical set $A$ is defined as a collection of well defined objects. The objects are called elements of $A$. A crisp set $A$, defined on the universal set $X$, can also be represented by $A = \{(x, \mu_A(x)); x \in X\}$

Where $\mu_A(x)$ is called characteristic function, that declares which element of universal set $X$ are member of a set and which are not. Set $A$ is defined by its characteristic function $\mu_A(x)$ as:

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

The characteristic function maps elements of set $X$ to elements of set $\{0,1\}$, which is formally expressed as:

$$\mu_A : X \rightarrow \{0,1\}.$$

**Definition 1.9**

The characteristic function $\mu_A(x)$ of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in $X$. This function can be generalized to a function $\mu_A(x)$ such that the value assigned to the element of the universal set $X$ fall within a specified range $[0,1]$ i.e. $\mu_A : X \rightarrow [0,1]$. The assigned values indicate the membership grade of the element in the set $A$.

The function $\mu_A$ is called the membership function and the set $\tilde{A} = \{x \in X; \mu_A(x) \neq 0\}$ defined by $\mu_A$ for each $x \in X$ is called a fuzzy set. $\mu_A(x)$ is the degree of membership of $x$ in $\tilde{A}$. The closer the value of $\mu_A(x)$ is to one the more $x$ belongs to $A$.

**Definition 1.10**
Let $\tilde{A}$ be a fuzzy set and $\alpha$ be a real number in the interval $[0,1]$. The crisp set $A_{\alpha}$ defined by $A_{\alpha} = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$ is called $\alpha$ cut of $\tilde{A}$.

### 1.7.2 Management and decision making

The subject of the field of decision making is, as the name suggests, the study both of how decisions are actually made and how they can be made better or more successfully. Much of the focus in this field has been in the area of management, in which the decision-making process is of key importance for functions such as inventory control, investments, Personnel actions, new-product development, allocation of resources, and many others. Decision making itself, however, is broadly defined to include any choice or selection of alternatives and is therefore of importance in many fields in both the “soft” social sciences and the “hard” natural and engineering sciences.

Applications of fuzzy sets within the field of decision making have, for the most part, consisted of extensions or “fuzzifications” of the classical theories of decision making. While decision making under conditions of risk and uncertainty have been modeled by probabilistic decision theories and by game theories, fuzzy decisions theories attempt to deal with the vagueness or fuzziness inherent in subjective or imprecise determinations of preferences, constraints, and goals. In this section, we briefly introduce some of the simple application of fuzzy sets in some selected areas of decision making.

Classical decision making generally deals with a set of alternatives comprising the decision space, a set of states of nature comprising the state space, a relation indicating the state or outcome to be expected from each alternative action, and, finally, a utility or objective function, which orders these outcomes according to their desirability. A decision is said to be made under conditions of certainty when the outcome for each action can be determined and ordered precisely, In this case, the alternative that leads to the outcome yielding the highest utility is chosen. A decision is made under conditions of risk, on the other hand when only available knowledge concerning the outcome states is their probability distributions. Again, this information can be used to optimize the utility function. When knowledge of the probabilities of the outcome states is unknown,
decisions must be made under conditions of uncertainty. In this case, fuzzy decision theories may be used to accommodate this vagueness.

There exist several major approaches within the theories of classical crisp decision making. Decisions may, for instance, be considered to occur in a single stage or in multiple stages. The decision maker may be a single person, or a collection of multiple decision makers may be involved. The decision may involve the simple optimization of a utility function, an optimization under constraints, or an optimization given multiple criteria. The first of these three types of decision problems corresponds to statistical decision theory, the second to mathematical (linear or nonlinear) programming, and the last to theories of multicriteria decision making.

Fuzziness can be introduced at several points in the existing models of decision making. For instance, under a simple optimization of utility, any uncertainty concerning the states of the system can be handled by modeling the states as well as the utility assigned to each state with fuzzy sets. Bellman and Zadeh (1970) suggest a fuzzy model of decision that must accommodate certain constraints $C$ and goals $G$. Here, both constraints and goals are treated as fuzzy sets characterized by membership functions.

\[
\begin{align*}
\mu_c : X &\to [0,1] \\
\mu_g : X &\to [0,1]
\end{align*}
\]

where $X$ is the universal set of alternative actions. The expected outcomes that result from these actions can, in many cases, be assumed to remain deterministic or probabilistic, thus restricting the introduction of fuzziness only to the goals and constraints themselves. This fuzziness allows the decision maker to frame the goals and constraints in vague, linguistic terms, which may more accurately reflect the actual state of Knowledge of preference concerning these. The membership function of the fuzzy goal in this case serves much the same purpose as a utility or objective function that orders the outcomes according to preferability. Unlike the classical theory of decision making under constraints, however, in which constraints are defined on the set $X$ to another space, the symmetry between the goals and constraints under this fuzzy model
allows them to be treated in exactly the same manner. This model can be extend to allow
goals and constraints to be defined on different universal sets, for instance, the set $X$ of
possible action and the set $Y$ of possible effects or outcomes. In this case, the fuzzy
constraints may be defined on the set $X$ and the fuzzy goals on the set $Y$ such that

\[
\begin{align*}
\mu_c : & X \rightarrow [0,1] \\
\text{and} & \\
\mu_g : & X \rightarrow [0,1]
\end{align*}
\]

A function $f$ can then be defined as a mapping from the set of actions $X$ to the set of
outcomes $Y$,

\[
f : X \rightarrow Y
\]

Such that a fuzzy goal $G$ defined on set $Y$ induces a corresponding fuzzy goal $G'$ on set
$X$, Thus,

\[
\mu_{G'}(x) = \mu_G(f(x)).
\]

A fuzzy decision $D$ may then be defined as the choice that satisfies both the goals $G$ and
the constraints $C$. If we interpret this as a logical “and” we can model it with the
intersection of the fuzzy sets $G$ and $C$.

\[
D = G \cap C,
\]

which can easily be extended for any number of goals and constraints. If the classical
fuzzy set intersection is used, the fuzzy decision $D$ is then specified by the membership
function

\[
\mu_{D(x)} = \min[\mu_G(x), \mu_C(x)]
\]

where $x \in X$. This definition of the intersection does not allow, however, for any
interdependence, Interaction, or trade-off between the goals and constraints under
consideration. For many decision applications, this lack of compensation may not be appropriate; the full compensation or trade-off offered by the union operation that corresponds to the logical “or” (the max operator) may be inappropriate as well. Therefore, an alternative fuzzy set intersection or aggregation operation may be used to reflect a situation in which some degree of positive compensation exists among the goals and constraints.

This fuzzy model can be further extended to accommodate the relative importance of the various goals and constraints by the use of weighting coefficients. In this case, the fuzzy decision \( D \) can be arrived at by a convex combination of the \( n \) weighted goals and \( m \) weighted constraints such that

\[
\mu_D(x) = \sum_{i=1}^{n} u_i \mu_{G_i}(x) + \sum_{j=1}^{m} v_j \mu_{C_j}(x) \quad \ldots 1.7.2.7
\]

Where \( u_i \) and \( v_j \) are weights attached to each fuzzy goal \( G_i(\in \mathbb{G}) \) and each fuzzy constraint \( C_j(\in \mathbb{C}) \) respectively, such that

\[
\sum_{i=1}^{n} u_i + \sum_{j=1}^{m} v_j = 1 \quad \ldots 1.7.2.8
\]

Once a fuzzy decision has been arrived at, it may be necessary to choose the “best” single crisp alternative from this fuzzy set. This may be accomplished in straightforward manner by choosing the alternative \( x \in X \) that attains the maximum membership grade in \( D \). Since this method ignores information concerning any of the other alternatives, it may not be desirable in all situations. Methods that calculate the mean or center of gravity of the fuzzy set \( D \) may therefore be used instead. These concepts have been effectively utilized to extend conventional crisp mathematical programming into methods of fuzzy mathematical programming.

The fuzzy model of decision making proposed by Bellman and Zadeh [1970] can be illustrated by a simple example. Suppose we must choose one of four different possible jobs \( a, b, c, \) and \( d \), salaries of which are given by the function \( f \) such that
\[ f(a) = 30,000 \]
\[ f(b) = 25,000 \]
\[ f(c) = 20,000 \]
\[ f(d) = 15,000 \]

Our goal is to choose the job that will give us a high salary given the constraints that the job is interesting and within close driving distance. This first constraint of interest value is represented by the fuzzy set \( C_1 \) defined on our universal set of alternative jobs as follows:

\[ C_1 = .4/a + .6/b + .8/b + .6/d. \]  

Our second constraint concerning the driving distance to each job is defined by the fuzzy set \( C_2 \) such that

\[ C_2 = .1/a + .9/b + .7/c + 1/d. \]

The fuzzy goal \( G \) of a high salary is defined on the universal set \( X \) of salaries by the membership function

\[
\mu_G(x) = \begin{cases} 
1 & \text{for } x > 40,000 \\
-0.0125\left(\frac{x}{100} - 40\right)^2 + 1 & \text{for } 13,000 \leq x \leq 40,000 \\
0 & \text{for } x < 13,000
\end{cases}
\]

where \( x \in X \), and the corresponding goal \( G' \) induced on the set of alternative jobs by the function \( f \) is given by

\[ G' = .875/a + .7b + .5/c + .2/d. \]

Taking the standard fuzzy set intersection of these three fuzzy sets, we obtain the fuzzy decision \( D \), where

\[ D = G' \cap C_1 \cap C_2 = .1/a + .6/b + .5c + .2/d. \]
Finally, we take the maximum of this set to obtain alternative b as the choice that seems best to satisfy our goal and constraints. Note that no real distinction exists here between a goal and a constraint; that is, the two concepts are symmetric.

When decision which are made by more than one person are modeled, two differences from the case of a single decision maker can be considered: first, the goals of the individual decision makers may differ such that each places a different ordering on the alternatives; second, the individual decision makers may have access to different information upon which to base their decision. Theories Known as n-person game theories deal with both of these considerations, team theories of decision making deal only with the second, and group-decision theories deal only with the first.

A fuzzy model of group decision was proposed by Blin [1974] and Blin and Whinston [1973]. Here, each member of a group of n individual decision maker is assumed to have to have a reflexive, antisymmetric, and transitive preference ordering $P_k$, $k \in \mathbb{N}$, which totally or partially orders a set X of alternatives. A “social choice” function must then be found which, given the individual preference orderings, produces the most acceptable overall group preference ordering. Basically, this model allows for the individual decision makers to passes different aims and values while still assuming that the overall purpose is to reach a common, acceptable decision. In order to deal with the multiplicity of opinion evidenced in the group, the social preference $S$ may be defined as a fuzzy binary relation with membership grade function.

$$\mu_i : X \times X \rightarrow [0, 1],$$

which assigns the membership grade $\mu_i(x_i, x_j)$ indicating the degree of group preference of alternative $x_i$ over alternative $x_j$. The expression of this group preference requires some appropriate means of aggregating the individual preferences. One simple method computers the relative popularity of alternative $x_i$ over $x_j$ dividing the number of persons preferring $x_i$ to $x_j$ denoted by $N(x_i, x_j)$, by the total number of decision makers, n. This scheme corresponds to the simple majority vote. Thus,
Other method of aggregating the individual preferences may be used to accommodate different degrees of influence exercised by the individuals in the group. For instance, a dictatorial situation can be modeled by the group preference relation $S$ for which

$$
\mu_s(x_i, x_j) = \begin{cases} 
1 & \text{if } x_i^k > x_j \text{ for some individual } k \\
0 & \text{Otherwise,}
\end{cases}
$$

where $>^k$ represents the preference ordering of the one individual $k$ who exercises complete control over the group decision.

Once the fuzzy relationship $S$ has been defined, the final nonfuzzy group preference can be determined by converting $S$ into its resolution form

$$S = \bigcup_{\alpha} S_\alpha$$

which is the union of the crisp relations $S_\alpha$ comprising the $\alpha$-cuts of the fuzzy relation $S$, $\alpha \in \Lambda$ (the level set of $S$) each scaled by $\alpha$. Each value $\alpha$ essentially represents the level of agreement between the individuals concerning the particular crisp ordering $S_\alpha$. One procedure that maximizes the final agreement level consists of intersecting the classes of crisp total orderings that are compatible with the pairs in the $\alpha$-cuts $S_\alpha$ for increasingly smaller values of $\alpha$ until a single crisp total ordering is achieved. In this process, any pairs $(x_i, x_j)$ that lead to an intransitivity are removed. The largest value $\alpha$ for which the unique compatible ordering on $X \times X$ is found represents the maximized agreement level of the group and the crisp ordering itself represents the group decision. This procedure is illustrated in the following example.

Assume that each individual of a group of eight decision makers has a total preference ordering $P_i, i \in \mathbb{I}$ on a set of alternatives $X = \{w, x, y, z\}$ as follows:

$$P_1 = (w, x, y, z)$$
\[ P_2 = P_5 = (z, y, x, w) \]

\[ P_3 = P_7 = (x, w, y, z) \]

\[ P_4 = P_8 = (w, z, x, y) \]

\[ P_6 = (z, w, x, y) \]

Using the membership function given in equation (1.7.2.15) for the fuzzy group preference ordering relation \( S \) (where \( n = 8 \)) we arrive at the following fuzzy social preference relation:

\[
S = \begin{bmatrix}
0 & .5 & .75 & .625 \\
.5 & 0 & .75 & .375 \\
.25 & .25 & 0 & .375 \\
.375 & .625 & .625 & 0
\end{bmatrix}
\]

The \( a \)-cuts of this fuzzy relation \( S \) are

\[ S_1 = \emptyset \]

\[ S_{.75} = \{(w, y), (x, y)\} \]

\[ S_{.625} = \{(w, z), (z, x), (z, y), (w, y), (x, y)\} \]

\[ S_{.5} = \{(x, w), (w, x), (w, z), (z, x), (z, y), (w, y), (x, y)\} \]

\[ S_{.375} = \{(z, w), (x, z), (y, z), (x, w), (w, x), (w, z), (z, x), (z, y), (w, y), (x, y)\} \]

\[ S_{.25} = \{(y, w), (y, x), (z, w), (x, z), (y, z), (x, w), (w, x), (w, z), (z, x), (z, y), (w, y), (x, y)\} \]

We can now apply the procedure to arrive at the unique crisp ordering which constitutes the group choice. All total orderings on \( X \times X \) are, of course, compatible with the empty set of \( S_1 \). The total orderings \( O_{.75} \) that are compatible with the pairs in the crisp relation \( S_{.75} \) are
\[ O_{.75} = \{(z, w, x, y), (w, x, y, z), (w, z, x, y), (w, x, z, y), (z, x, w, y), (x, w, y, z), (x, z, w, y), (x, w, z, y)\}. \]

Thus,

\[ O_{1} \cap O_{.75} = O_{.75} \]

The orderings compatible with \( S_{.625} \) are

\[ O_{.625} = \{(w, z, x, y), (w, z, y, x)\} \]

And

\[ O_{1} \cap O_{.75} \cap O_{.625} = \{(w, z, x, y)\} \]

Thus, the value .625 represents the group level of agreement concerning the social choice denoted by the total ordering \((w, z, x, y)\).