INEQUALITIES INVOLVING
THE ARITHMETICAL FUNCTIONS $\sigma_k, \sigma_k^*, \varphi_k$ AND $\varphi_k^*$

In this Chapter, we obtain inequalities involving $\sigma_k, \sigma_k^*, \varphi_k$ and $\varphi_k^*$. Two Examples are also given to give a good insight into the inequalities.

The contents of this chapter are essentially contained in SASTRY, SUKLA and PANDA [42]

§1. INTRODUCTION

1.1 A divisor $d$ of $n$ is said to be a unitary divisor of $n$ (COHEN [13]) if \( \left( \frac{d}{n} \right) = 1 \).

1.1.1 Let $n = p_1^{\alpha_1}p_2^{\alpha_2}......p_r^{\alpha_r}$ be the canonical representation of $n > 1$, $p_1, p_2, ......, p_r$ being distinct primes.

$\tau(n)$ and $\sigma(n)$ denote respectively the number and the sum of the divisor of $n$.

$\tau(1) = \sigma(1) = 1$ and it is well known (APOSTOL [1]) that both the functions are multiplicative. Further if $n$ is in the canonical representation (1.1.1), then

1.1.2 $\tau(n) = \prod_{i=1}^{r}(1+\alpha_i)$ and

1.1.3 $\sigma(n) = \prod_{i=1}^{r}\left\{\frac{p_i^{\alpha_i+1}-1}{p_i-1}\right\}$

1.1.4 Euler's totient function $\varphi(n)$ denotes the number of positive integers less than or equal to $n$ and which are relatively prime to $n$. 

(18)
\( \phi(1) = 1 \) and \( \phi(n) \) is multiplicative (APOSTOL [1]). Further if \( n \) is in the canonical representation (1.1.1), then

\[ \phi(n) = n \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i} \right) \]

1.1.6 The Mobius function \( \mu \) is defined as follows:

\[
\mu(1) = 1, \quad \mu(n) = (-1)^r \text{ if } n = p_1 p_2 \ldots p_r \\
(\text{primes being distinct}) \text{ and } \mu(n) = 0 \text{ if } n > 1 \text{ has a square factors.}
\]

1.1.7 The function \( \omega(n) \) (HARDY and WRIGHT [20]) is defined as follows:

\[
\omega(1) = 0 \text{ and if } n > 1, \quad \omega(n) = r, \text{ where } r \text{ is the number of distinct prime divisors of } n.
\]

1.1.8 The arithmetical function \( \sigma_k(n) \) (DICKSON [16], COHEN [10]) is defined as the number of positive integers \( \leq n^k \) and relatively \( k \)-prime (0.0.5) to \( n^k \).

1.1.9 \( \sigma_k(n) \) denotes the sum of the \( k \)th powers of divisors of \( n \). That is

\[
\sigma_k(n) = \sum_{d \mid n} d^k.
\]

It is well known (COHEN [11,12], DICKSON [16], SANDOR [41], SIVARAMA KRISHNAN [47]) that \( \sigma_k(n), \varphi_k(n) \) are multiplicative functions and hence if \( n \) is in the canonical form (1.1.1) we have,

\[
1.1.10 \quad \sigma_k(n) = \prod_{i=1}^{r} \left( \frac{p_i^{\omega(a_i+1)} - 1}{p_i^k - 1} \right) \text{ and}
\]

\[
1.1.11 \quad \varphi_k(n) = n^k \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i^k} \right)
\]

For \( k = 1 \), we obtain the arithmetical functions \( \sigma(n) \) and \( \varphi(n) \).

(19)
We also have the formula (COHEN [11])

1.1.12 \( \varphi_k(n) = \sum_{d|n} d^k \mu \left( \frac{n}{d} \right) \).

1.1.13 \( \varphi_k^*(n) \) is the number of k-tuples \((a_1, \ldots, a_k)\) such that \(1 \leq a_i \leq n\) with the property that \(\{a_1, a_2, \ldots, a_k, n\}^* = 1\), where \((m,n)^* (0.0.3)\) denotes the greatest divisor of \(m\) which is a unitary divisor of \(n\).

1.1.14 \( \sigma^*_k(n) \) is the sum of the kth powers of unitary divisors of \(n\).

1.1.15 \( \mu^*(n) \), the unitary analogue of the mobius \(\mu\) function is defined as \(\mu^*(n) = (-1)^{\omega(n)}\), where \(\omega(n)\) is defined in 1.1.7.

We have observed that \(\mu^*(n)\) is never zero, while \(\mu(n)\) is zero when \(n\) is a square.

We can write (SANDOR [39])

1.1.16 \( \varphi_k^*(n) = n^k \sum_{d|n} \frac{\mu^*(d)}{d^k} \) and

1.1.17 \( \sigma^*_k(n) = \sum_{d \parallel n} d^k \)

If \(n\) is in the canonical form (1.1.1), then we have (COHEN [13,14], SANDOR [39]).

1.1.18 \( \sigma^*_k(n) = \prod_{i=1}^{r} (p_i^{k \alpha_i} + 1) \) and

1.1.19 \( \varphi^*_k(n) = \prod_{i=1}^{r} (p_i^{k \alpha_i} - 1) \)

Consequently follows that \(\sigma^*_k\) and \(\varphi^*_k\) are multiplicative.

(20)
§2. INEQUALITIES INVOLVING $\sigma_k^*$ AND $\varphi_k^*$

In this section we obtain inequalities involving $\sigma_k^*$ and $\varphi_k^*$.

MAIN RESULTS: SANDOR [40] proved that

1.2.1 $m^t \sigma_k(n) \leq \sigma_k(mn) \leq \sigma_k(m) \sigma_k(n)$

and

1.2.2 $\varphi_k(m) \varphi_k(n) \leq \varphi_k(mn) \leq m^t \varphi_k(n)$.

where $m,n = 1,2,3,\ldots$

Now we obtain inequalities involving $\sigma_k^*$ and $\varphi_k^*$, analogous to (1.2.1) and (1.2.2)

1.2.3 Lemma: For any two positive integers $m$ and $n$,

1.2.4 $m^{t-1} \sigma_k^*(n) \leq \sigma_k^*(mn) \leq \sigma_k^*(m) \sigma_k^*(n)$

and

1.2.5 $\varphi_k^*(m) \varphi_k^*(n) \leq \varphi_k^*(mn) \leq m^{ks} \varphi_k^*(n)$

Proof: Since $\sigma_k^*$ and $\varphi_k^*$ are multiplicative, it is sufficient to prove the inequalities (1.2.4) and (1.2.5) when $m = p^\alpha$, $n = p^\beta$ where $p$ is a prime and $\alpha$ and $\beta$ are non-ve integers.

Since $p^{a_1(k-1)} \sigma_k^*(p^\beta) = p^{a_1(k-1)} (p^k + 1)$

(21)
Thus

\[ p^{(k-1)} \sigma_k^*(p^\alpha) \leq \sigma_k^*(p^{\alpha + \beta}). \]

and

\[ \sigma_k^*(p^{\alpha + \beta}) = p^{k(\alpha + \beta)} + 1 \]

\[ \leq (p^k + 1)(p^k + 1) = \sigma_k^*(p^\alpha)\sigma_k^*(p^\beta) \]

Thus

\( (1.2.4.2) \sigma_k^*(p^{\alpha + \beta}) \leq \sigma_k^*(p^\alpha)\sigma_k^*(p^\beta). \)

Hence (1.2.4) follows from (1.2.4.1) and (1.2.4.2).

Similarly,

\[ \varphi_k^*(p^\alpha)\varphi_k^*(p^\beta) = (p^\alpha - 1)(p^\beta - 1) \leq p^{k(\alpha + \beta)} - 1 = \varphi_k^*(p^{\alpha + \beta}) \]

Thus

\( (1.2.5.1) \varphi_k^*(p^\alpha)\varphi_k^*(p^\beta) \leq \varphi_k^*(p^{\alpha + \beta}) \)

For \( k \geq 1 \) and \( p \) prime we have \( k + 1 \leq p^k \). So that

\begin{align*}
1 + p^\alpha + \ldots + p^{ka} &\leq (k + 1)p^ka \leq p^k p^ka \leq p^k p^{ka} = p^{k(k+1)}a \\
\Rightarrow \frac{p^{(k+1)a} - 1}{p^a - 1} &\leq p^{k(\alpha + \beta)} \\
\Rightarrow (p^{(k+1)a} - 1) &\leq p^{k(\alpha + \beta)}(p^a - 1) = p^{(k+1)a + kb} - p^{k(\alpha + \beta)} \\
\Rightarrow p^{k(\alpha + \beta)} - 1 &\leq p^{(k+1)a}(p^kb - 1) \\
\end{align*}

\( (1.2.5.2) \varphi_k^*(p^{\alpha + \beta}) \leq (p^\alpha)^{k+1}\varphi_k^*(p^\beta) \)

and hence (1.2.5) follows from (1.2.5.1) and (1.2.5.2).
1.2.6 OBSERVATION: We observe that $n^k \leq \sigma_k^*(n)$ and $n^k \geq \varphi_k^*(n)$ and $\sigma_k^*(n) > n^{k-1}$ and $\varphi_k^*(n) < n^{k+1}$

SANDOR [40] proved that

1.2.7 $\left( \sigma_k(n) \right)^{\varphi_k(n)} < n^{k+1}$ for $n \geq 2$

and if $p^k \geq 5$ for every prime divisor $p$ of $n$

1.2.8 $\left( \varphi_k(n) \right)^{\sigma_k(n)} > n^{k-1}$

We prove similar inequalities involving $\sigma_k^*$ and $\varphi_k^*$ in the following theorem.

1.2.9 Theorem: For $n \geq 2$,

1.2.10 $\left( \sigma_k^*(n) \right)^{\varphi_k(n)} < n^{k+1}$

and if $p^k \geq 5$ for every prime divisor $p$ of $n$.

1.2.11 $\left( \varphi_k^*(n) \right)^{\sigma_k(n)} > n^{k-1}$

Proof: Observing that (SANDOR[40])

(1.2.10.1) $(a + 1)^{a-1} < a^a$ if $a \geq 2$

We shall prove (1.2.10) for $n = p^\alpha$, where $p$ is a prime and $\alpha$ is +ve integer.

$$\sigma_k^*(p^\alpha)^{\varphi_k(p^\alpha)} = \left[ p^{\alpha k} + 1 \right]^{-1} < p^{\alpha k \varphi_k^{p\alpha}}$$

(by 1.2.10.1)

Thus for $n = p^\alpha$ (1.2.10) is true. Now, assume the truth of (1.2.10) for all $n \in \mathbb{N}$. If $N$ is a prime power, (1.2.10) holds otherwise, there exists positive integers $a$ and $b$ each greater than 1, with $(a,b)=1$ and $N=ab$. Then

$$\sigma_k^*(ab)^{\varphi_k(ab)} = \left( \sigma_k^*(a) \sigma_k^*(b) \right)^{\varphi_k(a)\varphi_k(b)}$$

(since $\sigma_k^*$ and $\varphi_k^*$ are multiplicative)

(23)
Therefore, since 1 < a and b < N,
\[ a^{ka^k} \cdot b^{kb^k} \cdot \sigma_i(a)^{\alpha} \cdot \sigma_i(b)^{\alpha} = \left( a^k b^k \right)^{\alpha} = \left( N^k \right)^{\alpha} = N^{k\alpha} \quad \text{(by 1.2.6)} \]

Thus (1.2.10) is true for all \( n \geq 2 \).

Observing that (SANDOR[40])

(1.2.11.1) \( (a - 1)^{a+1} > a^a \quad \text{for} \quad a \geq 5 \)

We shall prove (1.2.11) for \( n = p^a \) where \( p \) is a prime and \( \alpha \) is +ve integer and \( p^k \geq 5 \).

\[ \phi_k^*(p^a) \cdot p^{\alpha} = \left[ p^{\alpha - 1} \right]^{a^{\alpha + 1}} > p^{akp^k} \quad \text{(by 1.2.11.1)} \]

Thus for \( n = p^a \) (1.2.11) is true. Now, assume the truth of (1.2.11) for all \( n < N \). If \( N \) is a prime power, (1.2.11) holds for \( N \) otherwise, there exists positive integers \( a \) and \( b \) each greater than 1, with \( (a, b) = 1 \) and \( N = ab \). Then

\[ \phi_k^*(ab) \cdot \sigma_i(ab) = \left( \phi_k^*(a) \cdot \phi_k^*(b) \right) \cdot \sigma_i(a)^{\alpha} \cdot \sigma_i(b)^{\alpha} \quad \text{(since} \quad \phi_k^* \quad \text{and} \quad \sigma_i^* \quad \text{are multiplicative)} \]

\[ = \left( \phi_k^*(a) \right)^{\alpha} \cdot \left( \phi_k^*(b) \right)^{\alpha} \cdot \sigma_i(a)^{\alpha} \cdot \sigma_i(b)^{\alpha} \]

\[ > a^{ka^k} \cdot b^{kb^k} \cdot \sigma_i(a) \quad \text{(since} \quad 1 < a \quad \text{and} \quad b < N \) \]

\[ > a^{ka^k} \cdot b^{kb^k} = \left( a^k b^k \right)^{\alpha} = \left( N^k \right)^{\alpha} = N^{k\alpha} \quad \text{(by 1.2.6)} \]

Thus (1.2.11) is true for \( p^k \geq 5 \)

Now we establish the following interesting inequality.
1.2.12 Theorem: If \( m > 1 \) and \( n \geq 1 \), then

\[
1 \leq \frac{\sigma_2^*(mn)}{m^{k-1} \sigma_k^*(n)} \leq \begin{cases} 
    m(\omega(m) - \omega((m,n))) & \text{if } \omega(m) - \omega((m,n)) \geq 2 \\
    2m & \text{if } \omega(m) - \omega((m,n)) = 1 \\
    m & \text{if } \omega(m) = \omega((m,n)) \n\end{cases}
\]

Proof: The first inequality is a consequence of (1.2.4).

To prove second part of the inequality we use the following lemma.

(1.2.12.1) Lemma: If \( a_1, a_2, \ldots, a_s \) are positive integers such that \( 2 \leq a_1 < a_2 < \ldots < a_s \), \( s \geq 2 \), then \((1+a_1)(1+a_2)\ldots(1+a_s) \leq s a_1 a_2 \ldots a_s\)

We shall prove this inequality by method of induction.

If \( 2 \leq a_1 < a_2 \), then

\[
(1+a_1)(1+a_2) = 1+a_1+a_1 a_2
\]

\[
\leq a_2+a_2+a_1 a_2
\]

\[
= 2a_2+a_1 a_2
\]

\[
\leq a_1 a_2 ( \text{since } 2 \leq a_1 )
\]

\[
= 2a_1 a_2
\]

Assume the truth for \( s \geq 2 \). Then

\[
(1+a_1)\ldots(1+a_s)(1+a_{s+1}) \leq s a_1 a_2 \ldots a_s (1+a_{s+1}) \quad \text{(by induction hypothesis)}
\]

\[
= s a_1 a_2 \ldots a_s + s a_1 a_2 \ldots a_s (\text{Since } s \leq a_{s+1})
\]

\[
= (s+1) a_1 a_2 \ldots a_{s+1}
\]

(25)
Now we prove the second part of the inequality (1.2.12)

Suppose $m, n > 1$ and

$$m = \prod_{i=1}^{r} p_{i}^{a_{i}} \prod_{j=1}^{s} q_{j}^{b_{j}}$$
$$n = \prod_{i=1}^{r} p_{i}^{c_{i}} \prod_{j=1}^{s} t_{j}^{d_{j}}$$

where $p_{i}, q_{j}, t_{j}$ are all distinct primes and $s \geq 2$.

We observe that $s = \omega(m) - \omega(m, n)$

Using the multiplicative property of $\sigma_{s}$, we get

$$\frac{\sigma_{s}(mn)}{\sigma_{s}(n)} = \frac{\prod_{i=1}^{r} \left( p_{i}^{\frac{a_{i} + d_{i}}{s}} + 1 \right) \prod_{j=1}^{s} \left( q_{j}^{b_{j}} + 1 \right)}{\prod_{i=1}^{r} \left( p_{i}^{c_{i}d_{i}} + 1 \right)}$$

$$= \prod_{i=1}^{r} \left( \frac{p_{i}^{k_{i}}}{p_{i}^{c_{i}d_{i}} + 1} \right) \prod_{j=1}^{s} \left( q_{j}^{b_{j}} + 1 \right)$$

$$< \prod_{i=1}^{r} \left( p_{i}^{k_{i}} \prod_{j=1}^{s} \left( q_{j}^{b_{j}} + 1 \right) \right) \quad \text{(since} \quad \frac{xy+1}{y+1} < x \quad \text{if} \quad x>1 \quad \text{and} \quad y>0)$$

$$< \prod_{i=1}^{r} p_{i}^{k_{i}} \prod_{j=1}^{s} q_{j}^{b_{j}}$$

from (1.2.12.1) if $s \geq 2$

$$< \left\{ \begin{array}{ll} 2m^{k} & \text{if } s = 1 \\ m^{k} & \text{if } s = 0 \end{array} \right.$$
1.2.13 Theorem: If $\omega(m) = \omega(n) = \omega((m,n))$, then

1.2.14 $\left( \sigma_k^*(mn) \right)^2 \leq \sigma_k^*(m^2)\sigma_k^*(n^2)$

and if $(m,n) = 1$, then

1.2.15 $\left( \sigma_k^*(mn) \right)^2 \geq \sigma_k^*(m^2)\sigma_k^*(n^2)$

Proof: Let $\omega(m) = \omega(n) = \omega((m,n))$.

Then the canonical representation of $m$ and $n$ are

$m = \prod_{i=1}^{r} p_i^{{\alpha}_i}$ and $n = \prod_{i=1}^{r} q_i^{{\beta}_i}$

where each $\alpha_i$ and $\beta_i \geq 1$

Now, $\left( \sigma_k^*(mn) \right)^2 = \left( \sigma_k^* \left( \prod_{i=1}^{r} p_i^{(a_i + \beta_i)} \right) \right)^2$

$= \left( \prod_{i=1}^{r} \left( p_i^{a_i + \beta_i} + 1 \right) \right)^2$ (using the multiplicative property of $\sigma_k^*$)

$\leq \prod_{i=1}^{r} \left( p_i^{2a_i} + 1 \right) \prod_{i=1}^{r} \left( q_i^{2\beta_i} + 1 \right)$

$= \sigma_k^*(m^2) \sigma_k^*(n^2)$ (since $(ab+1)^2 \leq (a^2+1)(b^2+1)$ if $a,b > 0$)

Let $(m,n) = 1$, then the canonical representation of $m$ and $n$ are

$m = \prod_{i=1}^{r} p_i^{{\alpha}_i}$, $n = \prod_{i=1}^{r} q_i^{{\beta}_i}$

where $p_i$ and $q_i$ are two different primes. Now,

$\left( \sigma_k^*(mn) \right)^2 = \left( \sigma_k^* \left( \prod_{i=1}^{r} p_i^{{\alpha}_i} \prod_{i=1}^{r} q_i^{{\beta}_i} \right) \right)^2$

(27)
\[\sigma_k^*(\prod_{i=1}^{r} p_i^{k_i}) \sigma_k^*(\prod_{i=1}^{r} q_i^{k_i})^2 \quad \text{(by using the multiplicative property of } \sigma_k^* \text{ )}\]

\[= \prod_{i=1}^{r} (p_i^{2k_i} + 1)^2 \prod_{i=1}^{r} (q_i^{2k_i} + 1)^2\]

\[\geq \prod_{i=1}^{r} (p_i^{2k_i} + 1)^2 \prod_{i=1}^{r} (q_i^{2k_i} + 1)^2 \quad \text{(since } (a+1)^2 \geq (a^2+1) \text{ if } a>0 )\]

\[= \sigma_k^*(m^2) \sigma_k^*(n^2)\]

Hence the inequality (1.2.15).

\[\text{§3. INEQUALITIES INVOLVING } \sigma_k, \sigma_k^*, \varphi_k, \varphi_k^*, \text{ AND } \varphi_k^*.\]

In section §2 of this Chapter-I, we obtained two inequalities (1.2.10) and (1.2.11) involving \(\sigma_k^*\) and \(\varphi_k^*\). Now, it is natural to ask whether the two inequalities (1.2.10) and (1.2.11) hold good if one of the stars is removed (if both are removed, they reduce to (1.2.7) and (1.2.8)).

In the following theorem, we give an answer to this:

1.3.1 Theorem: For \(n \geq 2\), we have

\[1.3.2 \sigma_k^*(n)^{\varphi_k^*(n)} < n^{b_k^*}\]

\[1.3.3 \sigma_k(n)^{\varphi_k(n)} < n^{b_k^{*+1}}\]

and if \(p^k \geq 5\) for every prime divisor \(p\) of \(n\),

\[1.3.4 \varphi_k^*(n)^{\sigma_k^*(n)} > n^{b_k^*}\]

\[1.3.5 \varphi_k^*(n)^{\sigma_k(n)} > n^{b_k^{*+1}}\]

(28)
To prove above theorem, we use the following lemma:

1.3.6 Lemma:

\[ p^{(a-1)(k-1)} \sigma_k^* (p) \leq \sigma_k^* (p^a) \leq (\sigma_k^* (p))^a \]

\[ \left( \varphi_k^* (p) \right)^a \leq \varphi_k^* (p^a) \leq p^{(a-1)(k+1)} \varphi_k^* (p) \]

Proof: Putting \( m = p^{a-1} \) and \( n = p \) in the inequalities (1.2.4) and (1.2.5) and by inductive argument (1.3.6.1) and (1.3.6.2) follows

Proof of (1.3.2)

Observing SANDER [40] (1.2.10.1), we first prove (1.3.2) for \( n = p^a \) where \( p \) is a prime and \( a \) is a +ve integer.

\[ \sigma_k^* (p^a) \varphi_k^* (p^a) \leq \left[ \sigma_k^* (p) \right]^{\sigma_k^* (p^a) \varphi_k^* (p)} \] (by 1.3.6.1 and 1.2.2)

\[ \leq [p^k + 1]^{\sigma_k^* (p^a) \varphi_k^* (p)} \] (by 1.1.11)

\[ = [p^k + 1]^{\sigma_k^* (p^a)} \]

\[ < p^{kp^{a+1}} \] (by 1.2.10.1)

\[ = p^{a \varphi_k^*} \]

Thus (1.3.2) is true when \( n \) is a prime power. The rest of the proof of (1.3.2) can be completed (analogous to that of (1.2.10)) using (1.2.10.1).

(29)
Proof of (1.3.3)

Suppose \( n \) is a prime power say \( p^\alpha \), where \( \alpha \) is a positive integer. Then

\[
\sigma_k(p^\alpha)^{\varphi_k(p^\alpha)} \leq \sigma_k(p^\alpha)^{p^{\alpha-1}(k+1)\varphi_k(p^\alpha)} \quad \text{(by 1.3.6.2)}
\]

\[
\leq [\sigma_k(p)]^{p^{\alpha-1}(k+1)(p^\alpha-1)} \quad \text{(by 1.3.6.1)}
\]

\[
= [p^k + 1]^{p^{\alpha-1}(k+1)(p^\alpha-1)}
\]

\[
= [p^k + 1]^{p^{\alpha-1}(k+1)} \quad \text{(by 1.2.10.1)}
\]

\[
< p^{\alpha k^{\alpha k^{\alpha k^{\alpha k^{\alpha k^{\alpha k^{\alpha k}}}}}}}
\]

Thus (1.3.3) is true for \( n \) is a prime power. The rest of the proof of (1.3.3) can be completed (analogous to that of (1.2.10) using (1.2.10.1) and the obvious inequality \( \varphi_k(n) < n^{k+1} \)

(1.3.4) can be proved (analogous to that of (1.2.11)) by using the fact that \( \varphi_k(n) \geq \varphi_1(n) \) and (1.2.11.1).

Proof of (1.3.5)

Observing that of SANDOR [40] (1.2.11.1), we prove (1.3.5) for \( n = p^\alpha \), where \( p \) is a prime and \( \alpha \) is a +ve integer and \( p^k \geq 5 \).

\[
\varphi_k(p^\alpha)^{\varphi_k(p^\alpha)} \geq [\varphi_k(p)]^{\varphi_k(p^\alpha)^{p^{\alpha-1}(k+1)\varphi_k(p^\alpha)}} \quad \text{(by 1.3.6.1 and 1.3.6.2)}
\]

(30)
\[ p^k - 1 \geq p^{\frac{k-1}{2}}(p^k + 1) \]

\[ = [p^k - 1]^{\frac{\phi(a-k-1)}{2}}(p^k + 1) \]

\[ > p^{k^2} p^{\phi(a-k-1)} (by \ 1.2.11.1) \]

Thus for \( n = p^a \) (1.3.5) is true. Now, assume the truth of (1.3.5) for all \( n < N \). If \( N \) is a prime power, (1.3.5) holds for \( N \), otherwise, there exist positive integers \( a \) and \( b \) each greater than 1, with \( (a, b) = 1 \) and \( N = ab \). Then

\[ \varphi_k(n)\sigma_k(n) = \varphi_k(ab)\sigma_k(ab) \]

\[ = [\varphi_k(a)\varphi_k(b)]\sigma_k(a)\sigma_k(b) \]

\[ = [\varphi_k(a)]^{\sigma_k(a)}\sigma_k(b) = [\varphi_k(a)]\sigma_k(b) \]

\[ > a^{k^{a-1}}b^{k^{b-1}}b^{\frac{k-1}{2}}(a) \]

\[ > a^{k^{a-1}}b^{k^{b-1}}b^{\frac{k-1}{2}}(a) \]

\[ = (a^{k^{a-1}}b^{k^{b-1}})(a) \]

\[ = [ab]^{k^{a-1}}(b) = (N^k)^{k^{b-1}} \]

Thus (1.3.5) is true for \( p^k \geq 5 \) for every prime divisor \( p \) of \( n \).
The following example shows that in (1.3.3) $n^{k+1}$ can not be replaced by $n^k$.

1.3.7 Example: Let $n = 4$ and $k = 3$ then $\sigma_3(4) = 73$ and $\varphi^*_3(4) = 63$.

Further,

$$n^{3\cdot 4^k} > \sigma_3(4)^{\varphi^*_3(4)} = 73^{63} > 4^{3\cdot 4^k} = n^{4^k}$$

The following example shows that in (1.3.5) $n^{k+1}$ can not be replaced by $n^k$.

1.3.8 Example: Take $n = 9$ and $k = 2$. Then, $\sigma_2(9) = 82$ and $\varphi_2(9) = 72$.

$$n^{2\cdot 9^k} < \varphi_2(9)^{\sigma_2(9)} = 72^{82} < 9^{2\cdot 9^k} = n^{9^k}.$$