Chapter 4

Scale-invariant Theory of Gravitation

4.1 Introduction

In the following section, we shall develop a scale-invariant theory of gravitation which, in addition to the general coordinate invariance imposed by Einstein, contains the invariance under scale or gauge transformation. These, not only are physical laws invariant for observers with different states of motion, but they are also invariant for observers with different measuring instruments. In practice, two kinds of units have been used. We shall call them gravitational units or 'Einstein units' and atomic units. We see that, within the scale-invariant theory of gravitation, the Large Number Hypothesis (LNH) finds a natural role. It yields the relation between atomic and Einstein units. In chapter II, we have seen, there are several scale-invariant theories of gravity in existence, the ones which are compatible with observation at some level being basically similar as regards their formal structure. But the formulation of scale-invariant theory of gravity is not yet found in literature in complete form. Recently Wesson[47, 48] formulated a new theory which is both coordinate and scale-invariant. He mentioned the two reasons for searching for an acceptable scale-invariant theory of gravity. The first reason is related to the observation that the matter in the Universe, as it is represented by galaxies, appears to be describable in a scale-free manner. The second reason related to recent progress in gauge theories, which may be the best way to describe all the interactions of Physics, including the gravitational interaction. In an attempt to unify the electromagnetic theory with Einstein's theory of gravitation, Weyl generalized the Riemannian geometry by allowing lengths to change under parallel displacement. Although the theory was rejected for being unphysical, the
mathematical operation of the theory, well-known as scale transformation, was introduced with some modification. A scale transformation represents a change of units of measurement and hence gives a general scaling of the physical system being investigated; this is pointed out by Eddington. Under a general scale transformation, the unit of measure is altered and the constant-rest-mass condition no longer seems physically necessary. In classical experiments concerning gravitational interaction, masses and lengths of arbitrary macroscopic objects have been used as units of measure, whether such units are in fact constant multiples of atomic units. Indeed it was the recognition of the possibility of a temporal dependence of the proportionality factor between atomic units and gravitational units of bulk matter that led Dirac to formulate the Large Number Hypothesis.

Previously, the gauge condition most commonly employed to simplify the scale-invariant gravity with the help of Dirac's hypothesis, which is originally proposed by Dirac. Wesson proposed a new hypothesis for the formulation of the scale-invariant theory of gravitation, which he called Conspiracy Hypothesis (CH): The matter parameters of a system (mass, density, pressure etc.), the 'constants' of Physics and the coordinates occur together in dimensionless combinations \( \eta - \text{numbers} \) in which the components may vary but in such a manner that the variations conspire to keep the \( \eta - \text{numbers} \) constant. So we see that LNH of Dirac and CH of Wesson are in opposite conception. In the Large Number Hypothesis, large dimensionless numbers of order \( 10^{40n} \) vary with time \( t^n \) (where \( n \) is a number). In the Conspiracy Hypothesis, dimensionless numbers of whatever size are constants in time. Canuto et al. [?] suggested in their scale covariant theory, Dirac's LNH for determination the gauge condition so that gravitational phenomena can be described in atomic units. However we shall present and develop the theory and show how naturally Wesson's CH can be fitted into the structure of the theory. We shall also consider the behaviour of Einstein's theory of gravitation under an arbitrary transformation of units.

### 4.2 Conformal transformation:

If the fundamental tensors \( g_{ij} \) and \( \bar{g}_{ij} \) of two spaces \( V_n \) and \( \bar{V}_n \) are in the relation

\[
\bar{g}_{ij} = \beta^2 g_{ij}
\]

(4.1)

where \( \beta \) is the function of \( x \)'s. Then from

\[
\bar{ds}^2 = \beta^2 g_{ij} dx^i dx^j,
\]

(4.2)

it follows that the magnitudes of the vectors of components \( dx^i \) at points of \( V_n \) and \( \bar{V}_n \) with the same coordinates are proportional at the angles between two
corresponding directions and corresponding points are equal. Accordingly we say
that the correspondence between $V_n$ and $V_n$ is conformal, and that $V_n$ and $V_n$ are
conformal spaces.

From (4.1) we have

$$\bar{g}^{ij} = \frac{1}{\beta^2} g^{ij}$$

(4.3)

Now the christoffel symbols transform as

$$\Gamma^i_{ij} = \bar{g}^{ik} \bar{\Gamma}^i_{jk} = \Gamma^i_{ij} + \frac{1}{\beta} \left( \delta^i_{j} \beta_j + \delta^i_{j} \beta_j - g^j_k \beta_k \right)$$

If $\beta_{ij}$ denotes the second covariant derivatives of $\beta$ with respect to the x's, we
write

$$\beta_{ij} = \beta_{ij} - \beta_j \beta_j.$$  (4.4)

We know that

$$R_{hijk} = \frac{\delta[hk, h]}{\delta x^i} - \frac{\delta[ij, h]}{\delta x^k} + \Gamma^l_{ij}[hk, l] - \Gamma^l_{ik}[hj, l]$$

then

$$\frac{1}{\beta^2} \bar{R}_{hijk} = R_{hijk} + g_{hk}(ln\beta)_{ij} + g_{ij}(ln\beta)_{hk} - g_{ij}(ln\beta)_{ik} - g_{ik}(ln\beta)_{ij}$$

$$+ (g_{hk} g_{ij} - g_{ij} g_{ik}) \Delta_1(ln\beta)$$

(4.5)

where

$$\Delta_1(ln\beta) = g^{\lambda \mu} \frac{\delta(ln\beta)}{\delta x^\lambda} \frac{\delta(ln\beta)}{\delta x^\mu} = g^{\lambda \mu} \beta_{\lambda} \beta_{\mu}.$$

By means of (4.3) and (4.5) we have for the expressions for the components of the
Ricci tensor for $V_n$

$$\bar{R}_{ij} = \bar{g}^{hk} \bar{R}_{hijk} = R_{ij} + (n - 2)(ln\beta)_{ij} + \Delta_2(ln\beta) + (n - 2)\Delta_1(ln\beta) g_{ij}$$

(4.6)

where

$$\Delta_2(ln\beta) = g^{\lambda \mu} (ln\beta)_{,\lambda \mu} = g^{\lambda \mu} \left[ \frac{\delta^2(ln\beta)}{\delta x^\lambda \delta x^\mu} - \frac{\delta (ln\beta)}{\delta x^k} \Gamma^k_{\lambda \mu} \right]$$

$$= \frac{1}{\beta^2} g^{\lambda \mu} \left[ \beta_{\lambda \mu} - \beta_k \Gamma^k_{\lambda \mu} \right] - \frac{g^{\lambda \mu} \beta_{\lambda} \beta_{\mu}}{\beta^2} = g^{\lambda \mu} \beta_{\lambda \mu} - \frac{g^{\lambda \mu} \beta_{\lambda} \beta_{\mu}}{\beta^2}$$

(4.7)

here, for a scalar $\beta_{\lambda} = \beta_{\lambda}$ and so that $\beta_{\lambda \mu} = (\beta_{\lambda})_{,\mu}$ and $\beta_{\lambda \mu} = \beta_{\lambda \mu} - \beta_k \Gamma^k_{\lambda \mu}$ and

$$\beta_{\lambda} = g^{\lambda \mu} \beta_{\mu} (\beta, \lambda = \beta) (,) \text{ means partial differentiation and (;} \text{ means conventional covariant differentiation using } g_{ij}.$$
Therefore, for $n = 4$,

$$
\tilde{R}_{ij} = R_{ij} + 2(\ln \beta)_{ij} + \left[ \frac{g^{\mu \nu} \beta_{\lambda \mu} \beta_{\lambda \nu}}{\beta^2} - \frac{g^{\lambda \mu} \beta_{\alpha \lambda} \beta_{\alpha \mu}}{\beta^2} + \frac{2g^{\lambda \mu} \beta_{\alpha \lambda} \beta_{\alpha \mu}}{\beta^2} \right] g_{ij}
$$

$$
= R_{ij} + 2 \left[ (\ln \beta)_{,ij} - (\ln \beta)_{,i}((\ln \beta)_{,j} + \frac{g^{\lambda \mu} \beta_{\alpha \lambda} \beta_{\alpha \mu}}{\beta^2} \right] g_{ij}
$$

$$
= R_{ij} + 2 \left[ \frac{\beta_{,ij}}{\beta} - \frac{\beta_{,i} \beta_{,j}}{\beta^2} + \frac{g^{\lambda \mu} \beta_{\alpha \lambda} \beta_{\alpha \mu}}{\beta^2} \right] g_{ij}
$$

$$
= R_{ij} + 2 \frac{\beta_{,ij}}{\beta} - 4 \frac{\beta_{,i} \beta_{,j}}{\beta^2} + \left[ \frac{g^{\lambda \mu} \beta_{\alpha \lambda} \beta_{\alpha \mu}}{\beta^2} \right] g_{ij}
$$

(4.8)

Similarly, for the invariant scalar curvature

$$
\tilde{R} = g^{ij} \tilde{R}_{ij} = \frac{1}{\beta^2} \left[ R + 6 \Delta_2 (\ln \beta) + 6 \Delta_1 (\ln \beta) \right]
$$

$$
= \frac{1}{\beta^2} \left[ R + \frac{6g^{\mu \nu} \beta_{\lambda \mu} \beta_{\lambda \nu}}{\beta^2} - \frac{6g^{\mu \nu} \beta_{\lambda \nu} \beta_{\mu \lambda}}{\beta^2} + \frac{6g^{\mu \nu} \beta_{\lambda \mu} \beta_{\lambda \nu}}{\beta^2} \right] = \frac{1}{\beta^2} \left[ R + \frac{6g^{\mu \nu} \beta_{\lambda \mu} \beta_{\lambda \nu}}{\beta} \right]
$$

(4.9)

and for the Einstein tensor

$$
\tilde{R}_{ij} - \frac{1}{2} \tilde{R} g_{ij} = R_{ij} + \frac{2\beta_{,ij}}{\beta} - \frac{4\beta_{,i} \beta_{,j}}{\beta^2} + \left[ \frac{g^{\lambda \mu} \beta_{\alpha \lambda} \beta_{\alpha \mu}}{\beta^2} + \frac{g^{\lambda \mu} \beta_{\alpha \lambda} \beta_{\alpha \mu}}{\beta^2} \right] g_{ij}
$$

$$
- \frac{1}{2} \left[ R + \frac{6g^{\mu \nu} \beta_{\lambda \mu} \beta_{\lambda \nu}}{\beta} \right] g_{ij}
$$

(4.10)

$$
\tilde{G}_{ij} = G_{ij} + \frac{2\beta_{,ij}}{\beta} - \frac{4\beta_{,i} \beta_{,j}}{\beta^2} + \left[ \frac{g^{\lambda \mu} \beta_{\alpha \lambda} \beta_{\alpha \mu}}{\beta^2} - \frac{2g^{\lambda \mu} \beta_{\alpha \lambda} \beta_{\alpha \mu}}{\beta} \right] g_{ij}
$$

(4.11)

### 4.3 Scale-invariant Action Principle

Now we shall construct a generalized variation principle from the ideas that led to the generalized field equation of scale-invariant theory of gravitation. Since, as we have seen, the equations are scale as well as coordinate invariant, the Lagrangian density must be an in-scalar. In Einstein units, the natural gauge $k_i = 0; \phi = \text{constant}$ is used. For any other system of units, the gauge must be changed, and the gauge induced by such a change of units. Thus, in general the metric potential $\phi$ must be written as

$$\phi = -\ln \beta$$
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and

\[ k_i = \frac{\beta}{\delta x^i} (\ln \beta) = -\frac{P_i}{\beta} \]

where \( \beta \) is the scale factor between the units being used and the Einstein units. Thus we can write the action principle of gravitational field with adding a constant term to the scalar in this form

\[ S_g = -\frac{c^3}{16\pi G} \int (\hat{R} + 2\Lambda)\sqrt{-\tilde{g}}\,d^4x \]  

(4.12)

where \( \Lambda \) is a new cosmological constant.

Thus we may write

\[ \delta S_g = -\frac{c^3}{16\pi G} \delta \int \hat{R}\sqrt{-\tilde{g}}\,d^4x - \frac{c^32\Lambda}{16\pi G} \delta \int \sqrt{-\tilde{g}}\,d^4x \]

(4.13)

We have, for the first integral of R.H.S of (4.13)

\[
\delta \int \hat{R}\sqrt{-\tilde{g}}\,d^4x = \int \left( \hat{R}_{ij} - \frac{1}{2} \hat{R}g_{ij} \right) \delta g^{ij}\sqrt{-\tilde{g}}\,d^4x \\
= \int \left( \hat{R}_{ij} - \frac{1}{2} \hat{R}g_{ij} \right) \delta \left( \frac{1}{\beta^2} g^{ij} \right) \sqrt{-\tilde{g}}\,d^4x \\
= \int d^4x \sqrt{-\tilde{g}} \left( \hat{R}_{ij} - \frac{1}{2} \hat{R}g_{ij} \right) \left( -2\beta \beta^1 \delta g^{ij} + 1 \frac{\delta \beta}{\beta^2} \delta g^{ij} \right) \beta^4 \\
= \int d^4x \sqrt{-\tilde{g}} \beta^2 \left[ \hat{R}_{ij} - \frac{1}{2} \hat{R}g_{ij} + \frac{2\beta \beta^1 \delta g^{ij}}{\beta^2} - \frac{4\beta \delta \beta^1 \delta g^{ij}}{\beta^2} + \left( \frac{g^{ij} \beta^1 \beta^1 \delta g^{ij}}{\beta^2} - \frac{2g^{\lambda \mu} \beta \beta^1 \delta g^{ij}}{\beta} \right) g_{ij} \right] \delta g^{ij} \\
- 2 \int d^4x \sqrt{-\tilde{g}} \beta^2 \left[ g^{ij} \hat{R}_{ij} - \frac{1}{2} \hat{R} + \frac{2g^{ij} \beta^1 \beta^1 \delta g^{ij}}{\beta^2} - \frac{4g^{ij} \beta \delta \beta^1 \delta g^{ij}}{\beta^2} + \frac{g^{ij} \beta^1 \beta^1 \delta g^{ij}}{\beta^2} - \frac{2g^{\lambda \mu} \beta \beta^1 \delta g^{ij}}{\beta} \right] \delta \beta \\

\text{(4.14)}
\]

Second integral of the R.H.S of (4.13)

\[
\delta \int \sqrt{-\tilde{g}}\,d^4x = \int d^4x \delta \left( \sqrt{-\tilde{g}}\beta^4 \right) \\
= \int d^4x \left[ -\frac{1}{2} \sqrt{-\tilde{g}} g_{ij} \delta g^{ij} \beta^4 + \sqrt{-\tilde{g}} \beta^2 \delta \beta \right] \\
= \int d^4x \sqrt{-\tilde{g}} \left\{ -\frac{1}{2} g_{ij} \beta^4 \delta g^{ij} + 4\beta^2 \delta \beta \right\} \\
\text{(4.15)}
\]

Therefore

\[ \delta S_g = -\frac{c^3}{16\pi G} \int d^4x \sqrt{-\tilde{g}}\beta^2 \left\{ G_{ij} + \frac{2\beta \beta^1 \delta g^{ij}}{\beta^2} - \frac{4\beta \delta \beta^1 \delta g^{ij}}{\beta^2} + \left( \frac{g^{ij} \beta^1 \beta^1 \delta g^{ij}}{\beta^2} - \frac{2g^{\lambda \mu} \beta \beta^1 \delta g^{ij}}{\beta} \right) g_{ij} - \Lambda \beta^2 g_{ij} \right\} \delta g^{ij} \]
For the variation of the action of the matter we can write immediately
\[ \delta S_m = \frac{1}{2c} \int T_{ij} \delta g^{ij} \sqrt{-g} d^4x \]
i.e.
\[ \delta S_m = \frac{1}{2c} \int d^4x \sqrt{-g} \beta^2 \left( -\frac{2\beta \delta}{\beta} g^{ij} + \delta g^{ij} \right) T_{ij} \] (4.17)
where \( T_{ij} \) is the energy momentum tensor as in conventional theory. Thus from the principle of least action
\[ \delta S_g + \delta S_m = 0 \]
Therefore, from (4.16) and (4.17)
\[
-\frac{c^3}{16\pi G} \int d^4x \sqrt{-g} \beta^2 \left[ G_{ij} + \frac{2\beta \delta}{\beta} - \frac{4\beta \delta}{\beta^2} + \left( \frac{g^{ij} \beta \delta \lambda}{\beta^2} - \frac{2g^{ij} \beta \delta \lambda}{\beta} \right) g_{ij} - \Lambda \beta^2 g_{ij} - \frac{8\pi G}{c^4} T_{ij} \right]
+ \frac{c^3}{8\pi G} \int d^4x \sqrt{-g} \beta \left( g^{ij} R_{ij} - \frac{1}{2} R + \frac{2\beta \delta}{\beta} - \frac{4\beta \delta}{\beta^2} + \frac{g^{ij} \beta \delta \lambda}{\beta^2} - \frac{2g^{ij} \beta \delta \lambda}{\beta} - \frac{2g^{ij} \beta \delta \lambda}{\beta} - 4\beta^2 + \frac{8\pi G}{c^4} \right) T_{ij} \] (4.18)
Since \( g_{ij} \) and \( \beta \) independently varying, we can write
\[ G_{ij} + \frac{2\beta \delta}{\beta} - \frac{4\beta \delta}{\beta^2} + \left( \frac{g^{ij} \beta \delta \lambda}{\beta^2} - \frac{2g^{ij} \beta \delta \lambda}{\beta} \right) g_{ij} + \Lambda_0 \beta^2 g_{ij} = \frac{8\pi G}{c^4} T_{ij} \] (4.19)
where \( \Lambda_0 = -\Lambda \), and
\[ \beta R - 12g^{ij} \beta \delta_{ij} - 8\beta^3 + \frac{16\pi G}{c^4} \beta g^{ij} T_{ij} = 0. \] (4.20)

4.4 Discussion

In the preceeding section, we have derived the Scale-invariant form of Einstein equations. We now perform a special scale-transformation on the length \( ds \) by
\[ ds_E = \beta(t) ds, \]
or equivalently a conformal transformation of the type
\[ \bar{g}_{ij} = \beta^2(t) g_{ij}, \]
where $\beta(t)$ is for the time being an arbitrary gauge function. Then Einstein equations have been shown to change to

$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi G(\beta) T_{ij} - f_{ij}(\beta)$$

(4.21)

where

$$f_{ij}(\beta) = \frac{2\beta_{ij}}{\beta} - \frac{4\beta_{ij}\beta_{ij}}{\beta^2} + \left( \frac{g^{\lambda\mu} \beta_{\lambda\mu}}{\beta^2} - \frac{2\beta^{\lambda\mu} \beta_{\lambda\mu}}{\beta} \right),$$

where $G$ is no longer is constant, but an unknown function of $\beta$. The function $f_{ij}(\beta)$ is a new term originating from the transformation of the Einstein tensor under a conformal transformation.

It can be shown that equation (4.21) is form invariant. That is, a new change in scale leaves the equation unaltered. In the preceding section, also it is shown how to derive equation (4.21) from an action principle, whose terms are constructed to be scale-invariant. The scale function that plays the role of scalar field is here represented by $j_3(t)$. Contrary to the case of Brans and Dicke, we can not have a dynamic equation yielding $\beta(t)$, since by definition it presents the freedom we have in choosing the gauge.

As seen from the previous treatment, the cosmological constant $\Lambda$ in the present theory is not constant at all. In fact, it must scale like $\beta^2$. At the same time, it has also provided the way to determine the scale factor $\beta(t)$ in connection with cosmological constant $\Lambda$, since $\Lambda$ is a function of both $G$ and microscopic constants.

4.5 Conclusion

We see that in the literature, repeated attempts to modify the Einstein's theory have in the long run encountered several difficulties and have been abandoned. Every time this happens, one becomes, and correctly so, more confident about General Theory of Relativity. So, we should be careful with observational facts. We are, however, concerned with cosmology, in which the situation is different. It is therefore, sensible to consider a theory which would be an amplification of Einstein's theory, only for the cosmological application.

Throughout this chapter, we gave some reasons to justify studing a scale-invariant theory of gravitation. The theory seems to be improve the cosmological aspects of Einstein's equations leaving, however, the local predictions unchanged. So we would like to propose the study of the scale-invariant theory of gravitation in the next chapter.