3.1 Introduction

As in Chapter II, all our matrices and sequences are real. Let $s$ be the set of all real sequences. We write, for $x \in s$,

$$I(x) = \lim_{n \to \infty} x_n,$$

$$Ax = \left\{ A_n(x) \right\}_{n=0}^{\infty},$$

where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k,$$

if the series on the right converges.

In Chapter II, we proved the following theorem:

**Theorem** $I(Ax) \leq w(x), \quad x \in s$

if and only if $A$ is almost positive and strongly regular, where, for $x \in s$

$$w(x) = \inf_{z \in m_0} \lim_{n \to \infty} (x_n + z_n),$$

where

$$m_0 = \left\{ x \in m : \sup_{n} \left| \sum_{i=0}^{n} x_i \right| < \infty \right\}.$$
In the present Chapter, we want to give a generalisation of the above theorem in Theorem 1 by introducing a new sublinear functional $\mathcal{K}$ and in Theorem 2 we give a generalisation of Knopp's Core theorem. In Theorem 3 we establish an inequality involving $\mathcal{K}$ and Banach's functional $w$. Lastly in Theorem 4, we have considered certain sets which arise in connection with $\mathcal{K}$.

3.2 We write

$$m_A = \{x \in s : Ax \in m\}$$

$$m_{A_0} = \{x \in s : Ax \in m_0\}$$

It is evident that $m_A$ is a linear space and $m_{A_0}$ is a subspace. Further if we define, for $x \in m_A$,

$$||x|| = \sup_{n>0} \left| \sum_{k} a_{nk} x_k \right|$$

(1)

then it is a seminorm on $m_A$. It is a norm if $A$ is invertible. It is familiar that

$$A : m \rightarrow s \iff \sum_{k} |a_{nk}| < \infty \quad (\text{for each } n)$$

$$A : m \rightarrow m \iff ||A|| = \sup_{n} \left( \sum_{k} |a_{nk}| \right) < \infty .$$

Let $\sigma_A$ be the summability field of $A$; that is,
\[ c_A = \{ x \in s : L(Ax) = -L(-Ax) \} \]

and we write, for \( x \in c_A \)

\[ \lim_{A_n} x = \lim_{n} A_n(x) . \]

It is evident that

\[ c_A \subseteq m_A , \quad (2) \]

It is also easily seen that

\[ m \cap m_A = m \iff \|A\| < \infty . \quad (3) \]

It is in order to quote the following theorem:

**Theorem**: Mazur-Orlicz (20). Let \( A \) be a regular matrix. Then

\[ c_A \cap c' \neq \emptyset \Rightarrow c_A \cap m' \neq \emptyset ; \]

that is,

\[ c_A \cap m' = \emptyset \Rightarrow c_A \cap c' = \emptyset . \]

In other words, if a regular matrix evaluates some divergent sequence, then it must evaluate an unbounded sequence; that is, if a regular matrix evaluates no unbounded sequence, then evaluates only convergent sequence.

From (3) we have

\[ \|A\| < \infty \Rightarrow m \subseteq m_A \]

and there are important cases when \( m \) is a proper subset
of $m_A$. For example, if $A$ is regular such that $A$ evaluates some divergent sequence (in fact, these cases are only important), then the above theorem gives that $c_A \cap m' \neq \emptyset$ and therefore from (2) we have $m_A \cap m' \neq \emptyset$.

3.3 Now suppose that $\|B\| < \infty$ and we write, for any real matrix $B$, and for $x \in m_B$

$$\sum_{B}(x) = \inf_{z \in m_{B_0}} I(B(x+z)) .$$

(4)

The functional $\sum_B : m_B \rightarrow \mathbb{R}$ is well-defined if we suppose that

$$\lim_{z \in m_{B_0}} Bz = 0 .$$

(5)

(see the observation in Chapter II, regarding the functional $q_Y$ before the statement of Theorem 1)

In the case

$$b_{nk} \rightarrow 0 \quad (n \rightarrow \infty, k \text{ fixed})$$

(6)

then, by Abel's transformation,

$$Bz = \sum_{k=0}^{\infty} \left( b_{nk} - b_{n,k+1} \right) y_k$$

where

$$y = \left\{ \sum_{k=0}^{\infty} z_k \right\} .$$

Further if
then, since, for $y \in m$ (that is, $z \in m_j$)

$$|Ez| \leq \|y\| \sum_k b_{nk} - b_{n,k+1}$$

it follows that, in the case (6) and (7) hold,

$$\lim_{n} Ez = 0 \quad (z \in m_j).$$

Hence in the case $m_j \subset m_o$ the requirement (5) is fulfilled.

We are now in a position to state our first theorem:

**Theorem 1.** Let $B$ be a normal matrix such that condition (5) holds. Then

$$L(Ax) \leq \sum B(x) \quad (x \in m_B) \quad (8)$$

if and only if $C = AB^{-1}$ is almost positive and strongly regular.

**Remark.** By taking $B = I$ (identity matrix) we obtain Theorem 3 of Chapter II.

For the proof of Theorem 1, we need to prove the following theorem which generalises the Knopp's Core theorem in the case $B = I$.

**Theorem 2.** Let $B$ be normal. Then

$$L(Ax) \leq L(Bx) \quad (x \in m_B) \quad (9)$$
if and only if $C = AB^{-1}$ is almost positive and regular.

For the proof of Theorem 2, we require the following lemmas:

**Lemma 1.** (Knopp's Core Theorem)

$$L(Ax) \leq L(x) \ (x \in m)$$

if and only if $A$ is almost positive and regular.

**Lemma 2.** (Simons (34) Corollary 12, Theorem 11).

If

1. $\sum_{k} |a_{nk}| < \infty \ (\text{for each } n)$,
2. $a_{nk} \to 0 \ (n \to \infty) \ \text{for fixed } k$, 

then there exists $y \in m: \|y\| \leq 1$ such that

$$\lim_{n} \sum_{k} a_{nk} y_{k} = \lim_{n} \sum_{k} |a_{nk}| .$$

**Proof of Theorem 2.** (Sufficiency).

Inequality (9) can be rewritten as follows:

$$L(CBx) \leq L(Bx) \ (Bx \in m) \quad (10)$$

Now the sufficiency follows from the sufficient part of Lemma 1.

**Necessity.** When (9) holds, we have

$$-L(-Bx) \leq -L(-Ax) \leq L(Ax) \leq L(Bx), \ (x \in m_B).$$
Hence it follows that
\[ L(Bx) = -L(-Bx) \Rightarrow L(Ax) = -L(-Ax), \]
that is,
\[ c_B \subseteq c_A, \]
and
\[ \lim_{A} x = \lim_{B} x, \]
and this is equivalent to the fact that the matrix \( C \) is regular.

Now since \( C \) is regular, the requirement of Lemma 2 is satisfied for the matrix \( C \). Hence there exists \( y \in m: \|y\| \leq 1 \) and
\[ L(Cy) = \lim_{n \to \infty} \sum_{k} |c_{nk}|. \]  
(11)

Now given \( y \) as above, define \( x \) by
\[ x = y^{-1}y \]  
(12)
so that \( \|Bx\| \leq 1. \)

Since
\[ L(Bx) \leq 1; \]
and hence from (9)
\[ L(Ax) = L(Cy) \leq 1. \]  
(13)

Now it follows from (11) and (13) that
\[ \lim_{n \to \infty} \sum_{k} |a_{nk}| \leq 1 \tag{14} \]

But since
\[ \lim_{n \to \infty} \sum_{k} |c_{nk}| \geq \lim_{n \to \infty} \sum_{k} c_{nk} = 1, \]

it follows from (14) that
\[ \lim_{n \to \infty} \sum_{k} |c_{nk}| = 1. \]

Hence \( C \) is almost positive. (Note that a regular matrix \( A \) is almost positive if and only if \( \lim_{n \to \infty} \sum_{k} |a_{nk}| = 1 \).)

This completes the proof of Theorem 2.

**Proof of Theorem 1.** (sufficiency). Suppose that \( C \) is almost positive and regular. Then by Theorem 2,

\[ L(A(x+z)) \leq L(B(x+z)) \]

for \( x \in m_B, z \in m_{B_o} \). Now taking the infimum with respect to \( z \in m_{B_o} \) in the above inequality, we have

\[ \bigcap_{A} (x) \leq \bigcap_{B} (x). \tag{15} \]

But, as \( L(Ax) \) is sublinear

\[ \bigcap_{A} (x) \geq \inf_{z \in m_{B_o}} \left\{ L(Ax) - L(-Az) \right\}. \tag{16} \]
But for \( z \in \text{m}_{B_0} \),

\[ A_0 z = C B_0 z = D y , \]

where

\[
D = \left( d_{nk} \right) = \left( c_{nk} - c_{n,k+1} \right)
\]

(17)

\[
y = \left\{ y_n \right\}_{n=0}^{\infty} = \left\{ \sum_{v=0}^{n} B_v(z) \right\}_{n=0}^{\infty} \subseteq m
\]

Since \( C \) is strongly regular and \( y \in m \), it follows that

\[ L(A_0 z) = L(D y) = 0 . \]

Hence it follows from (16) that

\[ \Omega_A(z) \geq \inf_{z \in m_{B_0}} L(A_0 z) = L(A z) . \quad (18) \]

Now the sufficiency follows from (15) and (18).

**Necessity.** Suppose that (8) holds. Since trivially

\[ \Omega_B(z) \leq L(B x) \quad ( z \in m_{B_0} ) \]

it follows from (8) that

\[ L(A x) \leq L(B x) \quad ( x \in m_{B} ) . \]
Hence by Theorem 2, $AB^{-1} = C$ is almost positive and regular.

Since (see Theorem 1(i), Chapter II)

$$\mathcal{N}_B(x) = 0 \quad (x \in m_B) ,$$

it follows from (3) that

$$L(Ax) \leq o \quad (x \in m_B) ;$$

that is,

$$L(CBx) \leq o \quad (x \in m_B) ;$$

that is,

$$L(Dy) \leq o \quad (y \in m) , \quad (19)$$

where $D$ and $y$ are given by (17).

Now since the matrix $D$ satisfies the conditions of Lemma 2 (as $C$ is regular) there exists $y_0 \in m : \|y_0\| \leq 1$ and

$$L(Dy_0) = \Re \sum_{n,k} d_{nk} \geq 0 . \quad (20)$$

Now define $x_0$ by

$$x_0 = B^{-1} \left( \sigma y_0 - y_0 \right) ,$$

so that

$$\sigma y_0 - y_0 = Bx_0 .$$
Hence

\[ y_0 \in m \iff Bx_0 \in m_0 \iff x_0 \in m_B. \]

now taking \( y \) to be \( y_0 \) in (19) together with relation (20), we have

\[ \lim_{n,k} \sum |d_{nk}| = \lim_{k} \sum |c_{nk} - c_{n,k+1}| = 0. \]

Hence \( C \) is strongly regular.

This completes the proof.

**Corollary 1.** Let the conditions of Theorem 1 hold. Then

\[ L(Ax) \leq \Omega_A(x) \leq \Omega_B(x) \leq L(Bx). \]

**Proof.** First inequality follows from (18), second inequality from (15) and the last one is trivial.

3.4 It is trivial that

\[ \Omega_A(x) \leq L(Ax) \quad (x \in m) \quad (21) \]

but by Theorem 3 (Chapter 2), we have

\[ L(Ax) \leq w(x) \quad (x \in m). \quad (22) \]

if and only if \( A \) is almost positive and strongly regular.

Hence combining (21) and (22) we have

**Theorem 3.** Let \( A \) be almost positive and strongly regular. Then
\[ \sum_A(x) \leq w(x) \quad (x \in m) \quad (23) \]

In other words
\[
\{ m, \sum_A \} \subseteq \beta.
\]

It is clear from the above theorem that
\[ \hat{\sigma} \subseteq S_1. \]

if \( A \) is almost positive and strongly regular, where
\[
\hat{\sigma} = \{ x : w(x) = -w(-x) \}, \]
\[ S_1 = \{ x \in m : \sum_A(x) = -\sum_A(-x) \}. \]

In what follows, we want to examine if the set \( S_1 \) can have a simpler characterisation.

We write
\[
S_0 = \{ x \in m : \sum_k a_{nk}(x_k + z_k) \text{ converges uniformly in } z \in m_0 \}\]
\[
S_2 = \{ x \in m : \sum_k a_{nk}(x_k + z_k) \text{ converges for all } z \in m_0 \}\]

We now prove

**Theorem 4.**

(i) \( S_0 \subseteq S_1 \)

(ii) \( S_1 \subseteq S_2 \)
\[
\sum_{k} |a_{n,k} - a_{n,k+1}| \rightarrow 0 \text{ as } n \rightarrow \infty,
\]

**Proof.** Given \( x \in S_{0} \) and \( \varepsilon > 0 \), there exists a positive integer \( n_{0} = n_{0}(\varepsilon) \):

\[
s - \varepsilon < \sum_{k} a_{n,k}(x_{k} + z_{k}) < s + \varepsilon \quad (24)
\]

for all \( z \in m_{0} \) and for all \( n \geq n_{0} \), where

\[
s = \lim_{n} \sum_{k} a_{n,k}(x_{k} + z_{k}),
\]

and \( s \) is independent of \( z \in m_{0} \). Taking \( \lim \sup \) over \( n \) and then the infimum over \( z \) in (24) we obtain:

\[
s - \varepsilon \leq -\bigcup_{A}(-x) \leq \bigcup_{A}(x) \leq s + \varepsilon. \quad (25)
\]

Since \( \varepsilon \) is arbitrary, it follows from (25) that

\[
\bigcup_{A}(x) = -\bigcup_{A}(-x) = s. \quad (26)
\]

This proves the first inclusion relation.

Next suppose that

\[
x \in S_{1}.
\]

Suppose that

\[
\bigcup_{A}(x) = -\bigcup_{A}(-x) = s,
\]
Hence from $\mathcal{N}_A(x) = s$, we obtain, given $\varepsilon > 0$, there exists $z' \in m_0$ and $n_1 \in \mathbb{N}$:

$$A_n(x+z') = \sum_{k} a_{nk}(x_k + z'_k) < s + \varepsilon, \quad (27)$$

for all $n \geq n_1$.

Now for $z \in m_0$,

$$A_n(x+z) = A_n(x+z') + A_n(z-z'). \quad (28)$$

Since $\sum_k |a_{nk} - a_{nk+1}| \to 0$ as $n \to \infty$, then,

$$A_n(z-z') \to 0 \quad \text{as} \quad n \to \infty,$$

that is, given $\varepsilon > 0$, there exists $n_2 \in \mathbb{N}$:

$$A_n(z-z') < \varepsilon \quad (n \geq n_2). \quad (29)$$

Now from (27), (28), (29) we have

$$A_n(x+z) < s + 2\varepsilon \quad \text{for all} \quad n \geq n_3 = \max(n_1, n_2).$$

Similarly we have

$$A_n(x+z) > s - 2\varepsilon \quad \text{for all} \quad n \geq n_4 \in \mathbb{N}$$

so that we have

$$|A_n(x+z) - s| < 2\varepsilon \quad \text{for all} \quad n \geq n_5 = \max(n_3, n_4).$$

Hence $x \in S_2$.
Remark. The set $S_0$ is usually empty. In fact it is only non-empty if $A$ has only a finite number of non-zero rows. In view of this it is evident that the inclusion (i) Theorem 4 is proper because when $A$ is almost positive and strongly regular then $\hat{c} \subseteq S_1$ (see below Theorem 3). Now the natural question arises as to what sublinear functional $\varphi$ will generate the set $S_0$ in the sense that

$$S_0 = \{x \in M : \varphi(x) = -\varphi(-x)\}.$$ 

Towards this end, we define $\varphi : M \to \mathbb{R}$ by

$$\varphi(x) = \lim_{n \to \infty} \sup_{z \in V} \sum_{k} a_{nk} (x_k + z_k)$$

where $V$ is a subspace of $M$.

Since

$$\varphi(x) \geq \lim_{n \to \infty} \sum_{k} a_{nk} x_k,$$

and if $x \in M$ and $\|A\| < \infty$ then $\varphi$ is bounded from below. $\varphi$ is also bounded from above if $V$ is a bounded subspace. In this case $\varphi$ is well-defined.

Now we have the following

**Theorem 5** Let $V$ be a bounded subspace of $M$ and let $\|A\| < \infty$. Write
\[ \hat{S}_0 = \{ x \in m : \sum_{k} a_{nk}(x_k + z_k) \to s \text{ uniformly in } z \in V \} \]

Then

\[ \hat{S}_0 = \{ x \in m : \varphi^+(x) = - \varphi^-(x) \} \]

**Proof.** Suppose that

\[ \sum_{k} a_{nk}(x_k + z_k) \to s \text{ uniformly in } z \in V. \]

Then \( \varepsilon > 0 \), there exists \( n_0 \):

\[ s - \varepsilon < \sum_{k} a_{nk}(x_k + z_k) < s + \varepsilon \quad \text{for all } z \in V, \; n > n_0. \]

Now taking \( \sup \) with respect to \( z \) and then taking \( \lim \) with respect to \( n \), we have

\[ s - \varepsilon \leq - \varphi^-(x) \leq \varphi^+(x) \leq s + \varepsilon \]

Since \( \varepsilon \) is arbitrary, we obtain

\[ \varphi^+(x) = - \varphi^-(x) = s. \]

Conversely suppose that

\[ \varphi^+(x) = s = - \varphi^-(x) \]

Then we shall have

\[ s - \varepsilon < \sum_{k} a_{nk}(x_k + z_k) < s + \varepsilon \]

for all \( z \in V, \; n \geq n_0 \).
from which follows that

\[ \sum_{k} a_{nk}(x_k + z_k) \to s \text{ uniformly in } z \in V. \]