8. LABELINGS IN EXTENDED TRIPlicate GRAPH OF A PATH Pn

In this chapter we introduce a new graph called extended triplicate graph (ETG) of a path \(P_n\) and prove the existence of graph labelings such as cordial, total cordial, product cordial, total product cordial, product E-cordial, total product E-cordial, E-cordial, total E-cordial, \(Z_3\)-magic, even graceful, odd graceful, mean, even and odd mean, prime cordial, prime E-cordial, \(4\)-cordial and vertex prime labelings.

**Definition 8.1** Let \(G\) be a path graph \(P_n\), where \(n\) is the length of the path with vertex set \(V = \{v_1, v_2, \ldots, v_{n+1}\}\) and the edge set \(E = \{e_1, e_2, \ldots, e_n\}\). A triplicate graph of \(G\) denoted by \(TG = (V_1, E_1)\) is defined by a set \(V_1\) of vertices such that
\[V_1 = \{v_i\} \cup \{v_i'\} \cup \{v_i''\}\]
and
\[\{v_i\} \cap \{v_i'\} \cap \{v_i''\} = \emptyset\]
where \(1 \leq i \leq n+1\) and \(v_i \in V\) and the edge set \(E_1\) of \(TG\) is defined as follows: The edge \(v_iv_j \in E\) if and only if
\[\{v_{j}v_i'\} \cup \{v_jv_i''\} \cup \{v_j'v_{i}\} \cup \{v_j''v_{i}'\}\]
where \(j = i+1\) are edges in \(E_1\). Clearly the triplicate graph \(TG\) of a path \(P_n\) is disconnected.

**Definition 8.2** The structure of triplicate graph \(TG(P_n)\) is defined as follows: By definition 8.1, it is clear that \(TG(P_n)\) has \(3(n+1)\) vertices and \(4n\) edges. Denote the vertex set as \(V = \{v_1, v_2, \ldots, v_{3(n+1)}\}\) and the edge set as \(E = \{e_1, e_2, \ldots, e_{4n}\}\) Clearly the triplicate graph \(TG(P_n)\) is disconnected. We present the following algorithm to make this a connected graph called the Extended Triplicate Graph (ETG) for a path \(P_n\).

**Algorithm 8.1**

**procedure** (extended triplicate graph of \(P_n\) with \(3(n+1)\) vertices and \(4n+1\) edges)

\[E_1 \leftarrow \emptyset\]

\[E_2 \leftarrow (v_{n}, v_{n+1}') \cup (v_n'', v_{n+1}') \cup (v_1', v_2) \cup (v_1', v_2'')\]
for $i = 2$ to $n$ do

$$E_3 \leftarrow \{ (v_{i-1}, v_i), (v_{i-1}, v_i'), (v_{i+1}, v_i'), (v_{i+1}, v_i') \}$$

$$E_1 \leftarrow E_3 \cup E_1$$

end for

if $(n \equiv 1 \pmod{2})$

$$E \leftarrow E_2 \cup E_1 \cup \{ (v_{n+1}, v_1) \}$$

else

$$E \leftarrow E_2 \cup E_1 \cup \{ (v_n, v_1) \}$$

end if

end procedure

**Definition 8.3:** The structure of Extended triplicate for a path $P_n$ denoted by ETG($P_n$) is defined as follows. From the construction of triplicate graph, $TG(P_n)$ has $3(n+1)$ vertices and $4n$ edges. From the algorithm 8.1, Extended triplicate ETG($P_n$) has the same vertex set as in $TG(P_n)$ and the edge set has $4n+1$ edges for all $n$ and it is denoted by $V = \{ v_1, v_2, \ldots, v_{3(n+1)} \}$ and $E = \{ e_1, e_2, \ldots, e_{4n+1} \}$ for all $n$.

**8.1 cordial and total cordial labeling**

In this section, we present algorithm and prove the existence of graph labeling such as cordial and total cordial labeling for the extended triplicate graph of a path $P_n$.

**Algorithm 8.1.1**

**procedure** (cordial labeling for extended triplicate graph)

$$V \leftarrow \{ v_1, v_2, \ldots, v_{3(n+1)} \}$$

$$E \leftarrow \{ e_1, e_2, \ldots, e_{4n+1} \}$$

if $(i \equiv 1 \pmod{2})$
for i = 1 to (n+1) do
    \( v_i \leftarrow 0 \)
    \( v_i'' \leftarrow 1 \)
end for

for i = 2 to n do
    if ( \( i \equiv 0 \pmod{2} \))
        \( v_i' \leftarrow 0 \)
    else
        \( v_i' \leftarrow 1 \)
    end if
end for

\( v_1' \leftarrow 0 \)
\( v_{n+1}' \leftarrow 1 \)

else

for i = 1 to (n+1) do
    \( v_i \leftarrow 0 \)
    \( v_i'' \leftarrow 1 \)
    if \( i \equiv 0 \pmod{2} \)
        \( v_i' \leftarrow 1 \)
    else
        \( v_i' \leftarrow 0 \)
    end if
end for
end if

end procedure
Theorem 8.1.1 Extended triplicate graph ETG($P_n$) is cordial.

Proof From the construction of Extended Triplicate graph ETG($P_n$), we have $3(n+1)$ vertices and $4n+1$ edges for all $n$.

Case (i) For $n \equiv 1 \pmod{2}$

Consider the arbitrary vertex $v_i \in V$. To label the vertices, using the algorithm 8.1.1, define the map $f : V \rightarrow \{0, 1\}$. Clearly, the number of vertices labeled ‘0’ is $(n + 1) + 1 + ((n -1)/2) = 3(n+1)/2$ and the number of vertices labeled ‘1’ is $(n+1) + 1 + ((n-1)/2) = 3(n+1)/2$. Thus the number of vertices labeled ‘0’ and the number of vertices labeled ‘1’ differ by atmost one.

In order to get the labels for the edges, define the induced map $f^* : E \rightarrow \{0, 1\}$ such that for any $v_iv_j \in E$, $f^*(v_iv_j) = (f(v_i) + f(v_j)) \pmod{2}$. Now

(i) For $2 \leq i \leq n$,

$$f^*(v_i', v_{i+1}'') = (f(v_i') + f(v_{i+1}'')) \pmod{2} = \begin{cases} 0, & i \equiv 1 \pmod{2} \\ 1, & i \equiv 0 \pmod{2} \end{cases}$$

(ii) For $2 \leq i \leq n$,

$$f^*(v_i', v_{i+1}'') = (f(v_i') + f(v_{i+1}'')) \pmod{2} = \begin{cases} 0, & i \equiv 1 \pmod{2} \\ 1, & i \equiv 0 \pmod{2} \end{cases}$$

(iii) For $1 \leq i \leq n-1$,

$$f^*(v_i, v_{i+1'}) = (f(v_i) + f(v_{i+1'})) \pmod{2} = \begin{cases} 0, & i \equiv 1 \pmod{2} \\ 1, & i \equiv 0 \pmod{2} \end{cases}$$

(iv) For $3 \leq i \leq n+1$,

$$f^*(v_i, v_{i-1'}) = (f(v_i) + f(v_{i-1'})) \pmod{2} = \begin{cases} 0, & i \equiv 1 \pmod{2} \\ 1, & i \equiv 0 \pmod{2} \end{cases}$$

(v) $f^*(v_1, v_{n+1}) = (f(v_1) + f(v_{n+1})) \pmod{2} = 0$

(vi) $f^*(v_n', v_{n+1'}) = (f(v_n') + f(v_{n+1'})) \pmod{2} = 0$
(vii) \( f^* (v_1'v_2') = (f (v_1') + f (v_2')) \mod 2 = 1 \)

(viii) \( f^* (v_1'v_2) = (f (v_1') + f (v_2)) \mod 2 = 0 \)

(ix) \( f^* (v_nv_{n+1}') = (f (v_n) + f (v_{n+1}')) \mod 2 = 1 \)

That is, the number of edges labeled ‘0’ is \((n-1)/2 + (n-1)/2 + (n-1)/2 + 3 = 2n + 1\) and the number of edges labeled ‘1’ is \((n-1)/2 + (n-1)/2 + (n-1)/2 + (n-1)/2 + 2 = 2n\)

**Case (ii) For \( n \equiv 0 \mod 2 \)**

Consider the arbitrary vertex \( v_i \in V \). To label the vertices, using algorithm 8.1.1, define a map \( f: V \rightarrow \{0, 1\} \).

Clearly the number of vertices labeled ‘0’ is \((n+1) + ((n/2) + 1) = 3(n/2) + 2 \) and the number of vertices labeled ‘1’ is \((n/2) + n+1 = 3(n/2) + 1 \). Thus the number of vertices labeled ‘0’ and the number of vertices labeled ‘1’ differ by atmost one.

In order to get the labels for the edges, define the induced map \( f^*: E \rightarrow \{0, 1\} \) such that for any \( v_iv_j \in E \), \( f^*(v_iv_j) = (f(v_i) + f(v_j)) \mod 2 \). Thus we get,

(i) For \( 2 \leq i \leq n+1 \), \( f^*(v_i'v_{i-1}') = (f (v_i') + f (v_{i-1}')) \mod 2 \)

\[ \begin{align*}
&= 1, \quad i \equiv 1 \mod 2 \\
&= 0, \quad i \equiv 0 \mod 2
\end{align*} \]

(ii) For \( 1 \leq i \leq n \), \( f^*(v_i'v_{i+1}) = (f (v_i') + f (v_{i+1}')) \mod 2 = \\
\[ \begin{align*}
&= 1, \quad i \equiv 1 \mod 2 \\
&= 0, \quad i \equiv 0 \mod 2
\end{align*} \]

(iii) For \( 2 \leq i \leq n+1 \), \( f^*(v_i v_{i-1}') = ((f (v_i) + f (v_{i-1}')) \mod 2 = \\
\[ \begin{align*}
&= 1, \quad i \equiv 1 \mod 2 \\
&= 0, \quad i \equiv 0 \mod 2
\end{align*} \]

(iv) For \( 1 \leq i \leq n \), \( f^*(v_i v_{i+1}') = (f (v_i) + f (v_{i+1}')) \mod 2 = \\
\[ \begin{align*}
&= 1, \quad i \equiv 1 \mod 2 \\
&= 0, \quad i \equiv 0 \mod 2
\end{align*} \]

(v) \( f^*(v_n v_1) = 0 \).

That is, the number of edges labeled ‘0’ is \( 4(n/2) + 1 = 2n + 1 \) and the number of edges labeled ‘1’ is \( 4(n/2) = 2n \).
Thus in both the cases the number of edges labeled ‘0’ and the number of edges labeled ‘1’ differ by atmost one.

Hence extended triplicate graph of $P_n$ admits cordial labeling.

**Theorem 8.1.2** Extended triplicate graph $ETG(P_n)$ admits total cordial labeling.

**Proof** By theorem 8.1.1, using the map $f$ on $V$ and there by the induced map $f^*$ on $E$, for $n \equiv 1 \pmod{2}$, we have the number of edges labeled ‘0’ is $2n+1$ and the number of vertices labeled ‘0’ is $3(n+1)$. Also, the number of edges labeled by ‘1’ is $2n$ and the number of vertices labeled by 1 is $3(n+1)$.

Thus the total number of one’s on vertices and edges taken together is $3(n+1) + 2n = 5n+3$ and the total number of zeroes on vertices and edges taken together is $3(n+1) + 2n + 1 = 5n+4$.

Also, for $n \equiv 0 \pmod{2}$, we have the number of edges labeled ‘0’ is $2n+1$ and the number of vertices labeled ‘0’ is $3(n+1)$. Also, the number of edges labeled by ‘1’ is $2n$ and the number of vertices labeled by ‘1’ is $3(n+1)$.

That is, the total number of one’s on vertices and edges taken together is $3(n+1) + 2n = 5n + 3$ and the total number of zeroes on vertices and edges taken together is $3(n+1) + 2n + 1 = 5n + 4$.

Thus in both the cases, the number of zeroes on the vertices and edges taken together differ by atmost 1 with the number of one’s on vertices and edges taken together.

Hence extended triplicate graph $ETG(P_n)$ admits total cordial labeling.

**Example 8.1** Consider the path $P_5$
The extended triplicate graph of the path $P_5$ and its cordial labeling is shown in figure 8.1.1. Consider the path $P_6$

The extended triplicate graph of the path $P_6$ and its cordial labeling is shown in figure 8.1.2.

8.2 Product cordial and total product cordial labelings

In this section we present an algorithm and prove the existence of product cordial and total product cordial labelings for the extended triplicate graph of a path $P_n$. 
Algorithm 8.2

procedure (product cordial labeling for the extended triplicate graph of a path \( P_n \))

\[ V \leftarrow \{ v_1, v_2, \ldots, v_{3(n+1)} \} \]

\[ E \leftarrow \{ e_1, e_2, \ldots, e_{4n+1} \} \]

for \( i = 1 \) to \( n+1 \)

if \( (i \equiv 1 \pmod{2}) \)

\[ v_i \leftarrow v_i'' \leftarrow 0 \]

\[ v_i' \leftarrow 1 \]

else

\[ v_i' \leftarrow 0 \]

\[ v_i \leftarrow v_i'' \leftarrow 1 \]

end if

end for

end procedure

Theorem 8.2.1 Extended Triplicate graph \( ETG(P_n) \) is product cordial.

Proof Clearly extended triplicate graph \( ETG(P_n) \) has \( p = 3(n+1) \) vertices and \( q = 4n + 1 \) edges for all \( n \).

Case (i) For \( n \equiv 1 \pmod{2} \):

Consider the arbitrary vertex \( v_i \in V \). To label the vertices, using algorithm 8.2, define a map \( f : V \rightarrow \{0,1\} \).

Clearly, the number of vertices labeled ‘0’ is \( 2((n+1)/2) + ((n+1)/2) = 3(n+1)/2 \) and the number of vertices labeled ‘1’ is \( ((n+1)/2) + ((n+1)/2) + ((n+1)/2) = 3(n+1)/2 \). Hence the number of vertices labeled ‘0’ and the number of vertices labeled ‘1’ differ by atmost one.
In order to get the labels for the edges, define the induced map \( f^*: E \to \{0,1\} \) such that \( f^*(v_i v_j) = (f(v_i) \times f(v_j)) \pmod{2} \) where \( v_i, v_j \in E \). Now

(i) For \( 2 \leq i \leq n+1 \), \( f^*(v_{i-1}v_i) = (f(v_{i-1}) \times f(v_{i})) \pmod{2} = \begin{cases} 0, & i \equiv 0 \pmod{2} \\ 1, & i \equiv 1 \pmod{2} \end{cases} \)

(ii) For \( 1 \leq i \leq n \), \( f^*(v_{i-1}v_i) = (f(v_{i-1}) \times f(v_{i})) \pmod{2} = \begin{cases} 0, & i \equiv 0 \pmod{2} \\ 1, & i \equiv 1 \pmod{2} \end{cases} \)

(iii) For \( 1 \leq i \leq n \), \( f^*(v_{i+1}v_{i-1}) = (f(v_{i+1}) \times f(v_{i-1})) \pmod{2} = \begin{cases} 0, & i \equiv 1 \pmod{2} \\ 1, & i \equiv 0 \pmod{2} \end{cases} \)

(iv) For \( 2 \leq i \leq n+1 \), \( f^*(v_{i-1}v_i) = (f(v_{i-1}) \times f(v_{i})) \pmod{2} = \begin{cases} 0, & i \equiv 1 \pmod{2} \\ 1, & i \equiv 0 \pmod{2} \end{cases} \)

(v) \( f^*(v_1 v_{n+1}) = (f(v_1) \times f(v_{n+1})) \pmod{2} = 0 \)

The number of edges labeled ‘0’ is \((n+1)/2 + (n-1)/2 + (n+1)/2 + (n-1)/2 + 1 = 2n + 1\). The number of edges labeled ‘1’ is \((n-1)/2 + (n+1)/2 + (n-1)/2 + (n+1)/2 = 2n\).

Thus the number of edges labeled ‘0’ and the number of edges labeled ‘1’ differ by atmost one.

**Case (ii) For \( n \equiv 0 \pmod{2} \):**

Consider the arbitrary vertex \( v_i \in V \). To label the vertices, using algorithm 8.2, define a map \( f : V \to \{0,1\} \). Clearly, the number of vertices labeled ‘0’ is \((n/2) + ((n/2)+1) + (n/2) = (3n/2) + 2\) and the number of vertices labeled ‘1’ is \((n/2) + 1 + (n/2) + (n/2) = (3n/2) + 1\) which is differ by atmost one.

In order to get the labels for the edges, define the induced map \( f^*: E \to \{0,1\} \) such that \( f^*(v_i v_j) = (f(v_i) \times f(v_j)) \pmod{2} \) where \( v_i, v_j \in E \).

Thus we get
(i) For $2 \leq i \leq n+1$, \[ f^*(v_i' v_{i-1}) = (f(v_i') \times f(v_{i-1}')) \pmod{2} = \begin{cases} 1, & i \equiv 1 \pmod{2} \\ 0, & i \equiv 0 \pmod{2} \end{cases} \]

(ii) For $1 \leq i \leq n$, \[ f^*(v_i' v_{i+1}) = (f(v_i') \times f(v_{i+1}')) \pmod{2} = \begin{cases} 1, & i \equiv 1 \pmod{2} \\ 0, & i \equiv 0 \pmod{2} \end{cases} \]

(iii) For $2 \leq i \leq n+1$, \[ f^*(v_i v_{i-1}') = (f(v_i) \times f(v_{i-1}')) \pmod{2} = \begin{cases} 0, & i \equiv 1 \pmod{2} \\ 1, & i \equiv 0 \pmod{2} \end{cases} \]

(iv) For $1 \leq i \leq n$, \[ f^*(v_i v_{i+1}') = (f(v_i) \times f(v_{i+1}')) \pmod{2} = \begin{cases} 0, & i \equiv 1 \pmod{2} \\ 1, & i \equiv 0 \pmod{2} \end{cases} \]

(v) $f^*(v_n v_1) = 0$.

Thus, the number of edges labeled ‘0’ is $4(n/2) + 1 = 2n + 1$ and the number of edges labeled ‘1’ is $4(n/2) = 2n$.

Thus in both the cases the number of vertices labeled ‘0’ and the number of vertices labeled ‘1’ differ by at most one and the number of edges labeled ‘0’ and the number of edges labeled ‘1’ differ by at most one.

Hence the extended triplet graph of $P_n$ is product cordial.

**Theorem 8.2.2** Extended Triplicate graph of $P_n$ admits total product cordial labeling.

**Proof** By theorem 8.2.1, using the map $f$ on $V$ and there by the induced map $f^*$ on $E$, for $n \equiv 1 \pmod{2}$, the number of edges labeled ‘0’ is $2n+1$ and the number of vertices labeled ‘0’ is $3(n+1)$. Also, the number of edges labeled by ‘1’ is $2n$ and the number of vertices labeled by ‘1’ is $3(n+1)$.

Thus the total number of one’s on vertices and edges taken together is $3(n+1) + 2n = 5n+3$ and the total number of zeroes on vertices and edges taken together is $3(n+1) + 2n + 1 = 5n + 4$. 

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Also, for \( n \equiv 0 \pmod{2} \), we have the number of edges labeled ‘0’ is \( 2n+1 \) and the number of vertices labeled ‘0’ is \( 3(n+1) \). Also, the number of edges labeled by ‘1’ is \( 2n \) and the number of vertices labeled by ‘1’ is \( 3(n+1) \).

Thus the total number of one’s on vertices and edges taken together is \( 3(n+1) + 2n = 5n + 3 \) and the total number of zeroes on vertices and edges taken together is \( 3(n+1) + 2n + 1 = 5n + 4 \).

Thus in both the cases, the number of zeroes on the vertices and edges taken together differ by at most one with the number of one’s on vertices and edges taken together. Hence extended triplicate graph of \( P_n \) admits total product cordial labeling.

**Example 8.2** The extended triplicate graph of the path \( P_5 \) and its product cordial labeling is shown in figure 8.2.1. The extended triplicate graph of the path \( P_6 \) and its product cordial labeling is shown in figure 8.2.2.
8.3 E-cordial labeling and total E-cordial labeling

Now we present an algorithm and prove the existence of E-cordial labeling for the extended triplicate graph ETG(P_n) for any finite n > 0.

**Algorithm 8.3**

**procedure** (E-cordial labeling for ETG(P_n))

for i = 2 to (n+1) do

    if \(i = 4m, m \in N\)
        \(v_i v_{i-1}'' \leftarrow 1\)
    else
        \(v_i v_{i-1}'' \leftarrow 0\)
    end if

end for

for i = 1 to n do

    if \(i = 4m, m \in N\)
        \(v_i v_{i+1}'' \leftarrow 0\)
    else
        \(v_i v_{i+1}'' \leftarrow 1\)
    end if

end for

for i = 1 to n do

    \(v_i v_{i+1} \leftarrow 1\)

end for

for i = 2 to (n+1) do

    \(v_i v_{i-1} \leftarrow 0\)

end for
end for

if \((n \equiv 0 \pmod{2})\)
  if \((n = (4k-2), \ k > 0)\)
    \(v_n \leftarrow 1\)
  else
    \(v_n \leftarrow 0\)
  end if
else
  if \((n = (4k-1), \ k > 0)\)
    \(v_1 \leftarrow 0\)
  end if
end if

end procedure

**Theorem 8.3.1** For any \(n \neq 4m-3\), where \(n > 0\), \(m \in \mathbb{N}\), the extended triplicate graph \(ETG(P_n)\) is \(E\)-cordial.

**Proof** Clearly the extended triplicate graph \(ETG(P_n)\) has \(3(n+1)\) vertices and \(4n+1\) edges. Denote the vertex set and edge set of \(ETG(P_n)\) as \(V = \{v_1, v_2, \ldots, v_{3(n+1)}\}\) and \(E = \{e_1, e_2, \ldots, e_{4n+1}\}\). Consider the arbitrary vertex \(v_i \in V\).

**Case (i)** Label the edges of \(ETG(P_n)\), using algorithm 8.3, by defining a map \(f : E \to \{0, 1\}\), for a finite \(n = 3, 7, 11, \ldots, 4k-1\), where \(k > 0\). Clearly the number of edges labeled ‘0’ is \(2n\) and the number edges labeled ‘1’ is \(2n+1\). Thus the number of edges labeled ‘0’ and the number of edges labeled ‘1’ differ by atmost one.
In order to get the labels for the vertices, define the induced map
\( f^* : V \to \{0, 1\} \) such that \( f^*(v_i) = \{\sum f(v_i v_j)\} \mod 2 \), where \( v_i \in V \) and \( v_j \) is adjacent
with \( v_i \). Thus

(i) for all \( 1 \leq i \leq n \), \( f^*(v_i) = 1 \)

(ii) \( f^*(v_{n+1}) = 0 \)

(iii) \( f^*(v_1) = 1 \)

(iv) for \( 2 \leq i \leq n+1 \), \( f^*(v_i) = 0 \)

(v) for \( 1 \leq i \leq (n+1) \)

\[
 f^*(v_i) = \begin{cases} 
 0, & i \equiv 1 \mod 2 \\
 1, & i \equiv 0 \mod 2 
\end{cases}
\]

That is, the number of vertices labeled ‘0’ is \( 3(n+1)/2 \) and the number of
vertices labeled ‘1’ is \( 3(n+1)/2 \) which is differ by atmost one.

**Case (ii)** Label the edges of \( ETG(P_n) \), using algorithm 8.3, by defining a map
\( f : E \to \{0, 1\} \), for a finite \( n = 2, 6, 10, \ldots, 4k-2 \), where \( k > 0 \). Clearly the number of
edges labeled ‘0’ is \( 2n \) and the number edges labeled ‘1’ is \( 2n+1 \). Thus the number of
edges labeled ‘0’ and the number of edges labeled ‘1’ differ by atmost one.

In order to get the labels for the vertices, define the induced map
\( f^* : V \to \{0,1\} \) such that \( f^*(v_i) = \{\sum f(v_i v_j)\} \mod 2 \) where \( v_i \in V \) and \( v_j \) is adjacent
with \( v_i \).

Thus we obtain,

(i) for \( 2 \leq i \leq (n-1) \), \( f^*(v_i) = 1 \) and \( f^*(v_1) = f^*(v_n) = f^*(v_{n+1}) = 0 \)

(ii) for \( 2 \leq i \leq n \), \( f^*(v_i) = 0 \), \( f^*(v_1) = f^*(v_{n+1}) = 1 \)

(iii) for \( 1 \leq i \leq n \), \( f^*(v_i) = \begin{cases} 
 0, & i \equiv 1 \mod 2 \\
 1, & i \equiv 0 \mod 2 
\end{cases} 
\]

and \( f^*(v_{n+1}) = \sum f(v_{n+1} v_j) = 1 \)
The number of vertices labeled ‘0’ is \((3n+4)/2\) and the number of vertices labeled ‘1’ is \((3n+2)/2\). Thus the number of vertices labeled ‘0’ and the number of vertices labeled ‘1’ differ by atmost one.

**Case (iii)** Label the edges of \(ETG(P_n)\), using algorithm 8.3, by defining a map \(f : E \rightarrow \{0, 1\}\), for a finite \(n = 4, 8, 12, \ldots, 4k\), where \(k > 0\). Clearly the number of edges labeled ‘0’ is 2n and the number edges labeled ‘1’ is 2n+1. Thus the number of edges labeled ‘0’ and the number of edges labeled ‘1’ differ by atmost one.

In order to get the labels for the vertices, define the induced map \(f^* : V \rightarrow \{0, 1\}\) such that \(f^*(v_i) = (\sum f(v_i v_j)) \pmod{2}\), where \(v_i \in V\) and \(v_j\) is adjacent with \(v_i\).

(i) for \(1 \leq i \leq n\), \(f^*(v_i) = 1\) and \(f^*(v_{n+1}) = 0\)

(ii) for \(2 \leq i \leq n\), \(f^*(v_i) = 0\) and \(f^*(v_1) = f^*(v_{n+1}) = 1\)

(iii) for \(1 \leq i \leq n+1\), \(f^*(v_i) = \begin{cases} 0, & i \equiv 1 \pmod{2} \\ 1, & i \equiv 0 \pmod{2} \end{cases}\)

That is, the number of vertices labeled ‘1’ is \((3n+4)/2\) and the number of vertices labeled ‘0’ is \((3n+2)/2\) which is differ by atmost 1.

Hence for all the above cases, for any \(n \neq 4m-3\), where \(n > 0\), \(m \in \mathbb{N}\), the extended triplicate graph \(ETG(P_n)\) is E-cordial.

**Theorem 8.3.2:** For any \(n \neq 4m-3\), where \(n > 0\), \(m \in \mathbb{N}\), the extended triplicate graph \(ETG(P_n)\) admits total E-cordial labeling.

**Proof** By case (i) of theorem 8.3.1, using the map \(f\) on \(E\) and there by the induced map \(f^*\) on \(V\), the number of edges labeled ‘0’ is 2n and the number of vertices labeled ‘0’ is \(3(n+1)/2\). Also, the number of edges labeled by ‘1’ is 2n+1 and the number of vertices labeled by ‘1’ is \(3(n+1)/2\).
Thus the total number of one’s on vertices and edges taken together is $3(n+1)/2 + 2n+1 = (7n+5)/2$ and the total number of zeroes on vertices and edges taken together is $3(n+1)/2 + 2n = (7n+3)/2$.

By case (ii) of theorem 8.3.1, using the map $f$ on $E$ and there by the induced map $f^*$ on $V$, we have the number of edges labeled ‘0’ is $2n$ and the number of vertices labeled ‘0’ is $3(n+4)/2$. Also, the number of edges labeled by ‘1’ is $2n+1$ and the number of vertices labeled by ‘1’ is $3(n+2)/2$.

Thus the total number of one’s on vertices and edges taken together is $3(n+2)/2 + 2n + 1 = (7n+4)/2$ and the total number of zeroes on vertices and edges taken together is $3(n+4)/2 + 2n = (7n+4)/2$.

Similarly by case (iii) of theorem 8.3.1, using the map $f$ on $E$ and there by the induced map $f^*$ on $V$, we have the number of edges labeled ‘0’ is $2n$ and the number of vertices labeled ‘0’ is $3(n+4)/2$. Also, the number of edges labeled by ‘1’ is $2n+1$ and the number of vertices labeled by ‘1’ is $3(n+2)/2$.

Thus the total number of one’s on vertices and edges taken together is $3(n+2)/2 + 2n + 1 = (7n+4)/2$ and the total number of zeroes on vertices and edges taken together is $3(n+4)/2 + 2n = (7n+4)/2$.

Thus in all the three cases, the number of zeroes on the vertices and edges taken together differ by atmost one with the number of one’s on vertices and edges taken together.

Hence for any $n \neq 4m-3$, where $n > 0$, $m \in \mathbb{N}$, the extended triplicate graph $ETG(P_n)$ admits total $E$-cordial labeling.
Example 8.3.1 Consider the path $P_7$.

The extended triplicate graph of the path $P_7$ and its E-cordial labeling is shown in figure 8.3.1. The extended triplicate graph of the path $P_6$ and its Product cordial labeling is shown in figure 8.3.2. Consider the path $P_8$.

The extended triplicate graph of the path $P_8$ and its E-cordial labeling is shown in figure 8.3.3.

8.4 Product E-cordial and total product E-cordial labeling

We present an algorithm to get product E-cordial labeling for extended triplicate graph $ETG(P_n)$. 
Algorithm 8.4

procedure (product E-cordial labeling for the ETG(P_n))

\[ V \leftarrow \{ v_1, v_2, \ldots, v_{3(n+1)} \} \]

\[ E \leftarrow \{ e_1, e_2, \ldots, e_{4n+1} \} \]

for \( i = 2 \) to \( (n+1) \) do

if \( i \equiv 0 \) (mod 2))

\[ v_i v_{i-1} \leftarrow 0 \]

\[ v_i v_{i-1} \leftarrow 1 \]

else

\[ v_i v_{i-1} \leftarrow 1 \]

\[ v_i v_{i-1} \leftarrow 0 \]

end if

end for

for \( i = 1 \) to \( n \) do

if \( i \equiv 0 \) (mod 2)

\[ v_i v_{i+1} \leftarrow 0 \]

\[ v_i v_{i+1} \leftarrow 1 \]

else

\[ v_i v_{i+1} \leftarrow 1 \]

\[ v_i v_{i+1} \leftarrow 0 \]

end if

end for

if \( n \equiv 1 \) (mod 2))

\[ v_1 v_{n+1} \leftarrow 1 \]

else
\[ v_n v_1 \leftarrow 1 \]

end if

\textbf{end procedure}

\textbf{Theorem 8.4.1} For any \( n > 0 \), the extended triplicate graph \( ETG(P_n) \) is product \( E\)-cordial.

\textbf{Proof} We know that \( ETG(P_n) \) has \( 3(n+1) \) vertices and \( 4n+1 \) edges. Denote the vertex set and edge set of \( ETG(P_n) \) as \( V = \{ v_1, v_2, \ldots, v_{3(n+1)} \} \) and \( E = \{ e_1, e_2, \ldots, e_{4n+1} \} \).

\textbf{Case(i)} For \( n \equiv 1 \pmod{2} \)

To label the edges, using algorithm 8.4, define a map \( f : E \to \{0, 1\} \). Clearly the number of edges labeled ‘0’ is \( 2n \) and the number edges labeled ‘1’ is \( 2n+1 \). Thus the number of edges labeled ‘0’ and the number of edges labeled ‘1’ differ by atmost one.

In order to get the labels for the vertices, define the induced map \( f^* : V \to \{0, 1\} \) such that \( f^*(v_i) = \{ f(v_i) \text{ (mod 2)} \} \), where \( v_i \in V \) and \( v_j \) is adjacent with \( v_i \).

(i) for \( 1 \leq i \leq n+1 \), \( f^*(v_i) = \begin{cases} 0, & i \equiv 1 \pmod{2} \\ 1, & i \equiv 0 \pmod{2} \end{cases} \)

(ii) for \( 1 \leq i \leq n+1 \), \( f^*(v_i) = \begin{cases} 1, & i \equiv 1 \pmod{2} \\ 0, & i \equiv 0 \pmod{2} \end{cases} \)

(iii) for \( 1 \leq i \leq n+1 \), \( f^*(v_i) = \begin{cases} 0, & i \equiv 1 \pmod{2} \\ 1, & i \equiv 0 \pmod{2} \end{cases} \)

That is, the number of vertices labeled ‘1’ = \( 3(n+1)/2 \) and the number of vertices labeled ‘0’ = \( 3(n+1)/2 \) which is differ by atmost one.
Case (ii) For \( n \equiv 0 \pmod{2} \)

To label the edges, using algorithm 8.4, define a map \( f : E \to \{0,1\} \). Clearly the number of edges labeled ‘0’ is \( 2n \) and the number edges labeled ‘1’ is \( 2n+1 \). Thus the number of edges labeled ‘0’ and the number of edges labeled ‘1’ differ by atmost 1.

In order to get the labels for the vertices, define the induced map \( f^* : V \to \{0, 1\} \) such that \( f^*(v_i) = \{ \prod f(v_i, v_j) \pmod{2} \} \) where \( v_i \in V \) and \( v_j \) is adjacent with \( v_i \). Thus we obtain,

(i) for \( 1 \leq i \leq n+1 \), \( f^*(v_i) = \begin{cases} 1, & i \equiv 1 \pmod{2} \\ 0, & i \equiv 0 \pmod{2} \end{cases} \)

(ii) for \( 1 \leq i \leq n+1 \), \( f^*(v_i') = \begin{cases} 1, & i \equiv 1 \pmod{2} \\ 0, & i \equiv 0 \pmod{2} \end{cases} \)

(iii) for \( 1 \leq i \leq n+1 \), \( f^*(v_i'') = \begin{cases} 0, & i \equiv 1 \pmod{2} \\ 1, & i \equiv 0 \pmod{2} \end{cases} \)

That is, the number of vertices labeled ‘1’ is \( 3(n+1)/2 \) and the number of vertices labeled ‘0’ is \( 3(n+1)/2 \) which is differ by atmost one.

Thus in both the cases the number of vertices labeled ‘0’ and the number of vertices labeled ‘1’ differ by atmost one.

Hence ETG(P_n) admits product E-cordial labeling for any \( n > 0 \).

**Theorem 8.4.2** For any \( n > 0 \), ETG(P_n) admits total product E-cordial labeling.

**Proof** By case (i) of theorem 8.4.1, using the map \( f \) on \( E \) and there by the induced map \( f^* \) on \( V \), the number of edges labeled ‘0’ is \( 2n \) and the number of vertices labeled ‘0’ is \( 3(n+1)/2 \). Thus the total number of ‘0’ on vertices and edges taken together is \( 3(n+1)/2 + 2n = (7n+3)/2 \). Also, the number of edges labeled by ‘1’ is \( 2n+1 \) and the number of vertices labeled by ‘1’ is \( 3(n+1)/2 \). Thus the total number of ‘1’ on vertices and edges taken together is \( 3(n+1)/2 + 2n +1 = (7n+5)/2 \).
Similarly by case (ii), using the map $f$ on $E$ and there by the induced map $f^*$ on $V$, the number of edges labeled ‘0’ is $2n$ and the number of vertices labeled ‘0’ is $(3n+4)/2$. Thus the total number of zeroes on vertices and edges taken together is $(3n+4)/2 + 2n = (7n+4)/2$. Also, the number of edges labeled by 1 is $2n+1$ and the number of vertices labeled by ‘1’ is $(3n+2)/2$. Thus the total number of one’s on vertices and edges taken together is $(3n+2)/2 + 2n +1 = (7n+4)/2$.

Thus in both the cases the number of zeroes on the vertices and edges taken together differ by atmost one with the number of 1’s on vertices and edges taken together.

Hence ETG($P_n$) admits total product E-cordial labeling for any $n$>0.

**Example 8.4** Product E-cordial labeling for ETG($P_3$) is shown in figure 8.4.1. Product E-cordial labeling for ETG($P_6$) is shown in figure 8.4.2.
8.5 $Z_3$- Magic Labeling

Now we present an algorithm to obtain the modified extended triplicate graph.

Algorithm 8.5

procedure (modified extended triplicate graph for $Z_3$-magic labeling)

$E_1 \leftarrow \emptyset$

$E_2 \leftarrow \{(v_n, v_{n+1}'), (v_{n+1}, v_n'), (v_1, v_2), (v_1', v_2')\}$

for $i = 2$ to $n$ do

$E_3 \leftarrow \{(v_{i-1}, v_i'), (v_{i+1}, v_i'), (v_i, v_{i+1}), (v_i, v_{i+1}')\}$

$E_1 \leftarrow E_1 \cup E_1$

end for

if $(n \equiv 1 \mod 2)$

$E \leftarrow E_2 \cup E_1 \cup (v_{n+1}, v_1, v_{n+1}''')$

else

$E \leftarrow E_2 \cup E_1 \cup (v_{n}, v_1) \cup (v_{n+1}'''', v_{i+1}'''', v_i''') \cup (v_{n+1}'''', v_{i+1}'''', v_i''')$

end if

end procedure

Now we present another algorithm and prove that the modified ETG($P_n$) admits $Z_3$-magic labeling for any finite $n > 0$.

Algorithm 8.5.1

procedure ($Z_3$-magic labeling for the modified extended triplicate graph)

if $(n \equiv 1 \mod 2)$

for $i = 2$ to $(n+1)$ do

if $(i \equiv 0 \mod 2)$


\[ v_i v_{i-1} \leftarrow 2 \]
\[ v_i v_{i+1} \leftarrow 1 \]
else
\[ v_i v_{i-1} \leftarrow 1 \]
\[ v_i v_{i+1} \leftarrow 2 \]
end if
end for

for \( i = 1 \) to \( n \) do
if \((i \equiv 0 \mod 2)\)
\[ v_i v_{i+1} \leftarrow 1 \]
\[ v_i v_{i+1} \leftarrow 2 \]
else
\[ v_i v_{i+1} \leftarrow 2 \]
\[ v_i v_{i+1} \leftarrow 1 \]
end if
end for

\[ v_1 v_{n+1} \leftarrow 2 \]
\[ v_1 v_{n+1} \leftarrow 1 \]
else

for \( i = 3 \) to \( n \) do
if \((i \equiv 1 \mod 2)\)
\[ v_i v_{i-1} \leftarrow 2 \]
else
\[ v_i v_{i-1} \leftarrow 1 \]
end if
end for

\[ v_2 \hat{v}_1 \hat{v} \leftarrow 2 \]

for \( i = 1 \) to \( n \) do

\( \text{if } i \equiv 1 \pmod{2} \)

\[ v_i \hat{v}_{i+1} \hat{v} \leftarrow 1 \]

\( \text{else} \)

\[ v_i \hat{v}_{i+1} \hat{v} \leftarrow 2 \]

end if

end for

for \( i = 2 \) to \( (n+1) \) do

\( \text{if } (i \equiv 1 \pmod{2}) \)

\[ v_i \hat{v}_{i-1} \hat{v} \leftarrow 1 \]

\( \text{else} \)

\[ v_i \hat{v}_{i-1} \hat{v} \leftarrow 2 \]

end if

end for

for \( i = 2 \) to \( (n-1) \) do

\( \text{if } (i \equiv 1 \pmod{2}) \)

\[ v_i \hat{v}_{i+1} \hat{v} \leftarrow 2, \ i \equiv 1 \pmod{2} \]

\( \text{else} \)

\[ v_i \hat{v}_{i+1} \hat{v} \leftarrow 1 \]

end if

end for

\[ v_1 \hat{v}_2 \leftarrow 1 \]

\[ v_n \hat{v}_1 \leftarrow v_1 \hat{v}_{n+1} \leftarrow 2 \]
end if

end procedure

Theorem 8.5  For any \( n > 0 \), modified extended triplicate graph admits \( \mathbb{Z}_3 \)-magic labeling.

Proof  Using algorithm 8.5, construct the modified extended triplicate graph \( \text{ETG}(P_n) \).

To label the edges, using algorithm 8.5.1, define the map \( f : E \rightarrow \{1, 2\} \).

In order to obtain the labels for the vertices, define the induced map \( f^* : V \rightarrow \{0, 1, 2\} \) such that for any vertex \( v_i \), \( f^*(v_i) = \left( \sum f(v_i v_j) \right) \pmod{3} \) where \( v_j \) is adjacent with \( v_i \).

Consider the arbitrary vertex \( v_i \in V \). Now \( f^*(v_i) = (1 + 2) \pmod{3} = 0 \), which is a constant for all \( i \).

Hence the modified extended triplicate graph admits \( \mathbb{Z}_3 \)-magic labeling for any \( n > 0 \).
**Z₃-MAGIC LABELING**

**Figure 8.5.1: ETG(P₃)**

**Figure 8.5.2: ETG(P₆)**

**Example 8.5** Z₃-magic labeling for the modified extended triplicate graph ETG(P₃) is shown in figure 8.5.1. Z₃-magic labeling for the modified extended triplicate graph ETG(P₆) is shown in figure 8.5.2.

### 8.6 Even graceful labeling

In this section we present an algorithm and prove the existence of even graceful labeling for the extended triplicate graph of a path Pₙ.

**Algorithm 8.6:**

**procedure** (even graceful labeling for ETG(Pₙ) )

if (n ≡ 0 (mod 2))

for i = 1 to (n + 1) do

if i ≡ 1 (mod 2)

vi ← 7n − i + 1
\[ v_i' \leftarrow 8n - i + 3 \]
\[ v_i'' \leftarrow 5n - i + 1 \]
else
\[ v_i \leftarrow 2n + i - 2 \]
\[ v_i' \leftarrow 3n + i - 2 \]
\[ v_i'' \leftarrow i - 2 \]
end if
end for
else
for \( i = 1 \) to \( (n + 1) \) do
if \( i \equiv 1 \) (mod 2)
\[ v_i \leftarrow 7n - i + 2, \ i \equiv 1 \) (mod 2) \]
\[ v_i' \leftarrow 8n - i + 3, \ i \equiv 1 \) (mod 2) \]
\[ v_i'' \leftarrow 5n - i + 2, \ i \equiv 1 \) (mod 2) \]
else
\[ v_i \leftarrow 2n + i - 2 \]
\[ v_i' \leftarrow 3n + i - 1 \]
\[ v_i'' \leftarrow i - 2 \]
end if
end for
end if
end procedure

**Theorem 8.6** Extended triplicate Graph \( ETG(P_n) \) is even graceful.

**Proof** Clearly the extended triplicate graph \( ETG(P_n) \) has \( p = 3(n+1) \) vertices and \( q = 4n + 1 \) edges for all \( n \).
Consider the arbitrary vertex \( v_i \in V \). To label the vertices, using algorithm 8.6, define a map \( f : V \to \{0, 1, 2, \ldots, 2q\} \).

To obtain the labels for edges, define the induced map \( f^* : E \to \{2, 4, \ldots, 2q\} \) such that for any \( v_iv_j \in E \), \( f^*(v_iv_j) = |f(v_i) - f(v_j)| \). Thus for all \( n \), we get

(i) For \( 2 \leq i \leq n+1 \), 
\[
\begin{align*}
  f^*(v_i'v_{i-1}') &= \begin{cases} 
    8n - 2i + 6, & i \equiv 1 \pmod{2} \\
    2n - 2i + 4, & i \equiv 0 \pmod{2}
  \end{cases}
\end{align*}
\]

(ii) For \( 1 \leq i \leq n \), 
\[
\begin{align*}
  f^*(v_i'v_{i+1}') &= \begin{cases} 
    8n - 2i + 4, & i \equiv 1 \pmod{2} \\
    2n - 2i + 2, & i \equiv 0 \pmod{2}
  \end{cases}
\end{align*}
\]

(iii) For \( 2 \leq i \leq n+1 \), 
\[
\begin{align*}
  f^*(v_i'v_{i-1}) &= \begin{cases} 
    4n - 2i + 4, & i \equiv 1 \pmod{2} \\
    6n - 2i + 6, & i \equiv 0 \pmod{2}
  \end{cases}
\end{align*}
\]

(iv) For \( 1 \leq i \leq n \), 
\[
\begin{align*}
  f^*(v_i'v_{i+1}'') &= \begin{cases} 
    4n - 2i + 2, & i \equiv 1 \pmod{2} \\
    6n - 2i + 4, & i \equiv 0 \pmod{2}
  \end{cases}
\end{align*}
\]

(v) \( f^*(v_{n+1}v_1) = 4n+2 \) if \( n \) is odd and \( f^*(v_nv_1) = 4n+2 \) if \( n \) is even.

Thus the edges are labeled from 2 to \( 2q \) for all \( n \).

Hence the Extended Triplicate Graph ETG(P\(_n\)) admits even graceful labeling for all \( n \).

**Example 8.6** The extended triplicate graph of the path \( P_5 \) and its even graceful labeling is shown in figure 8.6.1. The extended triplicate graph of the path \( P_6 \) and its
even graceful labeling is shown in figure 8.6.2.

Figure 8.6.1: ETG(P₃)  Figure 8.6.2: ETG(P₆)

8.7 Odd graceful labeling

In this section we present an algorithm and prove the existence of odd graceful labeling for the extended triplicate graph of a path Pₙ.

Algorithm 8.7

procedure (odd graceful labeling for ETG(Pₙ))

if \( n \equiv 0 \pmod{2} \)

for \( i = 1 \) to \( n + 1 \) do

if \( i \equiv 1 \pmod{2} \)

\( v_i \leftarrow 7n - i \)

\( v_i' \leftarrow 8n - i + 2 \)

\( v_i'' \leftarrow 5n - i \)

else

\( v_i \leftarrow 2n + i - 2 \)

\( v_i' \leftarrow 3n + i - 2 \)
\[ v_i^* \leftarrow i - 2 \]
end if
end for
else
for i = 1 to ( n + 1) do
if ( i \equiv 1 \pmod{2})
\[ v_i \leftarrow 7n - i + 1 \]
\[ v_i' \leftarrow 8n - i + 2 \]
\[ v_i'' \leftarrow 5n - i + 1 \]
else
\[ v_i \leftarrow 2n + i - 2 \]
\[ v_i' \leftarrow 3n + i - 1 \]
\[ v_i'' \leftarrow i - 2 \]
end if
end for
end if
end procedure

**Theorem 8.7** Extended triplicate graph ETG(P_n) is odd graceful.

**Proof** We know that ETG(P_n) has 3(n+1) vertices and 4n+1 edges for all n.

Consider the arbitrary vertex \( v_i \in V \). To label the vertices, using algorithm 8.7, define a map \( f : V \rightarrow \{0, 1, 2, \ldots, 2q-1\} \). In order to get the labels for the edges, define the induced map \( f^* : E \rightarrow \{1,3,\ldots, 2q-1\} \) such that for any \( v_i v_j \in E \), \( f^*(v_i v_j) = |f(v_i) - f(v_j)| \). Now for all n we get,

(i) For \( 2 \leq i \leq n+1, \quad f^*(v_i v_{i-1}^*) = \begin{cases} 8n - 2i + 5, & i \equiv 1 \pmod{2} \\ 2n - 2i + 3, & i \equiv 0 \pmod{2} \end{cases} \)
(ii) For $1 \leq i \leq n$, 
\[
f^*(v_i \ v_{i+1} \ ') = \begin{cases} 
8n - 2i + 3, & i \equiv 1 \pmod{2} \\
2n - 2i + 1, & i \equiv 0 \pmod{2} 
\end{cases}
\]

(iii) For $2 \leq i \leq n+1$, 
\[
f^*(v_i \ v_{i-1} \ ') = \begin{cases} 
4n - 2i + 3, & i \equiv 1 \pmod{2} \\
6n - 2i + 5, & i \equiv 0 \pmod{2} 
\end{cases}
\]

(iv) For $1 \leq i \leq n$, 
\[
f^*(v_i \ v_{i+1} \ ') = \begin{cases} 
4n - 2i + 1, & i \equiv 1 \pmod{2} \\
6n - 2i + 3, & i \equiv 0 \pmod{2} 
\end{cases}
\]

(v) $f^*(v_{n+1} \ v_1) = 4n+1$ if $n$ is odd and $f^*(v_n \ v_1) = 4n+1$ if $n$ is even.

Thus the edges are labeled from 1 to $2q-1$ for all $n$.

Hence the Extended Triplicate Graph ETG($P_n$) admits odd graceful labeling for all $n$.

**Example 8.7** The extended triplicate graph of the path $P_5$ and its odd graceful labeling is shown in figure 8.7.1. The extended triplicate graph of the path $P_6$ and its odd graceful labeling is shown in figure 8.7.2.

8.8 Odd mean labeling

Now we present an algorithm and prove that the extended triplicate graph of a path $P_n$ admits odd mean labeling.
Algorithm 8.8

Procedure (Odd mean labeling for the extended triplicate graph of a path)

\[ V \leftarrow \{v_1, v_2, \ldots, v_{3(n+1)}\} \]

\[ E \leftarrow \{e_1, e_2, \ldots, e_{4n+1}\} \]

if \( n \equiv 0 \pmod{2} \)

\[ \text{for } i = 1 \text{ to } (n + 1) \text{ do} \]

if \( i \equiv 0 \pmod{2} \)

\[ v_i \leftarrow 2n - 2i + 3 \]

\[ v_i' \leftarrow 2n + 2i + 1 \]

\[ v_i'' \leftarrow 8n - 2i + 3 \]

else

\[ v_i \leftarrow 2n + 2i + 1 \]

\[ v_i' \leftarrow 2n - 2i + 3 \]

\[ v_i'' \leftarrow 6n + 2i - 1 \]

end if

end for

else

\[ \text{for } i = 1 \text{ to } (n + 1) \text{ do} \]

if \( i \equiv 0 \pmod{2} \)

\[ v_i \leftarrow 2n - 2i + 3 \]

\[ v_i' \leftarrow 2n + 2i + 1 \]

\[ v_i'' \leftarrow 8n - 2i + 3 \]

else

\[ v_i \leftarrow 2n + 2i + 1 \]

\[ v_i' \leftarrow 2n - 2i + 3 \]

end if

end for
\[ v_i'' \leftarrow 6n + 2i + 1 \]
end if
end for
end if
end procedure

**Theorem 8.8** Extended triplicate graph ETG(P\(_n\)) admits odd mean labeling.

**Proof** Clearly ETG(P\(_n\)) has \( p = 3(n+1) \) vertices and \( q = 4n + 1 \) edges for all \( n \).

Consider the arbitrary vertex \( v_i \in V \). To label the vertices, using algorithm 8.8, define a map \( f : V \rightarrow \{1, 3, 5, \ldots, 2q-1\} \).

In order to obtain the labels for edges, define the induced map \( f^* : E \rightarrow \{1, 3, 5, \ldots, 2q-1\} \) such that for any \( v_i v_j \in E \), \( f^*(v_i v_j) = (f(v_i) + f(v_j))/2 \).

Case (i) For \( n \equiv 0 \) (mod 2), we obtain the edge labels as follows:

(i) For \( 2 \leq i \leq n+1 \), \( f^*(v_i v_{i-1}^\prime) = \begin{cases} 4n + 2i, & i \equiv 0 \text{ (mod 2)} \\ 5n - 2i + 4, & i \equiv 1 \text{ (mod 2)} \end{cases} \)

(ii) For \( 1 \leq i \leq n \), \( f^*(v_i v_{i+1}^\prime) = \begin{cases} 5n - 2i + 2, & i \equiv 1 \text{ (mod 2)} \\ 4n + 2i + 1, & i \equiv 0 \text{ (mod 2)} \end{cases} \)

(iii) For \( 2 \leq i \leq n+1 \), \( f^*(v_i v_{i-1}^\prime) = \begin{cases} 2n - 2i + 4, & i \equiv 0 \text{ (mod 2)} \\ 2n + 2i, & i \equiv 1 \text{ (mod 2)} \end{cases} \)

(iv) For \( 1 \leq i \leq n \), \( f^*(v_i v_{i+1}^\prime) = \begin{cases} 2n + 2i + 2, & i \equiv 1 \text{ (mod 2)} \\ 2n - 2i + 2, & i \equiv 0 \text{ (mod 2)} \end{cases} \)

(v) \( f^*(v_n v_1) = n+3 \)

Case (ii) For \( n \equiv 1 \) (mod 2), we obtain the edge labels as follows:

(i) For \( 2 \leq i \leq n+1 \), \( f^*(v_i v_{i-1}^\prime) = \begin{cases} 4n + 2i, & i \equiv 0 \text{ (mod 2)} \\ 5n - 2i + 4, & i \equiv 1 \text{ (mod 2)} \end{cases} \)
(ii) For $1 \leq i \leq n$, \( f^*(v_i, v_{i+1}) = \begin{cases} 
 5n - 2i + 2, & i \equiv 1 \pmod{2} \\
 4n + 2i + 2, & i \equiv 0 \pmod{2} 
\end{cases} \)

(iii) For $2 \leq i \leq n+1$, \( f^*(v_i, v_{i-1}) = \begin{cases} 
 2n - 2i + 4, & i \equiv 0 \pmod{2} \\
 2n + 2i, & i \equiv 1 \pmod{2} 
\end{cases} \)

(iv) For $1 \leq i \leq n$, \( f^*(v_i, v_{i+1}) = \begin{cases} 
 2n + 2i + 2, & i \equiv 1 \pmod{2} \\
 2n - 2i + 2, & i \equiv 0 \pmod{2} 
\end{cases} \)

(v) \( f^*(v_{n+1}, v_1) = n+2 \)

Thus each edge receives the labels $1, 2, 3, \ldots, 2q-1$ for all $n$.

Hence the extended triplicate graph $\text{ETG}(P_n)$ admits odd mean labeling for all $n$.

**Example 8.8** The extended triplicate graph of the path $P_5$ and its odd mean labeling is shown in figure 8.8.1. The extended triplicate graph of the path $P_6$ and its odd mean labeling is shown in figure 8.8.2.

![Figure 8.8.1: ETG(P5)](image1)

![Figure 8.8.2: ETG(P6)](image2)
8.9 Even mean labeling

In this section we present an algorithm and prove that the extended triplicate graph of a path $P_n$ admits even mean labeling.

**Algorithm 8.9**

**procedure** (even mean labeling for the extended triplicate graph of a path )

$V \leftarrow \{v_1, v_2, \ldots, v_{3(n+1)}\}$

$E \leftarrow \{e_1, e_2, \ldots, e_{4n+1}\}$

if $n \equiv 0 \pmod{2}$

for $i = 1$ to $(n + 1)$ do

if $i \equiv 0 \pmod{2}$

$v_i \leftarrow 2n - 2i + 4$

$v_i' \leftarrow 2n + 2i + 2$

$v_i'' \leftarrow 8n - 2i + 4$

else

$v_i \leftarrow 2n + 2i + 2$

$v_i' \leftarrow 2n - 2i + 4$

$v_i'' \leftarrow 6n + 2i$

end if

end for

else

for $i = 1$ to $(n + 1)$ do

if $i \equiv 0 \pmod{2}$

$v_i \leftarrow 2n - 2i + 4$

$v_i' \leftarrow 2n + 2i + 2$

$v_i'' \leftarrow 8n - 2i + 4$

end if

end for

end if
else
    \[ v_i \leftarrow 2n + 2i + 2 \]
    \[ v_i' \leftarrow 2n - 2i + 4 \]
    \[ v_i'' \leftarrow 6n + 2i + 2 \]
end if
end for
end if

end procedure

**Theorem 8.9** Extended triplicate graph ETG(P\(_n\)) admits even mean labeling.

**Proof** We know that extended triplicate graph ETG(P\(_n\)) has \( p = 3(n+1) \) vertices and \( q = 4n + 1 \) edges for all \( n \).

Consider the arbitrary vertex \( v_i \in V \). To label the vertices, using algorithm 8.9, define a map \( f : V \rightarrow \{2, 4, 6, \ldots, 2q\} \).

In order to get the labels for the edges, define the induced map \( f^* : E \rightarrow \{1, 2, \ldots, 2(4n+1)\} \) such that for any \( v_iv_j \in E \), \( f^*(v_iv_j) = (f(v_i)+ f(v_j))/2 \).

Case (i) For \( n \equiv 0 \pmod{2} \), we obtain the edge labels as follows:

(i) For \( 2 \leq i \leq n+1 \), \( f^*(v_i v_{i+1}) = \begin{cases} 4n + 2i, & i \equiv 0 \pmod{2} \\ 5n - 2i + 5, & i \equiv 1 \pmod{2} \end{cases} \)

(ii) For \( 1 \leq i \leq n \), \( f^*(v_i v_{i+1}) = \begin{cases} 5n - 2i + 3, & i \equiv 1 \pmod{2} \\ 4n + 2i + 2, & i \equiv 0 \pmod{2} \end{cases} \)

(iii) For \( 2 \leq i \leq n+1 \), \( f^*(v_i v_{i-1}) = \begin{cases} 2n - 2i + 5, & i \equiv 0 \pmod{2} \\ 2n + 2i + 1, & i \equiv 1 \pmod{2} \end{cases} \)

(iv) For \( 1 \leq i \leq n \), \( f^*(v_i v_{i+1}) = \begin{cases} 2n + 2i + 3, & i \equiv 1 \pmod{2} \\ 2n - 2i + 3, & i \equiv 0 \pmod{2} \end{cases} \)

(v) \( f^*(v_n v_1) = n+4 \)
Case (ii) For $n \equiv 1 \pmod{2}$, we obtain the edge labels as follows:

(i) For $2 \leq i \leq n+1$, 
$$f^*(v'_i, v_{i-1}) = \begin{cases} 
4n + 2i + 1, & i \equiv 0 \pmod{2} \\
5n - 2i + 5, & i \equiv 1 \pmod{2}
\end{cases}$$

(ii) For $1 \leq i \leq n$, 
$$f^*(v'_i, v_{i+1}) = \begin{cases} 
5n - 2i + 3, & i \equiv 1 \pmod{2} \\
4n + 2i + 3, & i \equiv 0 \pmod{2}
\end{cases}$$

(iii) For $2 \leq i \leq n+1$, 
$$f^*(v_i, v_{i-1}) = \begin{cases} 
2n - 2i + 5, & i \equiv 0 \pmod{2} \\
2n + 2i + 1, & i \equiv 1 \pmod{2}
\end{cases}$$

(iv) For $1 \leq i \leq n$, 
$$f^*(v_i, v_{i+1}) = \begin{cases} 
2n + 2i + 3, & i \equiv 1 \pmod{2} \\
2n - 2i + 3, & i \equiv 0 \pmod{2}
\end{cases}$$

(v) $f^*(v_{n+1}, v_1) = n+3$

Thus all the edges receive the labels $1, 2, 3, \ldots, 2q$ for all $n$.

Hence extended triplicate graph $ETG(P_n)$ admits even mean labeling for all $n$.

**Example 8.** The extended triplicate graph of the path $P_5$ and its even mean labeling is shown in figure 8.9.1. The extended triplicate graph of the path $P_6$ and its even mean labeling is shown in figure 8.9.2.
8.10 Mean Labeling

In this section we present an algorithm and prove the existence of mean labelings for the extended triplicate graph of a path \( P_n \).

Algorithm 8.10

Procedure (mean labeling of extended triplicate graph of a path)

\[ V \leftarrow \{ v_1, v_2, \ldots, v_{3(n+1)} \} \]

\[ E \leftarrow \{ e_1, e_2, \ldots, e_{4n+1} \} \]

if \( n \equiv 0 \pmod{2} \)

for \( i = 1 \) to \( n + 1 \) do

if \( i \equiv 0 \pmod{2} \)

\( v_i \leftarrow n - i \)

\( v_i' \leftarrow 4n - i + 2 \)

\( v_i'' \leftarrow 3n - i + 1 \)

else
\begin{align*}
    v_i & \leftarrow 4n - i + 2 \\
    v_i' & \leftarrow n - i + 2 \\
    v_i'' & \leftarrow 2n - i + 3
\end{align*}

end if

doi

end for

else

for \ i = 1 to (n + 1) do

    if \ (i \equiv 0 \ (\text{mod} \ 2))

        v_i \leftarrow n - i + 1 \\
        v_i' \leftarrow 4n - i + 2 \\
        v_i'' \leftarrow 3n - i + 2

    else

        v_i \leftarrow 4n - i + 2, \\
        v_i' \leftarrow n - i + 1 \\
        v_i'' \leftarrow 2n - i + 3

    end if

end for

end if

end procedure

\textbf{Theorem 8.10} Extended triplicate graph ETG(P_n) admits mean labeling.

\textbf{Proof} Clearly extended triplicate graph ETG(P_n) has \( p = 3(n+1) \) vertices and \( q = 4n+1 \) edges for all \( n \) where \( n \) is the length of the path \( P_n \).

Case (i) We prove this theorem for \( n \equiv 0 \ (\text{mod} \ 2) \)

Consider the arbitrary vertex \( v_i \in V \). To label the vertices, using algorithm 8.10, define a map \( f : V \rightarrow \{ 0, 1, 2, \ldots, q \} \).
In order to get the labels for the edges, define the induced map
\[ f^*: E \to \{1, 2, \ldots, q\} \]
such that for any \(v_i v_j \in E\),
\[ f^*(v_i v_j) = \frac{f(v_i) + f(v_j)}{2} \]
if \(f(v_i) + f(v_j)\) is even and \(f^*(v_i v_j) = \frac{f(v_i) + f(v_j) + 1}{2}\)
if \(f(v_i) + f(v_j)\) is odd. Thus we have

(i) For \(2 \leq i \leq n+1\),
\[ f^*(v_i' v_{i-1}) = \begin{cases} 3n - i + 3, & i \equiv 0 \pmod{2} \\ 2n - i + 2, & i \equiv 1 \pmod{2} \end{cases} \]

(ii) For \(1 \leq i \leq n\),
\[ f^*(v_i' v_{i+1}) = \begin{cases} 2n - i + 1, & i \equiv 1 \pmod{2} \\ 3n - i + 2, & i \equiv 0 \pmod{2} \end{cases} \]

(iii) For \(2 \leq i \leq n+1\),
\[ f^*(v_i v_{i-1}) = \begin{cases} 4n - i + 3, & i \equiv 1 \pmod{2} \\ n - i + 2, & i \equiv 0 \pmod{2} \end{cases} \]

(iv) For \(1 \leq i \leq n\),
\[ f^*(v_i v_{i+1}) = \begin{cases} 4n - i + 2, & i \equiv 1 \pmod{2} \\ n - i + 1, & i \equiv 0 \pmod{2} \end{cases} \]

(v) \(f^*(v_n v_1) = 2n+1\).

Case (ii) We prove this theorem for \(n \equiv 1 \pmod{2}\)

Consider the arbitrary vertex \(v_i \in V\). To label the vertices, using algorithm 8.10, define a map \(f: V \to \{0, 1, 2, \ldots, q\}\).

In order to obtain the labels for the edges for \(n\), define the induced map
\[ f^*: E \to \{1, 2, 3, \ldots, q\} \]
such that for any \(v_i v_j \in E\),
\[ f^*(v_i v_j) = \frac{f(v_i) + f(v_j)}{2} \]
if \(f(v_i) + f(v_j)\) is even and \(f^*(v_i v_j) = \frac{f(v_i) + f(v_j) + 1}{2}\) if \(f(v_i) + f(v_j)\) is odd.

Now we obtain the edge labels as follows:

(i) For \(2 \leq i \leq n+1\),
\[ f^*(v_i' v_{i-1}) = \begin{cases} 3n - i + 3, & i \equiv 0 \pmod{2} \\ 2n - i + 2, & i \equiv 1 \pmod{2} \end{cases} \]

(ii) For \(1 \leq i \leq n\),
\[ f^*(v_i' v_{i+1}) = \begin{cases} 2n - i + 1, & i \equiv 1 \pmod{2} \\ 3n - i + 2, & i \equiv 0 \pmod{2} \end{cases} \]
(iii) For \(2 \leq i \leq n+1\), \(f^*(v_i v_{i-1}) = \begin{cases} 
4n - i + 3, & i \equiv 1 \pmod{2} \\
2n - i + 2, & i \equiv 0 \pmod{2} \end{cases}\)

(iv) For \(1 \leq i \leq n\), \(f^*(v_i v_{i+1}) = \begin{cases} 
4n - i + 2, & i \equiv 1 \pmod{2} \\
2n - i + 1, & i \equiv 0 \pmod{2} \end{cases}\)

(vi) \(f^*(v_{n+1} v_1) = 2n+1\).

Thus all the edges receive the label from 1, 2, ..., \(q\) for all \(n\). Hence the extended triplicate graph \(ETG(P_n)\) admits mean labeling.

**Example 8.10** The extended triplicate graph of the path \(P_5\) and its mean labeling is shown in figure 8.10.1. The extended triplicate graph of the path \(P_6\) and its mean labeling is shown in figure 8.10.2.

![Figure 8.10.1: ETG(P_5)](image1)

![Figure 8.10.2: ETG(P_6)](image2)

### 8.11 Prime Cordial labeling

In this section we present an algorithm to obtain prime cordial labeling and prove that the extended triplicate graph \(ETG(P_n)\) admits prime cordial labeling.
Algorithm 8.11

procedure (prime cordial labeling for extended triplicate graph ETG(P_n))

if (n ≡ 1 (mod 2))
    for i = 1 to (n+1) do
        if i ≡ 1 (mod 2)
            \(v_i' \leftarrow 3i-2\)
            \(v_i \leftarrow 3i+1\)
            \(v_i^{''} \leftarrow 3i-1\)
        else
            \(v_i' \leftarrow 3i\)
            \(v_i \leftarrow 3i-1\)
            \(v_i^{''} \leftarrow 3i-3\)
        end if
    end for
else
    for i = 1 to (n+1) do
        if (i ≡ 1 (mod 2))
            \(v_i' \leftarrow 3i-2\)
            \(v_i^{''} \leftarrow 3i-1\)
        else
            \(v_i' \leftarrow 3i\)
            \(v_i^{''} \leftarrow 3i-3\)
        end if
    end for
end if

for i = 1 to n
if \( i \equiv 1 (\text{mod} \ 2) \)

\( v_i \leftarrow 3i+1 \)

else

\( v_i \leftarrow 3i-1 \)

end if

end for

\( v_{n+1} \leftarrow 3(n+1) \)

end if

end procedure

**Theorem 8.11** Extended triplicate graph \( \text{ETG}(P_n) \) admits prime cordial labeling.

**Proof** Clearly extended triplicate graph \( \text{ETG}(P_n) \) has \( p = 3(n+1) \) vertices and \( q = 4n + 1 \) edges for all \( n \).

**Case (i)** If \( n \equiv 1 (\text{mod} \ 2) \):

To label the vertices, using algorithm 8.11, define a map \( f \) from \( V \) to \( \{1, 2, \ldots, p\} \).

In order to obtain the labels for the edges, the induced function \( f^* \) on \( E \) is defined as 1 if \( \gcd (f(v_i), f(v_j)) = 1 \) and 0 if \( \gcd (f(v_i), f(v_j)) > 1 \). Thus we obtain,

(i) For \( 2 \leq i \leq n+1 \), \( f^*(v_i, v_{i+1}) = \begin{cases} 1, & i \equiv 1 (\text{mod} \ 2) \\ 0, & i \equiv 0 (\text{mod} \ 2) \end{cases} \)

(ii) For \( 1 \leq i \leq n \), \( f^*(v_i, v_{i+1}) = \begin{cases} 1, & i \equiv 1 (\text{mod} \ 2) \\ 0, & i \equiv 0 (\text{mod} \ 2) \end{cases} \)

(iii) For \( 2 \leq i \leq n+1 \), \( f^*(v_i, v_{i-1}) = \begin{cases} 0, & i \equiv 1 (\text{mod} \ 2) \\ 1, & i \equiv 0 (\text{mod} \ 2) \end{cases} \)

(iv) For \( 1 \leq i \leq n \), \( f^*(v_i, v_{i+1}) = \begin{cases} 0, & i \equiv 1 (\text{mod} \ 2) \\ 1, & i \equiv 0 (\text{mod} \ 2) \end{cases} \)

(v) \( f(v_{n+1}v_1) = 1 \)
The number of edges labeled ‘0’ is \((n+1)/2 + (n-1)/2 + (n-1)/2 + (n+1)/2 = 2n\)

The number of edges labeled ‘1’ is \((n-1)/2 + (n+1)/2 + (n+1)/2 + (n-1)/2 + 1 = 2n+1\)

**Case (ii)** If \(n \equiv 0 \pmod{2}\):

To label the vertices, using algorithm 8.11, define a map \(f\) from \(V\) to \(\{1, 2, \ldots, p\}\).

In order to get the labels for the edges, the induced function \(f^*\) on \(E\) is defined as 1 if \(\gcd(f(v_i), f(v_j)) = 1\) and as 0 if \(\gcd(f(v_i), f(v_j)) > 1\). Thus we obtain,

\[
\begin{align*}
\text{(i)} & \quad \text{For } 2 \leq i \leq n+1, \quad f^*(v_i v_{i-1}) = \begin{cases} 
1, & i \equiv 1 \pmod{2} \\
0, & i \equiv 0 \pmod{2}
\end{cases} \\
\text{(ii)} & \quad \text{For } 1 \leq i \leq n, \quad f^*(v_i v_{i+1}) = \begin{cases} 
1, & i \equiv 1 \pmod{2} \\
0, & i \equiv 0 \pmod{2}
\end{cases} \\
\text{(iii)} & \quad \text{For } 2 \leq i \leq n+1, \quad f^*(v_i v_{i-1}) = \begin{cases} 
0, & i \equiv 1 \pmod{2} \\
1, & i \equiv 0 \pmod{2}
\end{cases} \\
\text{(iv)} & \quad \text{For } 1 \leq i \leq n, \quad f^*(v_i v_{i+1}) = \begin{cases} 
0, & i \equiv 1 \pmod{2} \\
1, & i \equiv 0 \pmod{2}
\end{cases} \\
\text{(v)} & \quad f(v_n v_1) = 1
\end{align*}
\]

The number of edges labeled ‘0’ is \(n/2 + n/2 + n/2 + n/2 = 2n\)

The number of edges labeled ‘1’ is \(n/2 + n/2 + n/2 + n/2 + 1 = 2n+1\)

Thus in both the cases the number of edges labeled with ‘0’ and the number of edges labeled with ‘1’ differ by atmost one.

Hence extended triplicate graph \(ETG(P_n)\) admits prime cordial labeling

**Example 8.11** The extended triplicate graph of the path \(P_5\) and its prime cordial labeling is shown in figure 8.11.1. The extended triplicate graph of the path \(P_6\) and its prime cordial labeling is shown in figure 8.11.2.
8.12 Prime E-cordial labeling

Now we present an algorithm to obtain prime E-cordial labeling and prove that
the modified extended triplicate graph ETG(Pₙ) admits prime E-cordial labeling.

Algorithm 8.12

procedure (construction of modified extended triplicate graph)

E₁ ← φ

E₂ ← (vᵣ, vᵣ₊₁)∪(vᵣ, vᵣ₊₁)∪(v₁, v₂)∪(v₁, vₑ)

for i = 2 to n do

E₃ ← \{ (vᵢ₋₁, vᵢ), (vᵢ₋₁, vᵢ), (vᵢ, vᵢ), (vᵢ, vᵢ), (vᵢ, vᵢ) \}

E₁ ← E₃∪E₁

end for

if (n ≡ 1 (mod 2))

E ← E₂∪E₁∪(vᵣ₊₁, v₁)∪(vᵣ₊₁, v₁)∪(v₁, v₁)∪(v₁, v₁)

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else

\[ E \leftarrow E_2 \cup E_1 \cup (v_n v_1) \cup (v_{n+1} v_n) \cup (v_1 v_{n+1}) \cup (v_{n+1} v_n) \] 

end if

end procedure

Algorithm 8.12.1

procedure (prime E-cordial labeling for modified extended triplicate graph)

if \((n \equiv 1 \pmod{2})\) 

for \(i = 2\) to \((n+1)\) do

if \((i \equiv 0 \pmod{2})\)

\[ v_i v_{i-1}^{*} \leftarrow 4i-7 \]
\[ v_i v_{i-1}' \leftarrow 4i-4 \]

else

\[ v_i v_{i-1}^{*} \leftarrow 4i-6 \]
\[ v_i v_{i-1}' \leftarrow 4i-5 \]

end if

end for

for \(i = 1\) to \(n\) do

if \((i \equiv 0 \pmod{2})\)

\[ v_i v_{i+1}^{*} \leftarrow 4i-3 \]
\[ v_i v_{i+1}' \leftarrow 4i \]

else

\[ v_i v_{i+1}^{*} \leftarrow 4i-2 \]
\[ v_i v_{i+1}' \leftarrow 4i-1 \]

end if

end for
\[ \begin{align*}
  v_{n+1} &= 4n + 2 \\
  v_{2n+1} &= 4n + 4 \\
  v_1 &= 4n + 1 \\
  v_3 &= 4n + 3 \\
 &\text{else} \\
  \text{for } i = 2 \text{ to } (n+1) \text{ do} \\
  &\text{if } (i \equiv 0 \pmod{2}) \\
  &\quad v_i \leftarrow 4i - 7 \\
  &\quad v_i \leftarrow 4i - 4 \\
  &\text{else} \\
  &\quad v_i \leftarrow 4i - 6 \\
  &\quad v_i \leftarrow 4i - 5 \\
  &\text{end if} \\
  &\text{end for} \\
  \text{for } i = 1 \text{ to } n \text{ do} \\
  &\text{if } i \equiv 0 \pmod{2} \\
  &\quad v_i \leftarrow 4i - 3 \\
  &\quad v_i \leftarrow 4i \\
  &\text{else} \\
  &\quad v_i \leftarrow 4i - 2 \\
  &\quad v_i \leftarrow 4i - 1 \\
  &\text{end if} \\
  &\text{end for} \\
  v_n &= 4n + 2
\end{align*} \]
\texttt{v}_{n+1} \leftarrow 4n+1
\texttt{v}_1 \leftarrow 4n+3
\texttt{v}_n \leftarrow 4n+4

\textbf{end if}

\textbf{end procedure}

\textbf{Theorem 8.12.1} Modified extended triplicate graph admits prime E-cordial labeling.

\textbf{Proof} Construct the modified extended triplicate graph using algorithm 8.12. Clearly the modified extended triplicate graph has $p = 3(n+1)$ vertices and $q = 4n + 4$ edges for all $n$ where $n$ is the length of the path $P_n$.

\textbf{Case (i)} If $n \equiv 1 \text{ (mod 2)}$:

To label the edges, using algorithm 8.12.1, define a map $f$ from $E$ on to the set \{1, 2, ..., q\}. In order to obtain the labels for the vertices, the induced function $f^*$ on $V$ is defined as 1 if $\gcd(f(v_i v_j), f(v_k v_l)) = 1$ and 0 if $\gcd(f(v_i v_j), f(v_k v_l)) > 1$.

Thus, for all $v_i$ and $1 \leq i \leq n+1$

\[ f^*(v_i) = \begin{cases} 0, & i \equiv 0 \text{ (mod 2)} \\ 1, & i \equiv 1 \text{ (mod 2)} \end{cases} \]

\[ f^*( v_i' ) = \begin{cases} 1, & i \equiv 1 \text{ (mod 2)} \\ 0, & i \equiv 0 \text{ (mod 2)} \end{cases} \]

The number of vertices labeled ‘0’ is $3(n+1)/2$

The number of vertices labeled ‘1’ is $3(n+1)/2$

\textbf{Case (i)} If $n \equiv 0 \text{ (mod 2)}$:

To label the edges, using algorithm 8.12.1, define a map $f$ from $E$ on to the set \{1, 2, ..., q\}. In order to obtain the labels for the vertices, the induced function $f^*$ on $V$ is defined as 1 if $\gcd(f(v_i v_j), f(v_k v_l)) = 1$ and 0 if $\gcd(f(v_i v_j), f(v_k v_l)) > 1$.
Thus, for all $v_i$ and $1 \leq i \leq n+1$

$$f^*(v_i) = f^*(v_i^{''}) = \begin{cases} 0 & i \equiv 0 \pmod{2} \\ 1 & i \equiv 1 \pmod{2} \end{cases}$$

$$f^*(v_i^{'}) = \begin{cases} 0 & i \equiv 1 \pmod{2} \\ 1 & i \equiv 0 \pmod{2} \end{cases}$$

The number of vertices labeled ‘0’ is $(3n+1)/2$

The number of vertices labeled ‘1’ is $(3n+1)/2$

Thus in both the cases the number of vertices labeled 0 and the number of vertices labeled 1 differ atmost by one.

Hence the modified extended triplicate graph $ETG(P_n)$ admits prime E-cordial labeling.

**Example 8.12** The extended triplicate graph of the path $P_5$ and its prime E-cordial labeling is shown in figure 8.12.1. The extended triplicate graph of the path $P_6$ and its prime E-cordial labeling is shown in figure 8.12.2.
8.13 4-Cordial labeling

In this section we present an algorithm to obtain the 4-cordial labeling and we prove that the extended triplicate graph admits 4-cordial labeling.

Algorithm 8.13

procedure (4-cordial labeling for extended triplicate graph of $P_n$)

$V \leftarrow \{v_1, v_2, ..., v_{3(n+1)}\}$

$E \leftarrow \{e_1, e_2, ..., e_{4n+1}\}$

if ($i \equiv 1 \pmod{2}$)

for $i = 1$ to $(n+1)$ do

if ($i = 4m, m \in \mathbb{N}$)

Figure 8.12.1: ETG($P_5$)  
Figure 8.12.2: ETG($P_6$)
\[ v_i \leftarrow 3 \]
\[ v_i' \leftarrow 1 \]
\[ v_i'' \leftarrow 1 \]

else

if \ (i = 4m-1, m \in N) 

\[ v_i \leftarrow 0 \]
\[ v_i' \leftarrow 3 \]
\[ v_i'' \leftarrow 2 \]

else

if \ (i = 4m-2, m \in N) 

\[ v_i \leftarrow 2 \]
\[ v_i' \leftarrow 0 \]
\[ v_i'' \leftarrow 0 \]

else

if \ (i = 4m-3, m \in N) 

\[ v_i \leftarrow 1 \]
\[ v_i' \leftarrow 2 \]
\[ v_i'' \leftarrow 3 \]

end if

end if

end if

end if

end if

end for

else
for $i = 1$ to $(n + 1)$ do

if $(i = 4m, m \in \mathbb{N})$

$v_i \leftarrow 3$
$v_i' \leftarrow 0$
$v_i'' \leftarrow 1$

else

if $(i = 4m-1, m \in \mathbb{N})$

$v_i \leftarrow 0$
$v_i' \leftarrow 2$
$v_i'' \leftarrow 3$

else

if $(i = 4m-2, m \in \mathbb{N})$

$v_i \leftarrow 2$
$v_i' \leftarrow 1$
$v_i'' \leftarrow 0$

else

if $(i = 4m-3, m \in \mathbb{N})$

$v_i \leftarrow 1$
$v_i' \leftarrow 3$
$v_i'' \leftarrow 2$

end if

end if

end if

end if
end for

end if

end procedure

**Theorem 8.13** Extended triplicate graph ETG(Pₙ) admits 4-cordial labeling.

**Proof** Clearly extended triplicate graph ETG(Pₙ) has 3(n+1) vertices and 4n+1 edges for all n.

To label the vertices, using algorithm 8.13, define a map \( f : V \rightarrow \{0, 1, 2, 3\} \).

Clearly the number of vertices labeled with 0, 1, 2 and 3 differ by at most one to each other for all n.

In order to obtain the labels for the edges, define the induced map \( f^* : E \rightarrow \{0, 1, 2, 3\} \) such that for any \( v_i, v_j \in E \), \( f^*(v_i, v_j) = (f(v_i) + f(v_j)) \pmod{4} \)

**Case (i) For** \( n \equiv 1 \pmod{2} \).

Let \( m \in \mathbb{N} \) and

(i) For \( 2 \leq i \leq n+1 \), \( f^*(v_i, v_{i-1}^{'}) = 3 \)

(ii) For \( 1 \leq i \leq n \), \( f^*(v_i, v_{i+1}^{'}) = \begin{cases} 2, & i = 4m+2 & \text{or} & 4m+3 \\ 0, & i = 4m & \text{or} & 4m+1 \end{cases} \)

(iii) For \( 2 \leq i \leq n+1 \), \( f^*(v_i, v_{i+1}^{'}) = \begin{cases} 2, & i = 4m \text{ or } 4m+1 \\ 0, & i = 4m-1 \text{ or } 4m-2 \end{cases} \)

(iv) For \( 1 \leq i \leq n \), \( f^*(v_i, v_{i+1}^{'}) = 1 \)

(v) \( f^*(v_{n+1}, v_1) = \begin{cases} 0, & n = 4m-1 \\ 3, & n = 4m-3 \end{cases} \)

Hence the number of edges labeled with 0, 1, 2 and 3 differ by at most 1 to each other.

**Case (ii) For** \( n \equiv 0 \pmod{2} \)

Let \( m \in \mathbb{N} \) and
(i) For $2 \leq i \leq n+1$, 
$$f^*(v_i, v_{i+1}) = \begin{cases} 
3, & i = 2m \\
2, & i = 4m - 1 \\
0, & i = 4m + 1 
\end{cases}$$

(ii) For $1 \leq i \leq n$, 
$$f^*(v_i, v_{i+1}) = \begin{cases} 
3, & i = 2m - 1 \\
2, & i = 4m \\
0, & i = 4m - 2 
\end{cases}$$

(iii) For $2 \leq i \leq n+1$, 
$$f^*(v_i, v_{i+1}) = 1$$

(iv) For $1 \leq i \leq n$, 
$$f^*(v_i, v_{i+1}) = \begin{cases} 
2, & i = 4m & 4m - 3 \\
0, & i = 4n - 1 & 4m - 2 
\end{cases}$$

(v) 
$$f^*(v_n, v_1) = \begin{cases} 
0, & n = 4m \\
3, & n = 4m - 2 
\end{cases}$$

Hence the number of edges labeled with 0, 1, 2 and 3 differ by at most one to each other.

Thus in both the cases the number of vertices (respectively edges) labeled with 0, 1, 2 and 3 differ by at most one to each other.

Hence the extended triplicate graph ETG($P_n$) admits 4-cordial labeling.

**Example 8.13** Consider the path $P_3$

![Diagram of $P_3$]

The extended triplicate graph of the path $P_3$ and its 4-cordial labeling is shown in figure 8.13.1. The extended triplicate graph of the path $P_3$ and its 4-cordial labeling is shown in figure 8.13.2. Consider the path $P_4$

![Diagram of $P_4$]
The extended triplicate graph of the path $P_4$ and its 4-cordial labeling is shown in figure 8.13.3. The extended triplicate graph of the path $P_6$ and its 4-cordial labeling is shown in figure 8.13.4.
8.14 Vertex prime labeling

In this section we present an algorithm to get a modified extended triplicate graph and prove that ETG(P_n) admits vertex prime labeling.

Algorithm 8.14

procedure (construction of modified extended triplicate graph)

\[ E_1 \leftarrow \emptyset \]

\[ E_2 \leftarrow \{ (v_n, v_{n+1}) \cup (v_n^*, v_{n+1}) \cup (v_1', v_2) \cup (v_1', v_2^*) \} \]

for \( i = 2 \) to \( n \)

\[ E_3 \leftarrow \{ (v_{i-1}, v_i) \cup (v_{i-1}^*, v_i) \cup (v_{i+1}, v_i) \cup (v_{i+1}^*, v_i) \} \]

\[ E_i \leftarrow E_3 \cup E_i \]

end for

if \( (n \equiv 1 \text{ (mod } 2)) \)

\[ E \leftarrow E_2 \cup E_1 \cup (v_{n+1}, v_1) \cup (v_n^*, v_{n+1}) \cup (v_1^*, v_1) \]

else

\[ E \leftarrow E_2 \cup E_1 \cup (v_n, v_1) \cup (v_{n+1}^*, v_{n+1}) \cup (v_1^*, v_1) \]

end if

end procedure

Algorithm 8.14.1

procedure (vertex prime labeling for modified extended triplicate graph)

if \( (n \equiv 1 \text{ (mod } 2)) \)

for \( i = 2 \) to \( (n+1) \)

if \( (i \equiv 0 \text{ (mod } 2)) \)

\[ v_i' v_{i-1}^* \leftarrow 2n + i + 1 \]

end if

end for
\[ v_i v_{i-1}^{'} \leftarrow 2n - i + 3 \]

else

\[ v_i v_{i-1}^{'} \leftarrow 4n - i + 4 \]
\[ v_i^{'} v_{i-1}^{''} \leftarrow i - 1 \]

end if

end for

for \( i = 1 \) to \( n \) do

if \((i \equiv 0 \pmod{2})\)

\[ v_i^{'} v_{i+1}^{''} \leftarrow 2n + i + 2 \]
\[ v_i v_{i+1}^{'} \leftarrow 2n - i + 2 \]

else

\[ v_i^{'} v_{i+1}^{''} \leftarrow i \]
\[ v_i v_{i+1}^{'} \leftarrow 4n - i + 3 \]

end if

end for

\[ v_1 v_n+1 \leftarrow 4n + 3 \]
\[ v_n+1^{''} v_n+1 \leftarrow n+1 \]
\[ v_1^{''} v_1 \leftarrow 2n+2 \]

else

for \( i = 2 \) to \((n+1)\) do

if \((i \equiv 0 \pmod{2})\)

\[ v_i^{'} v_{i-1}^{''} \leftarrow 2n + i \]
\[ v_i v_{i-1}^{'} \leftarrow 2n - i + 2 \]

else

\[ v_i^{'} v_{i-1}^{''} \leftarrow i - 1 \]
\[
v_i v_{i+1} \leftarrow 4n - i + 4
\]

end if

end for

for \(i = 1\) to \(n\) do

if \((i \equiv 0 \pmod{2})\)

\[
v_i v_{i+1}'' \leftarrow 2n + i + 1
\]

\[
v_i v_{i+1}' \leftarrow 2n - i + 1
\]

else

\[
v_i v_{i+1}'' \leftarrow i
\]

\[
v_i v_{i+1}' \leftarrow 4n - i + 3
\]

end if

end for

\[
v_1 v_n \leftarrow 4n + 3
\]

\[
v_{n+1}'' v_{n+1} \leftarrow 3n + 2
\]

\[
v_1'' v_1 \leftarrow 2n + 1
\]

end if

end procedure

**Theorem 8.14** Modified extended triplicate graph admits vertex prime labeling.

**Proof** Using algorithm 8.14, construct the modified extended triplicate graph \(\operatorname{ETG}(P_n)\). Denote the vertex set of modified \(\operatorname{ETG}(P_n)\) as \(V = \{v_1, v_2, \ldots, v_{3(n+1)}\}\) and the edge set as \(E = \{e_1, e_2, \ldots, e_{4n+3}\}\).

To label the edges for all \(n\), using algorithm 8.14.1, define the map \(f\) from \(E\) to \(\{1, 2, \ldots, q\}\). In order to obtain the labels for the vertices, the induced function \(f^*\) on \(V\) is defined as 1 if \(\gcd(f(v_i v_j), f(v_k v_l)) = 1\).
Thus we have for all $v_i$ and $1 \leq i \leq n+1$, $f^*(v_i) = 1$. Hence the modified extended triplicate graph admits vertex prime labeling.

**Example 8.14** The modified extended triplicate graph of the path $P_5$ and its vertex prime labeling is shown in figure 8.14.1. The modified Extended triplicate graph of the path $P_6$ and its vertex labeling is shown in figure 8.14.2.

![Figure 8.14.1: ETG($P_5$)](image1)

![Figure 8.14.2: ETG($P_6$)](image2)