3.1 INTRODUCTION

In recent years several authors have generalized commuting condition of mapping introduced by Jungck. Sessa initiated the tradition of improving commutative condition in fixed point theorems by introducing the notion of weakly commuting, weak** commutative and reciprocal continuity of mappings. Pathak [163] defines weak* commuting and in [164] weak** commuting mapping in metric space and prove some theorem. Popa [182] proved theorem for weakly compatible non-continuous mapping using implicit relations. It was extended by Imdad et.al. [84] using coincidence commuting property. Singh and Jain [237] also extend the relation of Popa [182, 183] in fuzzy metric space. Very recently Jain et al. [93] prove a theorem for three mappings using implicit relation.

The main object of this chapter is to obtain some common fixed point theorems in fuzzy metric space using implicit relations. Our result differs from all above authors in the following way:

(i) We have taken four self maps,
(ii) Reciprocal continuity is used relaxing the weak** commuting properly.

Pant [155] introduced the notion of reciprocal continuity of mappings in metric spaces. Balasubramaniam et. al. [14] proved the problem on the existence of a contractive definition which generates a fixed point but does not force the mapping to be continuous at the fixed point.

### 3.2 PRELIMINERIES

Ever since the introduction of fuzzy sets by Zadeh [245], the fuzziness invaded almost all the branches of crisp mathematics. As Deng [52], Kaleva and Seikalla [109], and Kramosil & Michalek [124, 125] have introduced the notion of fuzzy metric spaces in different ways.

**Definition 3.2.1[233]:** A binary operation *: [0, 1] × [0, 1] → [0, 1] is a continuous t-norm if it satisfies the following conditions:

(i) * is associative and commutative,

(ii) * is continuous,

(iii) * (a, 1) = a for all a ∈ [0, 1],

(iv) a, * b ≤ c * d whenever a ≤ c and b ≤ d, for each a, b, c, d ∈ [0, 1].

Two typical examples of continuous t-norm are
\[ a \ast b = ab \quad \text{and} \quad a \ast b = \min(a, b). \]

**Definition 3.2.2[70]:** The 3–tuple \((X, M, \ast)\) is called a fuzzy metric space (shortly, FM-space) if \(X\) is an arbitrary set, \(\ast\) is a continuous \(t\)-norm and \(M\) is a fuzzy set in \(X^2 \times [0, \infty)\) satisfying the following conditions: for all \(x, y, z \in X\) and \(t > 0\).

- **(FM–1)** \(M(x, y, 0) = 0,\)
- **(FM–2)** \(M(x, y, t) = 1, \text{ for all } t > 0 \text{ if and only if } x = y,\)
- **(FM–3)** \(M(x, y, t) = M(y, x, t),\)
- **(FM–4)** \(M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s),\)
- **(FM–5)** \(M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]\) is left–continuous.

**Example 3.2.1 [70]:** Let \((X, d)\) be a metric space. Define \(a \ast b = ab\) (or \(a \ast b = \min\{a, b\}\)) and for all \(x, y \in X\) and \(t > 0,\)

\[
M(x, y, t) = \frac{t}{t + d(x, y)} \tag{3.2.1}
\]

then \((X, M, \ast)\) is a fuzzy metric space. We call this fuzzy metric \(M\) induced by the metric \(d\), the standard fuzzy metric. On the other hand, note that there exists no metric on \(X\) satisfying (3.2.1)

**Lemma 3.2.1[70]:** For \(x, y \in X, M(x, y, \cdot)\) is non-decreasing on \((0, \infty)\).
Proof: Suppose that \( M(x, y, s) < M(x, y, t) \) for some \( 0 < t < s \). Then we have
\[ M(x, y, t) * M(y, y, s - t) \leq M(x, y, s) < M(x, y, t), \]

By (FM-2), \( M(y, y, s - t) = 1 \) and thus \( M(x, y, t) \leq M(x, y, s) < M(x, y, t) \), which is a contradiction.

Let \( (X, M, *) \) be a fuzzy metric space with the following condition:
\[ \lim_{t \to \infty} M(x, y, t) = 1, \quad \text{for all } x, y \in X. \]

Definition 3.2.3 [70]: Let \( (X, M, *) \) be a fuzzy metric space. Then

(a) A sequence \( \{x_n\} \) in \( X \) is said to be convergent in \( X \) if for each \( \epsilon > 0 \) and each \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( M(x_n, x, t) > 1 - \epsilon \) for all \( n \geq n_0 \).

(b) A sequence \( \{x_n\} \) in \( X \) is said to be Cauchy if for each \( \epsilon > 0 \) and each \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( M(x_n, x_m, t) > 1 - \epsilon \) for all \( n, m \geq n_0 \).

(c) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Lemma 3.2.2 [233]: Let \( \{y_n\} \) be a sequence in a fuzzy metric space \( (X, M, *) \) with the condition (FM-6). If there exists \( k \in (0, 1) \) such that
\[ M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t) \quad (3.2.2) \]
for all \( t > 0 \) and \( n \in \mathbb{N} \), then \( \{y_n\} \) is a Cauchy sequence in \( X \).
Lemma 3.2.3: Let \( \{x_n\} \) be a sequence in a fuzzy metric space \((X, M, *)\) with (FM–6). If there is a number \( k \) where \( 0 < k < 1 \) such that

\[
M(x_{n+1}, x_{n+2}, kt) \geq M(x_n, x_{n+1}, t)
\]

(3.2.3)

for all \( t > 0 \) and \( n = 0,1,2,3, \ldots \), then \( \{x_n\} \) is a Cauchy sequence in \( X \).

Proof: Let \( p \) be any positive integer, then by repeated application of (FM–4) and in view of (3.1.3), we have

\[
M(x_n, x_{n+p}, t) \geq M(x_n, x_{n+1}, t/2)^* M(x_{n+p}, x_{n+1}, t/2)
\]

\[
\geq M(x_0, x_1, t/2k^n)^* M(x_{n+2}, t/4)^* M(x_{n+2}, t/4)^* M(x_{n+2}, t/4)^{n-1}
\]

Continuing this procedure, we obtain

\[
M(x_n, x_{n+p}, t) \geq M(x_0, x_1, t/2k^n)^* M(x_1, x_p, t/4k^{n-1})^* M(x_2, x_{n+p}, t/4k^{n+1})^* M(x_{n+2}, x_{n+p}, t/4)
\]

Since the \( t \)-norm \( * \) is continuous and \( M(x, y, \cdot) \) is continuous, letting \( n \to \infty \), we have

\[
\lim_{n \to \infty} M(x_n, x_{n+p}, t) \geq 1 * 1 * 1 * \ldots * 1 = 1.
\]

This shows that \( \{x_n\} \) is a Cauchy sequence and thus the lemma is proved.

Lemma 3.2.4: The limit of a sequence \( \{x_n\} \) in fuzzy metric space is unique.
**Proof:** Let \( x, y \) be two limit points of the sequence \( \{x_n\} \), then by the definition of limit \( \lim_{n \to \infty} M(x_n, x, t) = 1 \), and \( \lim_{n \to \infty} M(x_n, y, t) = 1 \) for all \( t > 0 \). Now, using (FM–4), we have for \( n = 0, 1, 2, \ldots \)

\[
M(x, y, t) \geq M(x_n, y, t/2) \cdot M(x_n, x, t/2)
\]

Since ‘*’ is continuous therefore, letting \( n \to \infty \), we have

\[
M(x, y, t) \geq 1 \cdot 1 = 1, \text{ for all } t > 0.
\]

Thus, (FM–2), yields \( x = y \). This completes the proof of lemma.

**Lemma 3.2.5:** Let \( \{y_n\} \) be a sequence in a fuzzy metric space \( (X, M, *) \)

If there exists a number \( q \in (0, 1) \) such that for all \( t > 0 \), and \( n = 1, 2, \ldots \),

\[
M(y_{n+2}, y_{n+1}, qt) \geq M(y_{n+1}, y_n, t), \text{ then } \{y_n\} \text{ is a Cauchy sequence in } X.
\]

**Lemma 3.2.6[233]:** If for all \( x, y \in X, t > 0 \) and for a number \( q \in (0, 1) \), in fuzzy metric space \( (X, M, *) \), then

\[
M(x, y, qt) \geq M(x, y, t), \text{ implies } x = y.
\]

**Proof:** Now, in view of the given condition, we have

\[
M(x, y, t) \geq M(x, y, t/k) \geq M(x, y, t/k^2)
\]

Proceeding in the same way, we obtain, for \( p = 1, 2, 3, \ldots \)

\[
M(x, y, t) \geq M(x, y, t/k^p)
\]
By noting $M(x, y, t/k^p) \rightarrow 1$ as $p \rightarrow \infty$, it follows that $M(x, y, t) = 1$ for all $t > 0$. Therefore by (FM– 2), $x = y$.

Analyzing the theorems on fuzzy metric spaces, it is concluded that the authors have tried to fuzzify some theorems on contraction in metric space e.g. Grabiec [75] presented the fuzzy version of Banach contraction principle as follows:

**Definition 3.2.4:** Let $(X, M, *)$ be a complete fuzzy metric space and $f$ a self map of $X$ such that

$$M(fx, fy, kt) \geq M(x, y, t)$$

for each $x, y \in X$, $t > 0$ and $k \in (0, 1)$. Then $f$ has a unique fixed point in $X$.

**Definition 3.2.5 [244]:** Two self mappings $A$ and $S$ of a fuzzy metric space $(X, M, *)$ are called compatible if $\lim_{n \to \infty} M(ASx_n, SAx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in $X$ such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x$$

for some $x$ in $X$.

**Definition 3.2.6 [244]:** Two self mappings $A$ and $S$ of a fuzzy metric space $(X, M, *)$ are called weakly commuting if

$$M(ASx, SAx, t) \leq M(Ax, Sx, t)$$

for all $x$ in $X$ and $t > 0$. 

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**Definition 3.2.7 [244]**: Two self mappings $A$ and $S$ of a fuzzy metric space $(X, M, \ast)$ are called point wise $R$–weakly commuting if there exists $R > 0$ such that

$$M(ASx, SAx, t) \sqsubseteq M(Ax, Sx, t/R) \text{ for all } x \in X \text{ and } t > 0.$$  

**Remark 3.2.1**: Clearly, point wise $R$–weakly commutativity implies weak commutativity only when $R \geq 1$.

We used here the following:

**Definition 3.2.8**: Two self mappings $A$ and $S$ of a fuzzy metric space $(X, M, \ast)$ is called weak** commuting if $A(X) \subseteq S(X)$ and for any $x$ in $X$.

$$M(A^2S^2x, S^2A^2x, t) \sqsubseteq M(A^2x, S^2x, t)$$

**Remark 3.2.3**: If $A$ and $S$ are idempotent maps ie $A^2 = A$ and $S^2 = S$. Then weak commutative reduced to weak commuting pair $(A, S)$

$$M(A^2S^2x, S^2A^2x, t) \sqsubseteq M(A^2Sx, S^2Ax, t) \sqsubseteq M(AS^2x, SA^2x, t) \sqsubseteq M(ASx, SAx, t) \sqsubseteq M(A^2x, S^2x, t)$$

However, point wise $R$–weakly commuting mappings need not to be compatible as shown in the example.

**Example 3.2.2**: Let $X = [2, 20)$ with the usual metric $d$ and define

$$M(x, y, t) = \frac{t}{t + |x - y|} \text{ for all } x, y \in X \text{ and } t > 0.$$  

Clearly $(X, M, \ast)$ is a...
complete fuzzy metric space where $\ast$ is defined by $a \ast b = ab$ for all $a, b \in [0, 1]$. Let $A$ and $S$ be self mappings of $X$ defined as

$$\begin{align*}
Ax &= \begin{cases} 
2, & x = 2 \text{ or } x > 5 \\
8, & 2 < x \leq 5
\end{cases} \quad \text{and} \quad Sx &= \begin{cases} 
2, & x = 2 \\
12 + x, & 2 < x \leq 5 \\
x - 3, & x > 5
\end{cases}
\end{align*}$$

It can be verified that $A$ and $S$ are point wise $R$–weakly commuting mappings but not compatible. Also, neither $A$ nor $S$ is continuous, not even at their coincidence points.

**Definition 3.2.9[11]:** Two self maps $A$ and $S$ which are idempotent maps i.e. $A^2 = A$ and $S^2 = S$, of a fuzzy metric space $(X, M, \ast)$ are called reciprocally continuous on $X$ if

$$\lim_{n \to \infty} A^2 S^2 x_n = A^2 x \quad \text{and} \quad \lim_{n \to \infty} S^2 A^2 x_n = S^2 x$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} A^2 x_n = \lim_{n \to \infty} S^2 x_n = x$ for some $x$ in $X$. That is

$$M(A^2 S^2 x_n, S^2 A^2 x_n, t) \geq M(A^2 x, S^2 x, t) \quad \text{and} \quad M(AS^2 x_n, SA^2 x_n, t) \geq M(A^2 x, S^2 x, t)$$

where $\{x_n\}$ is sequence in $x$ such that

$$\lim_{n \to \infty} M(A^2 x_n, S^2 x_n, t) = M(A^2 x, S^2 x, t), \text{ for all } t > 0$$

Thus if two self mapping are weak** commuting then they are reciprocally continuous as well.
Example 3.2.3: Let \((X, M, *)\) be the Fuzzy metric space with 
\[
M(x, y, t) = \exp\left(\frac{|x-y|}{t}\right)
\]
for all \(x, y \in X\) and \(t > 0\). Let \(X = [0, 1]\). Define \(A\) and \(T\) by \(Ax = x/(x+2)\) and \(Tx = x/2\) for all \(x \in X\), where \(Ax = [0, 1/3]\) and \(Tx = [0, 1/2]\), then 
\[
M(A^2T^2x, T^2A^2x, t) = \exp\left(\frac{x^2}{(3x+8)(4x+8)}\right)
\]
\[
\geq \exp\left(\frac{x^2}{(3x+8)(4x+8)}\right)
\]
\[
\geq M(A^2T^2x, T^2A^2x, t)
\]
\[
M(A^2Tx, T^2Ax, t) = \exp\left(\frac{x^2}{(3x+8)(4x+8)}\right)
\]
\[
\geq \exp\left(\frac{x^2}{(3x+8)(4x+8)}\right)
\]
\[
\geq M(AT^2x, TA^2x, t)
\]
\[ M(\text{AT}^2x, \text{TA}^2x, t) = \exp\left(\frac{x}{x+8} - \frac{x}{6x+8}\right) \geq \exp\left(\frac{x}{2x+4}\right) \geq M(\text{AT}x, \text{TA}x, t) \]

\[ M(\text{AT}x, \text{TA}x, t) = \exp\left(\frac{x}{x+4} - \frac{x}{2x+4}\right) \geq \exp\left(\frac{x}{3x+4}\right) \geq M(A^2x, T^2x, t) \]

Hence (A, T) is weak ** commuting map, which implies the reciprocally continuity of A and T.

Throughout this chapter, \((X, M, *)\) be a fuzzy metric space with the following condition:

(FM -6) \( \lim_{t \to \infty} M(x, y, t) = 1 \), for all \( x, y \in X \).

The following theorem was proved by Balasubramaniam et. al. [14]:

**Theorem 3.2.1:** Let \((A, S)\) and \((B, T)\) be pointwise \(R\)-weakly commuting pairs of self mappings of complete fuzzy metric space \((X, M, *)\) such that
(a) \( AX \subseteq TX, \ BX \subseteq SX, \)

(b) \( M(Ax, By, t) \leq M(x, y, ht); 0 < h < 1, x, y \in X \text{ and } t > 0. \)

Suppose that \((A, S)\) and \((B, T)\) is compatible pairs of reciprocally continuous mappings. Then \(A, B, S\) and \(T\) have a unique common fixed point.

The following theorem was proved by Pant and Jha [158]:

**Theorem 3.2.2:** Let \((A, S)\) and \((B, T)\) be pointwise \(R\)-weakly commuting pairs of self mappings of complete fuzzy metric space \((X, M, *)\) such that

(a)’ \( AX \subseteq TX, \ BX \subseteq SX, \)

(b)’ \( M(Ax, By, t) \leq M(x, y, ht); 0 < h < 1, x, y \in X \text{ and } t > 0. \)

Let \((A, S)\) and \((B, T)\) be compatible mappings. If any of the mappings in compatible pairs \((A, S)\) and \((B, T)\) is continuous then \(A, B, S\) and \(T\) have a unique common fixed point.

**Remark 3.2.4:** In [158], Pant and Jha proved that theorem 3.2.2 is an analogue of the theorem 3.2.1 by obtaining a connection between continuity and reciprocal continuity in fuzzy metric space.
3.3 IMPLICIT RELATIONS

Let \( \Phi \) be the set of all real continuous functions \( F: [0, 1]^6 \to \mathbb{R} \) is continuous function such that

(F\(_1\)): \( F \) is non-increasing in the fifth and sixth variable

(F\(_2\)) If for some \( k \in (0, 1) \), we have

(F\(_a\)) \( F\{u(kt), v(t), v(t), u(t), u(t/2) \ast v(t/2), 1\} \geq 1 \), or

(F\(_b\)) \( F\{u(kt), v(t), v(t), u(t), 1, u(t/2) \ast v(t/2)\} \geq 1 \)

for any fixed \( t > 0 \) and any non-decreasing function \( u, v: (0, \infty) \to [0, 1] \), with \( 0 < u(t), v(t) \leq 1 \) then there exists \( q \in (0, 1) \) with \( u(qt) \geq v(t) \ast u(t) \)

(F\(_3\)) If for some \( k \in (0, 1) \), we have

\( F\{u(kt), u(t), 1, 1, u(t), u(t)\} \geq 1 \), or \( F\{u(kt), 1, u(t), 1, u(t), 1\} \geq 1 \), or

\( F\{u(kt), 1, 1, u(t), 1, u(t)\} \geq 1 \), for some \( t > 0 \) and any non-decreasing function \( u : (0, \infty) \to [0, 1] \), then \( u(kt) \geq u(t) \).

Example 3.3.1: Let \( F(u_1, \ldots, u_6) = \frac{u_1}{\min\{u_2, u_3, u_4, u_5, u_6\}} \) and \( a \ast b = \min\{a, b\} \). Let \( t > 0, 0 < u(t), v(t) \leq 1, k \in (0, 1/2) \), where \( u, v : [0, \infty) \to [0, 1] \) are non-decreasing functions. Now, suppose that

\( F\{u(kt), v(t), v(t), u(t), 1, u(t/2) \ast v(t/2)\} \geq 1 \),

i.e. in (F\(_a\)), \( F\{u(kt), v(t), v(t), u(t), 1, u(t/2) \ast v(t/2)\} \geq 1 \)
Thus \( u(qt) \geq v(t) \ast u(t) \) if \( q = 2k \in (0, 1) \).

for \((F_b)\), \[ F\{u(kt), v(t), v(t), u(t), u(t/2) \ast v(t/2)\} \geq 1. \]

Thus \( u(qt) \geq v(t) \ast u(t) \) if \( q = 2k \in (0, 1) \).

Finally in \((F_3)\), suppose that \( t > 0 \) is fixed, \( u : [0, \infty) \rightarrow [0, 1] \) is a non-decreasing function and

\[
F\{u(kt), 1, u(t), 1, u(t)\} = \frac{u(kt)}{u(t)} \geq 1 \text{ or } F\{u(kt), 1, u(t), 1, u(t)\} = \frac{u(kt)}{u(t)} \geq 1 \text{ or } F\{u(kt), 1, 1, u(t), 1, u(t)\} = \frac{u(kt)}{u(t)} \geq 1.
\]

for some \( k \in (0, 1) \). Then we have \( u(kt) \geq u(t) \) and thus \( F \in \Phi \).

**Example 3.3.2:** Let \( F(u_1, \ldots, u_6) = \frac{u_i^2}{\min\{u_2 \ast u_3, u_3 \ast u_4, u_2 \ast u_4\} \max\{u_5, u_6\}} \)

and \( a \ast b = \min\{a, b\} \). Let \( t > 0, 0 < u(t), v(t) \leq 1, k \in (0, 1/2) \), where \( u, v : [0, \infty) \rightarrow [0, 1] \) are non-decreasing functions. Now, suppose that

\[ F\{u(kt), v(t), v(t), u(t), 1, u(t/2) \ast v(t/2)\} \geq 1, \]
i.e. in \((F_a)\), \(F\{u(kt), v(t), v(t), u(t), 1, u(t/2) \ast v(t/2)\}\)

\[
= \frac{u^2(kt)}{\min\{v(t) \ast v(t), v(t) \ast u(t), v(kt) \ast u(t)\} \max\{1, u(t/2) \ast v(t/2)\}}
\]

\[
= \frac{u^2(kt)}{\{u(t) \ast v(t)\}} \geq 1. \text{ Thus } u(qt) \geq v(t) \ast u(t) \text{ if } q = 2k \in (0, 1).
\]

For \((F_b)\) a similar argument works. Finally in \((F_3)\), suppose that \(t > 0, \) a fixed, \(u : [0, \infty) \to [0, 1]\) is a non-decreasing function and

\[
F\{u(kt), u(t), 1, u(t), u(t)\} = \frac{u(kt)}{u(t)} \geq 1 \text{ or }
\]

\[
F\{u(kt), 1, u(t), 1, u(t)\} = \frac{u(kt)}{u(t)} \geq 1 \text{ or }
\]

\[
F\{u(kt), 1, 1, u(t), 1, u(t)\} = \frac{u(kt)}{u(t)} \geq 1
\]

for some \(k \in (0, 1)\). Then we have \(u(kt) \geq u(t)\) and thus \(F \in \Phi\).

**Example 3.3.3:** Let \(F(u_1, \ldots, u_6) = \frac{u_1 \max\{u_2, u_3, u_4\}}{\min\{u_5, u_6\}}\) and \(a \ast b, a\) a continuous \(t\)-norm. Let \(t > 0, 0 < u(t), v(t) \leq 1, k \in (0, 1/2),\) where \(u, v : [0, \infty) \to [0, 1]\) are non-decreasing functions. Now, suppose that

\[
F\{u(kt), v(t), v(t), u(t), 1, u(t/2) \ast v(t/2)\} \geq 1,
\]

i.e. in \((F_a)\), \(F\{u(kt), v(t), v(t), u(t), 1, u(t/2) \ast v(t/2)\}\)

\[
= \frac{u(kt) \max\{v(t), v(t), u(t)\}}{\max\{1, u(t/2) \ast v(t/2)\}} = \frac{u(kt)u(t)}{[u(t/2) \ast v(t/2)]} \geq 1.
\]
Thus \( u(qt) \geq v(t) \cdot u(t) \) if \( q = 2k \in (0, 1) \). For \((F_b)\) a similar argument works. Finally in \((F_3)\), suppose that \( t > 0 \), a fixed, \( u : [0, \infty) \rightarrow [0, 1] \) is a non-decreasing function and

\[
F\{u(kt), u(t), 1, u(t), u(t)\} = \frac{u(kt)}{u(t)} \geq 1 \quad \text{or}
\]

\[
F\{u(kt), 1, u(t), 1, u(t)\} = \frac{u(kt)}{u(t)} \geq 1 \quad \text{or}
\]

\[
F\{u(kt), 1, 1, u(t), 1\} = \frac{u(kt)}{u(t)} \geq 1
\]

for some \( k \in (0, 1) \). Then we have \( u(kt) \geq u(t) \) and thus \( F \in \Phi \).

**Example 3.3.4:** Let \( F(u_1, \ldots, u_6) = \frac{(u_1)^3}{[u_2 \cdot u_3 \cdot u_4 \max\{u_5, u_6\}]} \) and \( a \cdot b = b \).

Let \( t > 0, 0 < u(t), v(t) \leq 1, k \in (0, 1/2) \), where \( u, v : [0, \infty) \rightarrow [0, 1] \) are non-decreasing functions. Now, suppose that

\[
F\{u(kt), v(t), v(t), u(t), 1, u(t/2) \cdot v(t/2)\} \geq 1,
\]

i.e. in \((F_a)\),

\[
F\{u(kt), v(t), v(t), u(t), 1, u(t/2) \cdot v(t/2)\} = \frac{(u(kt))^3}{(v(t))^2 u(t)} \geq 1.
\]

Thus \( u(qt) \geq v(t) \cdot u(t) \) if \( q = 2k \in (0, 1) \). For \((F_b)\) a similar argument works. Finally in \((F_3)\), suppose that \( t > 0 \), a fixed, \( u : [0, \infty) \rightarrow [0, 1] \) is a non-decreasing function and

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\[ F\{u(kt), u(t), 1, 1, u(t), u(t)\} = \frac{u(kt)}{u(t)} \geq 1 \text{ or} \]
\[ F\{u(kt), 1, u(t), 1, u(t), 1\} = \frac{u(kt)}{u(t)} \geq 1 \text{ or} \]
\[ F\{u(kt), 1, 1, u(t), 1, u(t)\} = \frac{u(kt)}{u(t)} \geq 1 \]

for some \( k \in (0, 1) \). Then we have \( u(kt) \geq u(t) \) and thus \( F \in \Phi \).

**Lemma 3.3.1:** Let \( (X, M, *) \) be a complete fuzzy metric space with \( t \ast t \geq t \) for all \( t \in [0, 1] \) and the condition (FM-6). Let \( (A, S) \) and \( (B, T) \) be point wise R-weakly commuting pairs of self mappings of \( X \) such that

(a)'' \( A(X) \subseteq T(X) \) and \( B(X) \subseteq S(X) \)

(b)'' there exists \( k \in (0, 1) \), such that
\[
F\{M(A^2x, B^2y, kt), M(S^2x, T^2y, t), M(T^2y, B^2y, t), M(S^2x, A^2x, t), M(S^2x, B^2y, t), M(T^2y, A^2x, t)\} \geq 1
\]

for all \( x, y \in X, t > 0 \). Then the continuity of one mapping in the compatible pair \( (A, S) \) or \( (B, T) \) implies the reciprocal continuity.

**Proof:** We suppose that in the compatible pair \( (A, S) \), \( S \) is continuous.

We will show that \( A \) and \( S \) are reciprocally continuous. Here we take that \( A \) and \( S \) are idempotent maps i.e. \( A^2 = A \) and \( S^2 = S \). Let \( \{x_n\} \) be a
sequence in X such that \(Ax_n \to u\) and \(Sx_n \to u\) for some \(u \in X\) as \(n \to \infty\).

Since \(S\) is continuous so we have

\[
S^2Ax_n \to S^2u \quad \text{and} \quad S^2Sx_n \to S^2u \quad \text{as} \quad n \to \infty.
\]

Since \((A, S)\) is compatible pair so that from definition 3.2.7

\[
\lim_{n \to \infty} M(A^2S^2x_n, S^2A^2x_n, t) = 1 \quad \Rightarrow \quad \lim_{n \to \infty} M(A^2Sx_n, S^2Ax_n, t) = 1
\]

\[
\Rightarrow \quad \lim_{n \to \infty} M(A^2Sx_n, S^2u, t) = 1 \quad \Rightarrow \quad A^2Sx_n \to S^2u, \quad \text{as} \quad n \to \infty.
\]

Again since in \((a)''\) \(A(X) \subset T(X)\) i.e. there exists a sequence \(\{y_n\}\) in X such that \(A^2u = A^2Sx_n = T^2y_n\). Thus we have

\[
S^2Sx_n \to S^2u, \quad S^2Ax_n \to S^2u, \quad A^2Sx_n \to S^2u \quad \text{and} \quad T^2y_n \to S^2u
\]

as \(n \to \infty\) we have \(A^2Sx_n = T^2y_n\). Again by \((a)''\) \(B(X) \subset S(X)\) so we claim that \(B^2y_n \to S^2u\) as \(n \to \infty\).

Suppose \(B^2y_n \to S^2u\) as \(n \to \infty\), then there exists subsequence \(\{B^2y_m\}\) of \(\{B^2y_n\}\) such that for given \(t > 0\), there exists \(\varepsilon > 0\) and a positive integer \(n_0\) such that for all \(m > n_0\), \(M(B^2y_m, S^2u, t) \leq \varepsilon\) and \(M(A^2Sx_m, B^2y_m, t) \leq \varepsilon\).

Now from \((b)''\) for \(\alpha = 1\),

\[
F\{M(A^2Sx_m, B^2y_m, kt), M(S^2Sx_m, T^2y_m, t), M(T^2y_m, B^2y_m, t), M(S^2Sx_m, A^2Sx_m, t), M(S^2Sx_m, B^2y_m, t) M(T^2y_m, A^2Sx_m, t) \} \geq 1
\]

or

\[
F\{M(S^2u, B^2y_m, kt), M(S^2u, S^2u, t), M(S^2u, B^2y_m, t), M(S^2u, S^2u, t), M(S^2u, S^2u, t),
\]

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\[ M(S^2u, B^2y_m, t), M(S^2u, S^2u, t) \geq 1 \]

or \[ F\{M(S^2u, B^2y_m,kt), 1, M(S^2u, B^2y_m,t), 1, M(S^2u, B^2y_m, t), 1\} \geq 1 \]

from (F_3), we have \[ M(S^2u, B^2y_m, kt) \geq M(S^2u, B^2y_m, t). \]

Thus from lemma 3.1.6, \[ B^2y_m \rightarrow S^2u \quad \text{i.e.} \quad B^2y_n \rightarrow S^2u \quad \text{as} \quad n \rightarrow \infty. \]

Again by (b)', use \[ x = u, y = y_n \]

\[ F\{M(A^2u, B^2y_n, kt), M(S^2u, T^2y_n, t), M(T^2y_n, B^2y_n, t), \]
\[ M(S^2u, A^2u, t), M(B^2y_n, S^2u, t), M(T^2y_n, A^2u, t)\} \geq 1 \]

now letting \[ n \rightarrow \infty, \] we have

\[ F\{M(A^2u, S^2u, kt), M(S^2u, S^2u, t), M(S^2u, S^2u, t), M(S^2u, A^2u, t)\}, \]
\[ M(S^2u, S^2u, t), M(A^2u, A^2u, t)\} \geq 1 \]

or \[ F\{M(A^2u, S^2u, kt), 1, 1, M(A^2u, S^2u, t), 1, M(A^2u, S^2u, t)\} \geq 1 \]

From (F_3), we have \[ M(A^2u, S^2u, kt) \geq M(A^2u, S^2u, t). \]

Hence by lemma 3.2.6, \[ A^2u = S^2u. \]

Thus \[ S^2Ax_n \rightarrow S^2u \] and \[ A^2Sx_n \rightarrow A^2u \] as \[ n \rightarrow \infty. \] Therefore A and S are reciprocally continuous on X. Similarly we can show that in compatible pair (B, T), B and T are reciprocally continuous on X.
3.4 MAIN RESULT

Theorem 3.4.1: Let \((X, M, *)\) be a complete fuzzy metric space with \(t \ast t \sqsubseteq t\) for all \(t \in [0, 1]\) and the condition (FM - 6). Let \((A, S)\) and \((B, T)\) be point wise R-weakly commuting pairs of self maps on \(X\) satisfying

(i) \(AX \subseteq TX\) and \(BX \subseteq SX\);

(ii) \((A, S)\) and \((B, T)\) are compatible pairs and one of the mapping in each pair is continuous;

(iii) there exists \(k \in (0, 1)\) such that

\[
F\{M(A^2x, B^2y, kt), M(S^2x, T^2y, t), M(T^2y, B^2y, t), M(S^2x, A^2x, t), M(S^2x, B^2y, t), M(T^2y, A^2x, t)\} \geq 1
\]

for all \(x, y \in X, t > 0\) and where \(F \in \Phi\) then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

Proof: Let \(x_0\) be any point in \(X\), then by (i) there exists points \(x_1, x_2 \in X\) such that \(A^2x_0 = T^2x_1 = y_0\) and \(B^2x_1 = S^2x_2 = y_1\), where \(A, S, B\) and \(T\) are idempotent maps then we define sequence \(\{x_n\}\) and \(\{y_n\}\) such that

\(y_{2n} = A^2x_{2n} = T^2x_{2n+1}\) and \(y_{2n+1} = B^2x_{2n+1} = S^2x_{2n+2}\) for \(n = 0, 1, 2, \ldots \).

When \(y_n \neq y_{n+1}\) for all \(n = 0, 1, 2, \ldots\), we put \(x = y_{2n}\) and \(y = y_{2n+1}\), in (iii), we have
\[ F\{ M(A^2x_{2n}, B^2x_{2n+1}, kt), M(S^2x_{2n}, T^2x_{2n+1}, t), M(T^2x_{2n+1}, B^2x_{2n+1}, t), M(A^2x_{2n}, S^2x_{2n}, t), M(S^2x_{2n}, B^2x_{2n+1}, t), M(A^2x_{2n}, T^2x_{2n+1}, t) \} \]

1

i.e. \[ F\{ M(y_{2n}, y_{2n+1}, kt), M(y_{2n}, y_{2n-1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n}, t) \} \]

1

or \[ F\{ M(y_{2n}, y_{2n+1}, kt), M(y_{2n}, y_{2n-1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n}, t) \} \]

1

Using condition \((F_b)\) in \((F_2)\), we have

\[ M(y_{2n}, y_{2n+1}, qt) \leq M(y_{2n-1}, y_{2n}, t) \]

which implies \((a \ast b = \min \{a, b\})\) that \( M(y_{2n}, y_{2n+1}, qt) \leq M(y_{2n-1}, y_{2n}, t) \).

Again by \((F_2)\), we have \( M(y_{2n-1}, y_{2n}, qt) \leq M(y_{2n-2}, y_{2n-1}, t) \).

In general, we have for \( m = 1, 2, \ldots \) and \( t > 0 \),

\[ M(y_m, y_{m+1}, qt) \leq M(y_{m-1}, y_m, t) \]

(3.4.1)

We shall prove that \( \{y_n\} \) is a Cauchy sequence. for which

\[ M(y_n, y_{n+1}, t) \leq M(y_{n-1}, y_n, q^{-1}t) \leq M(y_{n-2}, y_{n-1}, q^{-2}t) \]
it follows that \( \lim_{n \to \infty} M(y_n, y_{n+1}, t) = 1 \) for all \( n \in \mathbb{N} \), i.e. result holds for \( m = 0 \).

By induction hypothesis suppose that the result holds for \( m = r \), i.e.

\[
M(y_n, y_{n+r}, t) \to 1 \quad \text{as} \quad n \to \infty,
\]

for all \( n \in \mathbb{N} \), i.e. result holds for \( m = 1 \).

Now \( M(y_n, y_{n+r+1}, t) \geq M(y_n, y_{n+r}, t/2) \ast M(y_{n+r}, y_{n+r+1}, t/2) \to 1 \).

Thus the result holds for \( m = r + 1 \) and so by induction it holds for all \( n \in \mathbb{N} \). Hence \( \{y_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, then there exists a point \( z \) in \( X \) such that \( y_n \to z \) as \( n \to \infty \). Moreover its subsequences \( \{A^2x_{2n}\}, \{T^2x_{2n+1}\}, \{B^2x_{2n+1}\} \) and \( \{S^2x_{2n+2}\} \) also converges to \( z \). i.e. \( y_{2n} = A^2x_{2n} = T^2x_{2n+1} \to z \), and \( y_{2n+1} = B^2x_{2n+1} = S^2x_{2n+2} \to z \).

Since \( A \) and \( S \) are compatible and reciprocally continuous mappings, then

\[
A^2S^2x_{2n} \to A^2z \quad \text{and} \quad S^2A^2x_{2n} \to S^2z
\]

By compatibility of \( A^2 \) and \( S^2 \), it yields \( \lim_{n \to \infty} M(A^2S^2x_{2n}, S^2A^2x_{2n}, t) = 1 \), i.e. \( M(A^2z, S^2z, t) = 1 \Rightarrow A^2z = S^2z \). Since \( A(X) \subseteq T(X) \), then there exists a point \( u \in X \) such that \( A^2z = T^2u \). Now using (iii), we have

\[
F\{M(A^2z, B^2u, kt), M(S^2z, T^2u, t), M(T^2u, B^2u, t), M(S^2z, A^2z, t), M(S^2z, B^2u, t), M(T^2u, A^2z, t) \} \geq 1
\]

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i.e. \( F\{M(A^2z, B^2u, kt), M(A^2z, A^2z, t), M(A^2z, B^2u, t), M(A^2z, A^2z, t), \\
M(A^2z, B^2u, t), M(A^2z, A^2z, t)\} \neq 1 \)

or \( F\{M(A^2z, B^2u, kt), 1, M(A^2z, B^2u, t), 1, M(A^2z, B^2u, t), 1\} \neq 1 \)

From (F3), we have \( M(A^2z, B^2u, kt) \neq M(A^2z, B^2u, t) \), which implies by lemma 3.2.6, \( A^2z = B^2u. \)

Thus \( A^2z = S^2z = B^2u = T^2u \) \hspace{1cm} (3.4.2)

Now again using the R-weakly conditions of mappings there exists \( R > 0 \) such that

\[
M(A^2S^2z, S^2A^2z, t) \neq M(A^2Sz, S^2A^2z, t) \neq M(AS^2z, SA^2z, t) \\
\neq M(ASz, SAz, t) \neq M(A^2z, S^2z, t) \neq M(A^2z, S^2z, t/R)
\]

i.e. \( A^2z = S^2z \) and \( A^2Az = A^2Sz = S^2Az = S^2Sz. \)

Similarly since \( B \) and \( T \) are point wise R-weakly commuting mappings, we have

\[
M( B^2T^2z, T^2B^2z, t) \neq M( B^2Tz, T^2Bz, t) \neq M(BT^2z, TB^2z, t) \\
\neq M(BTz, TBz, t) \neq M( B^2z, T^2z, t) \neq M(B^2z, T^2z, t/R)
\]

which implies \( B^2z = T^2z. \) i.e. \( B^2u = B^2Tu = T^2Bu = T^2u. \)
Again putting $x = x_{2n}$ and $y = z$ in (iii) we get

$$F\{M(A^2x_{2n}, B^2z, kt), M(S^2x_{2n}, T^2z, t), M(T^2z, B^2z, t), M(S^2x_{2n}, A^2x_{2n}, t),$$

$$M(S^2x_{2n}, B^2z, t), M(T^2z, A^2x_{2n}, t)\} \geq 1$$

taking $n \rightarrow \infty$, $F\{M(A^2z, B^2z, kt), M(S^2z, T^2z, t), M(T^2z, B^2z, t),$ $M(S^2z, A^2z, t), M(S^2z, B^2z, t), M(T^2z, A^2z, t)\} \geq 1$

i.e. $F\{M(A^2z, B^2z, kt), M(A^2z, B^2z, t), 1, 1, M(A^2z, B^2z, t),$ $M(A^2z, B^2z, t) \geq 1,$

From $(F_3)$, we have, $M(A^2z, B^2z, kt) \geq M(A^2z, B^2z, t)$,

i.e. from lemma 3.2.6, $A^2z = B^2z$. (3.4.3)

Similarly, $B^2z = S^2z$. Thus $A^2z = S^2z = B^2z = T^2z$. (3.4.4)

Again we suppose that $z = A^2z$ is a common fixed point of $A$, $B$, $S$ and $T$. for this we have by (iii) putting, $x = x_{2n}$, $y = z$

$$F\{M(A^2x_{2n}, B^2z, kt), M(S^2x_{2n}, T^2z, t), M(T^2z, B^2z, t), M(S^2x_{2n}, A^2x_{2n}, t),$$

$$M(S^2x_{2n}, B^2z, t), M(T^2z, A^2x_{2n}, t)\} \geq 1$$

taking $n \rightarrow \infty$, $F\{M(z, B^2z, kt), M(T^2z, z, t), M(T^2z, B^2z, t), M(z, z, t),$ $M(z, B^2z, t), M(T^2z, z, t)\} \geq 1$
i.e. \( F\{M(z, B^2z, kt), M(B^2z, z, t), 1, 1, M(z, B^2z, t), M(B^2z, z, t)\} \geq 1 \)

From (F_3), we have, \( M(z, B^2z, kt) \geq M(B^2z, z, t) \)

i.e. from lemma 3.2.6, \( z = B^2z \). It follows from (3.4.4) that \( z = A^2z = S^2z = T^2z \) is a common fixed point of A, B, S and T.

For uniqueness, we put \( x = Az, y = w (\neq z) \) in (iii), we have

\[
F\{M(A^2Az, B^2w, kt), M(S^2Az, T^2w, t), M(T^2w, B^2w, kt), \\
M(S^2Az, A^2Az, t), M(S^2Az, B^2w, t), M(T^2w, A^2Az, t)\} \geq 1
\]

As the continuity of one of the mapping is compatible pair (A, S) implies the reciprocal continuity, i.e.

\[
M(A^2S^2z, S^2A^2z, t) \geq M(A^2Sz, S^2Az, t) \geq M(AS^2z, SA^2z, t) \\
\geq M(A^2Az, S^2Sz, t) \geq M(ASz, SAz, t) \geq M(A^2z, S^2z, t)
\]

i.e. \( F\{M(A^2z, w, kt), M(A^2z, w, t), M(w, w, kt), M(A^2z, A^2z, t), \\
M(A^2z, w, t), M(w, A^2z, t)\} \geq 1 \)

\[
F\{M(z, w, kt), M(z, w, t), 1, 1, M(w, z, t), M(z, w, t)\} \geq 1
\]
From (F₃), we have \( M(z, w, kt) \geq M(z, w, t) \). That is from lemma 3.1.6, \( z = w \). Thus \( z \) is a unique common fixed point of \( A, B, S \) and \( T \). Hence the theorem.

**Corollary 3.4.1:** Let \( (X, M, *) \) be a complete fuzzy metric space and let \( (A, S) \) and \( (B, T) \) be self mapping of \( X \) satisfying:

(i) \( * \) \( AX \subseteq TX \) and \( BX \subseteq SX \); 

(ii) \( * \) \( (A, S) \) and \( (B, T) \) are compatible pairs and one of the mapping in each pair is continuous; 

(iii) \( * \) there exists \( k \in (0, 1) \) such that 

\[
F\{M(Ax, By, kt) , M(Sx, Ax, t), M(Ty, Sx, t), M(Ty, By, t), M(Sx, By, 2t), M(Ax, Ty, t)\} \geq 1
\]

for all \( x, y \in X, t > 0 \), then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \), where \( F(t_1 \ldots \ldots t_6) : [0, 1]^6 \rightarrow R \in F^* \).

**Proof:** From the definition \( F(t_1 \ldots \ldots t_6) : [0, 1]^6 \rightarrow R \in F^* \), and \( t^*t \not\subseteq \), we have

\[
F\{M(Ax, By, kt) , M(Sx, Ax, t), M(Ty, Sx, t), \\
M(Ty, By, t), M(Sx, By, 2t) ,M(Ax, Ty, t)\} \geq 1
\]

\[
\geq F\{M(Ax, By, kt) , M(Sx, Ax, t), M(Ty, Sx, t), M(Ty, By, t), \\
M(Sx, Ty, t)*M(Tx, By, t) ,M(Ax, Ty, t)\}
\]

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\[ \geq F\{M(Ax, By, kt), M(Sx, Ax, t), M(Ty, Sx, t), M(Ty, By, kt), M(Ax, Ty, t)\} \geq 1 \]

and hence from theorem 3.4.1; A, B, S and T have a unique common fixed point.

**Corollary 3.4.2:** Let S and T be self mappings on a complete metric space \((X, d)\) of \(p\) and \(q\) be positive integers and there exists a number \(0 < k < 1\), such that

\[ F\{M(S^p x, T^q y, kt), M(S, y, t), M(x, S^p x, t), M(y, T^q y, t), M(y, S^p x, t), M(x, T^q y, t)\} \geq 1 \]

for all \(x, y \in X, t \geq 0\), Further suppose that

(i)’ \(S^p(X) \subseteq T^q(X)\),

(ii)’ \(T^q(X)\) is complete ,

(iii)’ S and T are same as in Theorem.

Then S and T have a unique common fixed point.

**Proof:** By theorem 3.4.1, we put \(A = B = I\) (identity map) and S and T are idempotent maps that is \(S^2 = S\) and \(T^2 = T\), then \(S^p\) and \(T^q\) have a unique common fixed point \(u\). Then we have \(S^p u = u\) and \(T^q u = u\),

Now using (iii)’ \(S^p(Su) = S(S^p u) = Su\) and \(T^q(Su) = S(T^q u) = Su\)
Thus $S u$ is also a common fixed point of $S^p$ and $T^q$. But by the uniqueness of $u$ implies $S u = u$. Similarly $T u = u$. Thus $S$ and $T$ have a unique common fixed point.

**Corollary 3.4.3:** Let $(X, M, *)$ be a complete fuzzy metric space and let $(A, S)$ and $(B, T)$ be self mapping of $X$ satisfying (i)$^*$, (ii)$^*$ of corollary 3.4.1 and

(iii)$^{**}$ there exists $k \in (0, 1)$ such that

$$F\{M(Ax, By, kt) , M(Sx, Ax, t), M(Ty, Sx, t), M(Ty, By, t) \} \geq 1$$

for all $x, y \in X, t > 0$, then $A, B, S$ and $T$ have a unique common fixed point in $X$.

**Proof:** We have the following example

$$F\{t_1, \ldots, t_5\} = 18t_1 - 16t_2 + 8t_3 - 10t_4 - t_5 + 1$$

then

$$F\{t_1, \ldots, t_4\} > F\{t_1, \ldots, t_5\}$$

i.e.

$$F\{M(Ax, By, kt) , M(Sx, Ax, t), M(Ty, Sx, t), M(Ty, By, t) \} \geq F\{M(Ax, By, kt) , M(Sx, Ax, t), M(Ty, Sx, t), M(Ty, By, kt), M(Ax, Ty, t) \} \geq 1$$

The result proved easily. 

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