CHAPTER -2
HOMOMORPHISMS ON SEMI GROUP OF ARITHMETIC FUNCTIONS ASSOCIATED WITH A GENERALIZED BASIC SEQUENCE

In this chapter, we introduce notions of a generalized basic sequence, generalized additive function and a generalized convolution associated with a generalized basic sequence and semi groups of arithmetic functions are shown to be homomorphic through a mapping induced by a particular generalized basic sequence and a generalized additive function.

The contents of the Chapter-II are essentially contained in SASTRY, SUKLA and PANDA [43]

§1. INTRODUCTION

2.1 As already stated in Chapter - 0, section §2, \((A_0, +)\) is an abelian group and \((A_1, \times)\) is a sub group of \((P, \times)\), \(\times\) being the unitary convolution. \((P, *)\) and \((M, \ast)\) are sub group of \((Q, \ast)\), \(\ast\) being Dirichlet convolution. We also recall (Chapter - 0) that \((A_0, +)\) is a sub group of \((A, +)\).

Throughout this Chapter - II unless otherwise specified \(\alpha\) stands for an additive function.

REARICK [37] has proved that the groups \((A, +), (P, \ast), (M, \ast)\), \((P, \times)\) and \((M, \times)\) are all isomorphic. Infact for each \(f \in P\) he has defined the function \(L_f \in A\) by

\[
(L_f)(1) = Log(1)
\]

\[
(L_f)(n) = \sum_{d \mid n} f(d) f^{-1}\left(\frac{n}{d}\right) \log d, \quad \text{if} \quad n > 1 \quad \text{and has shown that}
\]

\(L:(p, \ast) \rightarrow (A, +)\) is an isomorphism.

(33)
Later KRISHNA GANDHI [22] defined $L_a : (A_1, \ast) \to (A_0, +)$ as

$$\text{2.1.2} \quad (L_a f)(n) = \sum_{d \mid n} \alpha(d)f(d)f_0^{-1}\left(\frac{n}{d}\right)$$

corresponding to a completely additive function $\alpha$, and claimed that $L_a$ is an isomorphism onto.

P.L.V NAGESWARA RAO [32] found that while $L_a$ is a homomorphism it need not always be an isomorphism and obtained conditions on $\alpha$, under which $L_a$ is an isomorphism.

GOLDSMITH [17,18] introduced the notion of a basic sequence as defined in (0.2.9). Associated with a basic sequence $B$ GOLDSMITH [18] also introduced the notion of generalised convolution $\circ_B$ as defined in (0.2.10). Further GOLDSMITH [18] has shown that $(A_1, \circ_B)$ is an abelian group with $\delta$ (0.2.3) as the identity. The inverse of $f \in A_1$ with respect to $\circ_B$ being denoted by $f_B^{-1}$.

Later RAJASEKHAR [34] introduced the notion of a $B$-additive function (0.2.11) with respect to a basic sequence $B$ as defined in (0.2.9). Given a basic sequence $B$ and a $B$-additive function $\alpha$, RAJASEKHAR [34] defined the operator $L_{\beta a}$ as defined in (0.2.12) and has shown that $L_{\beta a}$ is always a homomorphism onto and investigated for condition on $\alpha$ under which $L_{\beta a}$ is an isomorphism.

Motivated by the work of GOLDSMITH [18], in the present Chapter-II of this dissertation, we introduce the notion of (2.2.1) generalized basic sequences, and (2.3.1) generalised additive functions associated with generalised basic sequence and (2.4.1) generalised convolutions associated with a generalised basic sequence, give an example of a generalised basic sequence $F$ and show that $(A_1, \circ_F)$ is a commutative semi group with
cancellation law, define $L_\alpha : (A, o_F) \to (A_0, +)$ suitably and show that $L_\alpha$ is a semi group homomorphism, where $o_F$ is the generalized convolution associated with the generalized basic sequence $F$ and $\alpha$ is a generalized additive function associated with generalized basic sequence $F$. Further we show that $L_\alpha : (A, o_F) \to (A_0, +)$ is never onto and never one-one (Examples (2.4.12) and (2.4.14)).

§2. GENERALIZED BASIC SEQUENCES

2.2 In this section, we define generalized basic sequences and give an example of a generalized basic sequence.

2.2.1 Let $\tau$ be a subset of $Z^+ \times Z^+$ (where $Z^+$ is the set of positive integers) satisfying

(2.2.1.1) $(a, b) \in \tau \Rightarrow (b, a) \in \tau$

(2.2.1.2) If $(b, c) = 1$, then $(a, bc) \in \tau \iff (a, b) \in \tau$ and $(a, c) \in \tau$

(2.2.1.3) $(1, k) \in \tau \forall k \in Z^+$

Then $\tau$ is called a generalised basic sequence.

We observe that every basic sequence is a generalised basic sequence.

We recall that (0.0.3), for any two positive integers $a, b$ $(a, b)^*$ denotes the largest divisor of $a$ which is a unitary divisor of $b$ (COHEN [13]).

and $r \| a$ means (0.0.2) $r$ is a unitary divisor of $a$.

We observe that, in general, $(a, b)^* \neq (b, a)^*$. 

(35)
2.2.2 Example \( F = \left\{ (a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid (a,b)^* = (b,a)^* \right\} \)

Then \( F \) satisfies (2.2.1.1) and (2.2.1.3) and in theorem (2.2.7), we show that \( F \) also satisfies (2.2.1.2), hence is a generalized basic sequence. We observe that \( F \) does not satisfy (0.2.9.2) (Example 2.2.8) and hence is not a basic sequence.

In the following theorem we characterize the set \( F \).

2.2.3 Theorem: \((a,b) \in F \iff \exists r,s,t \in \mathbb{Z}^+ \exists a = rs, b = rt \)

and \((r,s) = (r,t) = (s,t) = 1\)

( This triad \((r,s,t)\) is uniquely determined by the pair \((a,b)\) since \((a,b) = r\).)

Proof: Suppose \((a,b)^* = (b,a)^* = r\) (say)

Then \( s,t \in \mathbb{Z}^+ \exists a = rs, b = rt, r \nmid a, r \nmid b \)

So that \((r,s) = 1 = (r,t)\)

Now \((s,t) = x \Rightarrow \text{either } \left(x, \frac{s}{x}\right) = 1 \text{ or } \left(x, \frac{t}{x}\right) = 1\).

Suppose, without loss of generality, that \( \left(x, \frac{s}{x}\right) = 1 \), then \( rx \nmid a \) and \( rx \nmid b \)

so that \( r = (b,a)^* \geq rx \Rightarrow x = 1 \). Thus \((s,t) = 1\).

Conversely, assume that \( a = rs, b = rt \) and \((r,s) = (r,t) = (s,t) = 1\).

Clearly \( r \mid a \) and \( r \mid b \)

Now suppose \( e \mid a \) and \( e \mid b \). Then \( e \mid rs \)

If \((e,s) > 1\), let \( p \) be a prime divisor of \((e,s)\) so that \((p,r) = 1\) and \( p \mid b \) and hence \( p \mid rt \). Consequently \( p \mid t \) which contradicts that \((s,t) = 1\). Hence \( (e,s) = 1\) so that \( e \mid r \).

Thus \((a,b)^* = r\). By symmetry, follows that \((a,b)^* = r = (b,a)^*\).
The following example shows that \( a \cdot b = r^2 st, (r,s,t) = 1 \) need not imply that \((a,b) \in F\).

2.2.4 Example: Let \( ab = 2^4 \cdot 3 \cdot 5 \), \( (2,3) = (2,5) = (3,5) = 1 \)

Let \( a = 2^3 \cdot 3 \) and \( b = 2 \cdot 5 \)
Then \( (a,b)^* = 2 \neq 1 = (b,a)^* \)
Hence \( (a,b) \notin F \).

2.2.5 Theorem: If \((a,b) \in F\) and \((a,c) \in F\) then \(((a,b)^*, (a,c)^*) \in F\)

Proof: Let \((a,b)^* = d, (a,c)^* = e\)

Suppose \((a,b) \in F\) and \((a,c) \in F\). Then, by Theorem 2.2.3, \( \exists d,f,g,e,l,m \in \mathbb{Z}^+ \)

\[ a = df, b = dg \] and \( (d,f) = (d,g) = (f,g) = 1 \) and

\[ a = el, c = em \] and \( (e,l) = (e,m) = (l,m) = 1 \)

Now, \( e \mid a \Rightarrow e \mid df \Rightarrow \exists d_i, f_i \) such that \( e = d_i f_i, d_i \mid d \) and \( f_i \mid f \)

\( d_i \mid a \Rightarrow d_i \mid el \Rightarrow \exists e_i, l_i \) such that \( d = e_i l_i, e_i \mid e \) and \( l_i \mid l \).

\[ e_i \mid e \Rightarrow e_i \mid d_i l_i \Rightarrow e_i = x_i y_i \text{ where } x_i \mid d_i \text{ and } y_i \mid f_i \Rightarrow y_i \mid d_i y_i \mid f \text{ since } y_i \mid e_i \text{ and } e_i \mid d \text{ and } y_i \mid f_i \text{ and } f_i \mid f. \]

Thus \( y_i \mid d_i \) and \( y_i \mid f \Rightarrow y_i = 1 \) since \( (d,f) = 1 \Rightarrow \)

Similarly \( d_i \mid e_i \) so that \( e_i = d_i \)

Clearly \( (d_i, f_i) = 1 = (e_i, l_i) = (d_i, l_i) \)

Also \( l_i \mid d_i, f_i \mid f \) and \( (d,f) = 1 \Rightarrow (l_i, f_i) = 1 \)

Thus \( d = d_i l_i, e = d_i f_i, (d_i, l_i) = 1 = (d_i, f_i) = (l_i, f_i) \)

Hence by theorem 2.2.3, \((d,e) \in F\)

(37)
Note: The converse of the above result is not true in view of the following example.

2.2.6 Example: \((2, 2^2)^* = 1, (2,1)^* = 1\) and \((1,1) \in F\) but \((2, 2^2) \notin F\) while \((2,1) \in F\).

The following theorem shows that \(F\) satisfies (2.2.1.2) and hence \(F\) is a generalized basic sequence.

2.2.7 Theorem: If \(a, b, c\) are positive integers such that \((b,c) = 1\) then \((a, bc) \in F\) if and only if \((a,b) \in F\) and \((a,c) \in F\)

Proof: Suppose \((a,b) \in F\) and \((a,c) \in F\) then by theorem 2.2.3,

\[(2.2.7.1) \exists \ d, f, g \ a = df, b = dg, (d,f) = (d,g) = (f,g) = 1\]

and

\[(2.2.7.2) \exists e, l, m \ a = el, c = em, (e,l) = (e,m) = (l,m) = 1\]

\[(2.2.7.3) (b,c) = 1 \Rightarrow (d,e) = (d,m) = (g,e) = (g,m) = 1\]

\[e | a \Rightarrow e | df \Rightarrow e | f \text{ (by (2.2.7.3))} \Rightarrow \exists f_1 \text{ such that } f = ef_1\]

\[d | a \Rightarrow d | el \Rightarrow d | l \text{ (by (2.2.7.3))} \Rightarrow \exists l_1 \text{ such that } l = dl_1\]

Hence

\[(2.2.7.4) \ de, f_1 = df = a = el = d, l_1 \Rightarrow f_1 = l_1\]

and \((d,f) = 1 \Rightarrow (df_1) = 1 \ (e,f_1) = (e,l_1) = 1\) since \((e,f) = 1\) and \(l_1 \parallel l\)

Thus

\[(2.2.7.5) (de, f_1) = 1\]

Similarly we can show, from (2.2.7.4), that

\[(38)\]
(2.2.7.6) \( (gm, f_1) = 1 \)

(2.2.7.1), (2.2.7.2) and (2.2.7.3) imply that

(2.2.7.7) \( (de, gm) = 1 \)

Since \( a = def_1 \) and \( bc = degm \), from theorem 2.2.3, using (2.2.7.5), (2.2.7.6) and (2.2.7.7) we concluded that \( (a, bc) \in F \).

Conversely, assume that \( (a, bc) \in F \). Then by theorem (2.2.3)

(2.2.7.8) \( \exists d, x, y \) such that \( a = dx \), \( bc = dy \) and \( (d, x) = (d, y) = (x, y) = 1 \)

(2.2.7.9) \( b | dy \Rightarrow \exists d, y_1 \) such that \( b = d y_1 \) and \( d | d, y_1 | y \)

Let \( d_2 = \frac{d}{d_1} \) and \( y_2 = \frac{y}{y_1} \). Then

\( d_2, y_1 \ c = bc = dy = d_1 d_2 y_1, y_2 \) so that \( c = d_2 y_2 \)

This shows that

(2.2.7.10) \( (d_1, d_2) = 1 = (y_1, y_2) \)

We have, from (2.2.7.8) and (2.2.7.9),

(2.2.7.11) \( (d_1, d_2 x) = 1 \) and \( (d_2 x, y_1) = 1 \) and \( (d_1, y_1) = 1 \)

Since \( a = d_1 d_2 x \) and \( b = d_1 y_1 \) from theorem 2.2.3 and (2.2.7.10), follows that

\( (a, b) \in F \). Similarly \( (a, c) \in F \).

The following example shows that \( F \) does not satisfy (0.2.9.2) and hence \( F \) is not a basic sequence.

2.2.8 Example : Let \( a = 2.3, b = 3.5 \) and \( c = 3.5^2 \)

\( (a, b)^* = 3 = (b, a)^* \) and \( (a, c)^* = 3 = (c, a)^* \)

\( (a, bc)^* = 1 \neq 3 = (bc, a)^* \)
§3. GENERALIZED ADDITIVE FUNCTIONS

In this section we define a generalized additive function, give some examples and characterize generalized additive function (Theorem 2.3.4).

2.3.1. Definition: Suppose \( \alpha \) is an arithmetical function satisfying

\[
\begin{align*}
(2.3.1.1) \quad \alpha(st) &= \alpha(s) + \alpha(t) \quad \text{if } (s,t) = 1 \\
(2.3.1.2) \quad \alpha(r^3 s) &= \alpha(s) \quad \text{if } (r,s) = 1
\end{align*}
\]

Then \( \alpha \) is called a generalized additive function associated with generalized basic sequence \( F \).

2.3.2 Observation: \( \alpha(1) = 0 \) if \( \alpha \) is a generalized additive function.

Infact \( \alpha(n^2) = 0 \) for \( n = 1, 2, 3, \ldots \).

2.3.3. Examples: We give below three examples of generalized additive functions associated with \( F \).

(2.3.3.1) Define \( \alpha(p^k) = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases} \)

If \( n = \prod_{i=1}^{r} p_i^{k_i} \), define \( \alpha(n) = \sum_{i=1}^{r} \alpha(p_i^{k_i}) \)

(2.3.3.2) Define \( \alpha(p^k) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases} \)

and \( \alpha(n) = \sum_{i=1}^{r} \alpha(p_i^{k_i}), \text{where} \quad n = \prod_{i=1}^{r} p_i^{k_i} \)

(2.3.3.3) Define \( \alpha(p^k) = \begin{cases} k & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases} \)

and \( \alpha(n) = \sum_{i=1}^{r} \alpha(p_i^{k_i}), \text{where} \quad n = \prod_{i=1}^{r} p_i^{k_i} \)

(40)
OBSERVATION: To define a generalized additive function, it is sufficient to define at prime powers. Further, from the definition follows that \( \alpha (p^{2^k}) = 0 \) for \( k = 0,1,2,... \).

2.3.4. Theorem: \( \alpha \) is a generalized additive function if and only if

- (2.3.1.1) \( \alpha (st) = \alpha (s) + \alpha (t) \) whenever \( (s,t) = 1 \) and
- (2.3.4.1) \( \alpha (n) = 0 \) whenever \( n \) is a square.

Proof: Suppose \( \alpha \) is a generalized additive function. Then \( \alpha (1) = 0. \)

\[ n = m^2 \Rightarrow \alpha (n) = \alpha (m^2) = \alpha (1) = 0 \text{ by (2.3.1.2). Thus (2.3.4.1) is satisfied.} \]

Conversely, assume that (2.3.1.1) and (2.3.4.1) hold.

Then \( (r,s) = 1 \Rightarrow (r^2,s) = 1 \Rightarrow \alpha (r^2 s) = \alpha (r^2) + \alpha (s) \text{ (by (2.3.1.1))} \)

\[ = \alpha (s) \text{ by (2.3.4.1)} \]

Thus (2.3.1.2) holds.

§4. GENERALIZED CONVOLUTIONS

2.4 In this section we define generalized convolution induced by the generalized basic sequence \( F \) of §2 and obtain some properties of this generalized convolution.

Throughout this section

\[ F = \{ \ (a,b) \ | \ (a,b)^* = (b,a)^* \ \} \]
Notation: Suppose positive integers \( r_1, r_2, \ldots, r_k \) are pairwise coprime, then we write \((r_1, r_2, \ldots, r_k) = 1\).

2.4.1 Definition: For any two arithmetic functions \( f \) and \( g \) we define \( f \circ_F g \) as follows:

\[
(f \circ_F g)(n) = \sum_{\substack{d \mid n, e \mid n \\text{ and } (d, e) \in F}} f\left(\frac{d}{(d, e)}\right)g\left(\frac{e}{(d, e)}\right)
\]

we observe that (by theorem 2.2.3)

\[
(2.4.1.1) \quad (f \circ_F g)(n) = \sum_{n=rs, t \mid n, (r, t) = 1} f(s)g(t)
\]

(If we write \( d = rs \) and \( e = rt \), then \( n = de \) and \( (d, e) \in F \))

2.4.2 Lemma: The convolution \( \circ_F \) is commutative and associative on the set \( A \) of arithmetic function.

Proof: Let \( f, g \in A \). Then, for \( n \geq 1 \),

\[
(f \circ_F g)(n) = \sum_{\substack{n=rs, t \mid n, (r, t) = 1}} f(s)g(t)
\]

\[= \sum_{\substack{n=rs, t \mid n, (r, t) = 1}} g(t)f(s) = (g \circ_F f)(n)\]

So that \( \circ_F \) is a commutative in \( A \).

Let \( f, g, h \in A \). Then, for \( n \geq 1 \)

\[
(2.4.2.1) \quad ((f \circ_F g) \circ_F h)(n) = \sum_{\substack{n=rs, t \mid n, (r, t) = 1}} (f \circ_F g)(s)h(t)
\]

(42)
\[
\sum_{n=r^*st} h(t)(f \circ r \circ g)(s)
\]
\[
= \sum_{n=r^*st} h(t) \sum_{n=r^*st} f(b)g(c)
\]
\[
= \sum_{n=r^*st} f(b)g(c)h(t)
\]

and

\[
(2.4.2.2) \quad (f \circ g \circ h)(n) = \sum_{n=r^*st} f(b)(g \circ h)(s)
\]
\[
= \sum_{n=r^*st} f(b) \sum_{n=r^*st} g(c)h(t)
\]
\[
= \sum_{n=r^*st} f(b)g(c)h(t)
\]

From (2.4.2.1) and (2.4.2.2) follows that \( f \circ g \circ h = (f \circ g) \circ h \)

So that \( \circ \) is associative on \( A \).

2.4.3 Corollary : \( (A_1, \circ_F) \) is a commutative semi group .

Proof : This is a simple consequence of lemma 2.4.2

since \( f, g \in A_1 \Rightarrow f \circ_F g \in A_1 \).

2.4.4 Lemma : If \( f, g, h \in A \), then \( (f + g) \circ_F h = (f \circ_F h) + (g \circ_F h) \)

Proof : \( ((f + g) \circ_F h)(n) = \sum_{n=r^*st} (f + g)(s)h(t) \)
Hence lemma follows.

Now we show that cancellation law is valid in \((A, f, g)\).

2.4.5 Lemma: Suppose \(f \in A\) and \(g, h \in A\). Then \(f \circ h = f \circ g \Rightarrow g = h\)

Proof: Clearly \(g(1) = h(1)\) since \(f(1) = 1\).

Assume \(g(m) = h(m)\) for \(1 \leq m < n\). Then

\[
(f \circ h)(n) = \sum_{n=r2st \atop (r,s,t)=1} f(s)h(t) = f(1)g(n) + \sum_{n=r2st \atop (r,s,t)=1} f(s)g(t)
\]

\[
= f(1)g(n) + \sum_{n=r2st \atop (r,s,t)=1} f(s)h(t) \quad \text{(by induction hypothesis)}
\]

\[
= f(1)g(n) + (f \circ h)(n) - f(1)h(n)
\]

\[
\Rightarrow g(n) = h(n) \text{ since } f(1) = 1.
\]

Hence \((A, f, g)\) is a commutative semi group with cancellation law.

2.4.6 Theorem: Let \(\alpha\) be a generalized additive function. Then

\(f \in A \Rightarrow \exists\) unique \(g \in A\) such that \(g \circ f = \alpha f\)

Proof: We define \(g\) inductively.

Define \(g(1) = 0\) and assume that \(g\) is defined for \(s < n\).

Define \(g(n) = \alpha(n)f(n) - \sum_{n=r2st \atop scn,(r,s,t)=1} g(s)f(t)\)

(44)
Then,
\[ \alpha(n)f(n) = g(n) + \sum_{\substack{(r, t) \\
(r, t) \in F}} g(s)f(t) \]
\[ = \sum_{\substack{(r, t) \\
(r, t) \in F}} g(s)f(t) = (g \circ F)(n) \]

So that \((\alpha f)(n) = (g \circ F)(n)\)

Consequently \(g \circ F = \alpha f\)

Now we show that \(g\) is unique.

\(h \in A_0\) is such that \(h \circ F = \alpha f\)

Then, \(g \circ F = h \circ F\)

\[ \Rightarrow g = h \quad (\text{by cancellation law 2.4.5}) \]

\[ \Rightarrow g \text{ is unique.} \]

2.4.7 Definition: Suppose \(\alpha\) is a generalized additive function associated with generalized basic sequence \(F\). Define

(2.4.7.1) \(L_\alpha : (A_1, \circ_F) \to (A_0, +)\) by \(L_\alpha f = g\) where \(g \circ F = \alpha f\)

(which is determined uniquely in view of cancellation law)

and define \(\lambda_\alpha : A_1 \to A_0\) by

(2.4.7.2) \((\lambda_\alpha f)(n) = \alpha(n)f(n)\) for every positive integer \(n\).

2.4.8 Observation: We observe that \(L_\alpha f \circ F = \lambda_\alpha f\)

In the rest of the chapter \(\alpha\) stands for the generalized additive function associated with \(F\).

(45)
2.4.9 Theorem: For $f, g \in A_1$, we have
\[ \lambda_\alpha (f \circ_F g) = \lambda_\alpha f \circ_F g + f \circ_F \lambda_\alpha g. \]

Proof: For any positive integer $n$, we have
\[ (\lambda_\alpha (f \circ_F g))(n) = \alpha(n)(f \circ_F g)(n) \quad \text{(by 2.4.7.2)} \]
\[ = \alpha(n) \sum_{\substack{r, s, t \in \mathbb{N}^3 \\ (r, s, t) = 1}} f(s)g(t) \]
\[ = \sum_{\substack{r, s, t \in \mathbb{N}^3 \\ (r, s, t) = 1}} \alpha(r^2 s^2) \alpha(s^2 t^2) \alpha(t^2) \phi(r^2 s^2) \phi(s^2 t^2) \phi(t^2) \]
\[ = \sum_{\substack{r, s, t \in \mathbb{N}^3 \\ (r, s, t) = 1}} \alpha(r^2 s^2) \phi(r^2 s^2) + \sum_{\substack{r, s, t \in \mathbb{N}^3 \\ (r, s, t) = 1}} \alpha(s^2 t^2) \phi(s^2 t^2) + \sum_{\substack{r, s, t \in \mathbb{N}^3 \\ (r, s, t) = 1}} \alpha(t^2) \phi(t^2) \]
\[ = (\lambda_\alpha f \circ_F g)(n) + (f \circ_F \lambda_\alpha g)(n) \]

2.4.10 Theorem: For $f, g \in A_1$, we have
\[ L_\alpha (f \circ_F g) = L_\alpha f + \lambda_\alpha g \]

Proof: $L_\alpha (f \circ_F g) \circ_F (f \circ_F g) = \lambda_\alpha (f \circ_F g) \quad \text{(by 2.4.8)}$
\[ = \lambda_\alpha f \circ_F g + f \circ_F \lambda_\alpha g \quad \text{(by theorem 2.4.9)} \]
\[ = (L_\alpha f \circ_F g) \circ_F (f \circ_F g) + f \circ_F (L_\alpha g \circ_F g) \quad \text{(by 2.4.8)} \]
\[ = (L_\alpha f) \circ_F (f \circ_F g) + (L_\alpha g) \circ_F (f \circ_F g) \]
\[ = (46) \]
2.4.11 Note: The above theorem shows that \( L_a : (A_1, \circ_F) \rightarrow (A_0, +) \) is always a semi group homomorphism.

In the following example we show that \( L_a \) is never on-to.

2.4.12 Example: Define \( g(n) = \begin{cases} 0 & \text{if } n = 1 \\ 1 & \text{otherwise} \end{cases} \)

Suppose \( L_a \) is onto. Then \( \exists f \in A_1 \) such that \( L_a f = g \)

\[ \Rightarrow g \circ_F f = L_a f \circ_F f = \alpha f \]

(2.4.12.1) \[ (\alpha f)(4) = \alpha(4)f(4) = 0 \quad (\text{since } \alpha(4) = 0) \]

(2.4.12.2) \[ (g \circ_F f)(4) = \sum_{(r,t) = 1} g(s)f(i) \]

\[ = g(1)f(1) + g(4)f(1) + g(1)f(4) \]

\[ = g(4) \quad (\text{since } g(1) = 0 \text{ and } f(1) = 1) \]

From (2.4.12.1) and (2.4.12.2) we get \( g(4) = 0 \) a contradiction.

The following theorem characterizes the set of all \( f \in A_1 \) which are mapped into the zero element of \( A_0 \) under \( L_a \).
2.4.13 Theorem: Suppose \( f \in A_1 \). Then \( L_\alpha f = 0 \) if and only if \( \alpha f = 0 \)

Proof: \( L_\alpha f = 0 \Rightarrow 0 = L_\alpha f \circ f = \alpha f \Rightarrow \alpha f = 0 \)

Conversely, suppose \( \alpha f = 0 \). Clearly \( (L_\alpha f)(1) = 0 \)

Suppose \( (L_\alpha f)(m) = 0 \) for all \( m < n \). Then

\[
0 = (\alpha f)(n) = (L_\alpha f \circ f)(n) = \sum_{(r,s,t)=1}^{n \in \mathbb{N}} (L_\alpha f)(s)f(t)
\]

\[
= (L_\alpha f)(n)f(1) + \sum_{(r,s,t)=1}^{n \in \mathbb{N}} (L_\alpha f)(s)f(t) = (L_\alpha f)(n)
\]

(by induction hypothesis and since \( f(1) = 1 \))

Thus \( L_\alpha f = 0 \)

The following example shows that \( L_\alpha \) is never one-one.

2.4.14 Example: Let \( f(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} \)

and \( g(n) = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise} \end{cases} \)

Then, for any generalized additive function \( \alpha \), by theorem 2.4.13,

\( (L_\alpha f)(n) = 0 \) since \( (\alpha f)(n) = 0 \) for all \( n \geq 1 \)

and \( (L_\alpha g)(n) = 0 \).

Since \( (\alpha g)(n) = \alpha(n)g(n) = \begin{cases} 0 & \text{if } n \text{ is a square} \text{ by } 232 \\ 0 & \text{if } n \text{ is not a square} \text{ by definition of } g \end{cases} \)

= 0 always

Thus \( L_\alpha f = L_\alpha g \) but \( f \neq g \)

(48)
In view of examples 2.4.12 and 2.4.14, $L_a$ is never onto and never one-one. Hence it is natural to investigate for the range of $L_a$, for a given generalized additive function $\alpha$. This is carried in chapter - III.